Viewing the Proton Through "Color"-Filters

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While the form factors and parton distributions provide separately the shape of the proton in coordinate and momentum spaces, a more powerful imaging of the proton structure can be obtained through phase-space distributions. Here we introduce the Wigner-type quark and gluon distributions which depict a full-3D proton at every fixed light-cone momentum, like what is seen through momentum("color")-filters. After appropriate phase-space reductions, the Wigner distributions are related to the generalized parton distributions (GPD's) and transverse-momentum dependent parton distributions, which are measurable in high-energy experiments. The new interpretation of GPD's provides a classical way to visualize the orbital motion of the quarks, which is known to be the key to the spin and magnetic moment of the proton.

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What is the shape of the proton? The textbook answer is that it is round [1]. Indeed, the proton is a spin-1/2 particle, and the strong interactions are invariant under time-reversal symmetry. Hence there cannot be any dipole, quadrupole or higher-multiple deformations, and the quark and gluon density distributions in space have to be spherically symmetric. It is well known that the Fourier transformation of the proton's electric form factor yields the *spherical* charge distribution when the relativistic recoil effects are neglected [1].

The classical answer is correct, but might not be complete. The quarks and gluons are not static inside the proton, they are moving relativistically to produce the magnetic moment and part of the angular momentum (spin) of the proton. The orbital motion has an axial, rather than spherical, symmetry when the proton spin is polarized along a fixed direction [3]. To be sure, the Feynman distributions do provide a momentum space picture of partons, but there is no spatial density information involved— they are just *correlation* functions in the position space. To gain a more complete understanding of the proton's structure, it is useful to know the quarks' and gluons' joint-position-and-momentum (or phase-space) distributions. Indeed, in classical physics, a state of a particle is specified by knowing both its position x and momentum p. In a gas of classical identical particles, the single-particle properties are described by a phase-space distribution $f(\vec{r},\vec{p})$ representing the density of particles at phase-space positions $(\vec{r},\vec{p})$. Time-dependence of the distribution is governed by the Boltzmann equation.

In quantum mechanics, however, the notion of a phase-space distribution directly contradicts one of its fundamental principles: the momentum and position of a particle cannot be determined simultaneously. Nonetheless, over the years, physicists have introduced various quantum phase-space distributions which reduce to $f(\vec{r},\vec{p})$ in the classical limit. These distributions have proven extremely useful in wide-spread areas like heavy-ion collisions, quantum molecular dynamics, signal analysis, quantum information, non-linear dynamics, optics, image processing, etc. One of the most frequently used is the Wigner distribution [2]

$$W(x,p) = \int \psi^*(x-\eta/2)\psi(x+\eta/2)e^{ip\eta}d\eta \ , \qquad (1)$$

where $\psi(x)$ is a quantum wave function ($\hbar = 1$). When integrating out the coordinate x, one gets the momentum density $|\psi(p)|^2$, and when integrating out p, the coordinate space density $|\psi(x)|^2$ follows. For arbitrary p and x, the Wigner distribution is not positive definite and does not have, strictly speaking, a probability interpretation. However, there is no doubt that it contains much more information than the usual quantum mechanical observables. Modifed Wigner distributions with positivity are known in the literature and are better interpreted as densities, but we leave them for a further study.

In this paper, we introduce the quark and gluon Wigner-type phase-space distributions in hadrons such as the proton. After certain phase-space reductions, these distributions reduce to the generalized parton distributions (GPD's) and the transverse-momentum dependent parton distributions which are measurable in high-energy experiments. [For reviews of these distributions and experiments to measure them, see [4, 5].] Their relation to the GPD's reveals a new interpretation of the latter— the Fourier transformation of GPD's yields full-3D quark and gluons images of the proton at every fixed Feynman momentum x, like what's seen through momentum (or better yet "color" or x-) filters. The skewness parameter ξ in GPD's now has a natural interpretation: a measure of the proton deformation along the direction selected by high-energy probes. From the sign variation of the images, we might have a practical way to determine how "classical" the quarks and gluons are in the proton.

Disregarding the renormalization scale μ, the GPD's depend on three kinematic variables x (Feynman momentum), ξ (skewness), and q^2 (t-channel momentum transfer, also called t). Introduced as quantum interference amplitudes, their original interpretation as parton and their angular momentum densities was limited to $t = \xi = 0$ [6, 7]. In a seminal paper by M. Burkardt [8], the probability interpretation was successfully extended to the GPD's at $t \neq 0$: the Fourier transformation of the parton densities in the two-dimensional transverse plane (impact-parameter space). A number of subsequent papers have studied the physical significance of the $\xi \neq 0$ case [9], without seeking a classical or quasi-classical interpretation. The present work expands the 2D-pictures of Burkardt along the longitudinal direction, relating the GPD's to full-3D distributions at a fixed light-cone momentum, interpretable as densities in the classical limit.

We begin by introducing the Wigner distribution operator for quarks in QCD

$$\hat{\mathcal{W}}_\Gamma(\vec{r}, k) = \int \overline{\Psi}(\vec{r} - \eta/2)\Gamma\Psi(\vec{r} + \eta/2)e^{ik\cdot\eta}d^4\eta \ , \quad (2)$$

where $\vec{r}$ is the quark phase-space position and k the phase-space four-momentum. Γ is a Dirac matrix defining the types of quark denstities. Ψ is a *gauge-invariant* quark field in non-singular gauges (gauge potentials vanish at the spacetime infinity [10]),

$$\Psi(\eta) = \exp\left(-ig\int_0^\infty d\lambda n \cdot A(\lambda n + \eta)\right)\psi(\eta) \ , \quad (3)$$

where n^μ is a constant four-vector. Thus $\hat{\mathcal{W}}_\Gamma$ is gauge-invariant but depends on a choice of n^μ. The actual Wigner distribution can be define as the expectation value of $\hat{\mathcal{W}}_\Gamma$ in the hadron states. For example,

$$\begin{aligned} W_\Gamma(\vec{r}, k) &= \frac{1}{2M}\int \frac{d^3\vec{q}}{(2\pi)^3}\left\langle \vec{q}/2\left|\hat{\mathcal{W}}(\vec{r}, k)\right| - \vec{q}/2\right\rangle \quad (4) \\ &= \frac{1}{2M}\int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left\langle \vec{q}/2\left|\hat{\mathcal{W}}(0, k)\right| - \vec{q}/2\right\rangle \ , \end{aligned}$$

where M is the hadron mass and the states are normalized relativistically. The initial and final hadrons are taken with different center-of-mass momenta, otherwise translational invariance results in a trivial dependence on the phase-space position $\vec{r}$ [3]. For non-relativistic quantum systems, the above definition readily reduces to the known Wigner distribution. For relativistic systems such as the proton, the above interpretation as a density neglects relativistic effects when the momenta are comparable to the proton mass [8]. Fortunately, since the proton mass is of order 1 GeV and charge radius 0.86 fm, the relativistic correction is about %10.

For the proton, because of the high energy involved in experiments and because of the color confinement, the $k^- = (k^0 - k^z)/\sqrt{2}$ energy distribution is difficult to measure, where the z-axis refers to the momentum direction of a probe. Moreover, the leading observables are associated with the "good" components of the quark (gluon) fields in the sense of light-cone quantization [11], which can be selected by $\Gamma = \gamma^+$, $\gamma^+\gamma_5$, or $\sigma^{+\perp}$ where $\gamma^+ = (\gamma^0+\gamma^z)/\sqrt{2}$. The direction of the gauge link, n^μ, is then determined by the trajectories of high-energy partons traveling along the light-cone $(1, 0, 0, -1)$. Therefore, from now on, we restrict ourselves to the reduced Wigner distributions integrating over k^-,

$$W_\Gamma(\vec{r}, \vec{k}) = \int \frac{dk^-}{(2\pi)^2}\mathcal{W}_\Gamma(\vec{r}, k) \ , \quad (5)$$

with a light-cone gauge link. Eq. (5) defines the most general (master) phase-space quark distributions relevant in high-energy processes. Unfortunately, there is no known experiment at the moment capable of measuring them in the full 6-dimensional phase space.

However, certain phase-space reductions of the distributions are measurable through known processes. For example, integrating out the transverse momentum of partons, we find

$$\begin{aligned} \tilde{f}_\Gamma(\vec{r}, k^+) &= \int \frac{d^2\vec{k}_\perp}{(2\pi)^2} \ W_\Gamma(\vec{r}, \vec{k}) \\ &= \frac{1}{2M}\int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\int \frac{d\eta^-}{2\pi}e^{i\eta^- k^+} \\ &\quad \times \left\langle \vec{q}/2\left|\overline{\Psi}(-\eta^-/2)\Gamma\Psi(\eta^-/2)\right| - \vec{q}/2\right\rangle \quad (6) \end{aligned}$$

The matrix element under the integrals is what defines the GPD's. More precisely, if one replaces k^+ by Feynman variable xP^+ ($P^+ = E_q/\sqrt{2}$, proton energy $E_q = \sqrt{M^2 + \vec{q}^2/4}$) and η^- by λ/P^+, the reduced Wigner distribution becomes the Fourier transformation of the GPD $F_\Gamma(x, \xi, t)$

$$f_\Gamma(\vec{r}, x) = \frac{1}{2M}\int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}F_\Gamma(x, \xi, t) \ . \quad (7)$$

In the present context, the relation between kinematic variables are $\xi = q_z/(2E_q)$ and $t = -\vec{q}^{\ 2}$. Taking $\Gamma = \sqrt{2}\gamma^+$, the corresponding GPD has the expansion [6]

$$\begin{aligned} & F_{\gamma^+}(x, \xi, t) \\ &= \int \frac{d\lambda}{2\pi}e^{i\lambda x}\left\langle \vec{q}/2\left|\overline{\psi}(-\lambda n/2)\mathcal{L}\sqrt{2}\gamma^+\psi(\lambda n/2)\right| - \vec{q}/2\right\rangle \\ &= H(x, \xi, t)\overline{U}(\vec{q}/2)\sqrt{2}\gamma^+U(-\vec{q}/2) \\ &\quad + E(x, \xi, t)\overline{U}(\vec{q}/2)\frac{i\sigma^{+i}q_i}{\sqrt{2}M}U(-\vec{q}/2) \ , \quad (8) \end{aligned}$$

where $\mathcal{L}$ is the shorthand for the light-cone gauge link.

The distribution $f_{\gamma^+}(\vec{r}, x)$ can be interpreted as the 3D density in the rest frame of the proton, for the quarks with a selected light-cone momentum x. Integrating over the z coordinate, the GPD's are set to $\xi \sim q^z = 0$, and the resulting two-dimensional density $f_{\gamma^+}(\vec{r}_\perp, x)$ is just the impact-parameter space distribution in the Burkardt paper [8].

To understand better the physical content of the density, let us examine its spin structure. Working out the matrix element in Eq. (8),

$$\begin{aligned} \frac{1}{2M}F_{\gamma^+}(x, \xi, t) &= [H(x, \xi, t) - \tau E(x, \xi, t)] \\ &\quad + i(\vec{s} \times \vec{q})^z\frac{1}{2M}\left[H(x, \xi, t) + E(x, \xi, t)\right] \ , \quad (9) \end{aligned}$$

where $\tau=(1+\xi)\vec{q}^2/4M^2+\xi$. The first term is independent of the proton spin, and the related quark density is

$$\rho_+(\vec{r},x)=\int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}[H(x,\xi,t)-\tau E(x,\xi,t)] \ . \quad (10)$$

Just like charge density, it is not positive-definite at any x. The second term depends on the proton spin, the corresponding density is physically the 3rd-component of a current

$$j^z_+(\vec{r},x) \ = \ \int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}i(\vec{s}\times\vec{q})^z \times\frac{1}{2M}\left[H(x,\xi,t)+E(x,\xi,t)\right] \ . \quad (11)$$

The H-term current generates the proton's Dirac moment, and the E-term generates a convection current due to the orbital angular momentum of massless quarks and vanishes when all quarks are in the s-orbit[12]. The physics in separating f^+_γ into ρ_+ and j^z_+ can be seen from the Dirac matrix γ^+ selected by the high-energy probes, which is a combination of time and space components. Because the current distribution has no spherical symmetry, the quark density seen in the infinite momentum frame, $\rho_+ + j^z_+$, is deformed in the impact parameter space [13]. This is the kinematic effect of Lorentz transformations.

We can also study the x-moments of the quark and current densities. If we integrate over x directly, we obtain the electric charge and current densities,

$$\begin{aligned}\rho^e_+(\vec{r}) \ &= \ \int dx\rho_+(\vec{r},x)\\ &= \ \int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}(F_1(q^2)-\tau F_2(q^2)) \ ,\\ \vec{j}(\vec{r}) \ &= \ \int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}i(\vec{s}\times\vec{q})\frac{1}{2M}(F_1(q^2)+F_2(q^2)) \ .\end{aligned}$$

Notice that ρ^e_+ is not spherically symmetric due to the ξ dependence in τ, reminiscent of the γ^+-probe. The electric current density has a donut shape and is responsible for the magnetic moment of the proton [3]. Indeed, if we calculate the proton's magnetic moment using the classical formula, $\vec{\mu}=(1/2)\int\vec{r}\times\vec{j}$, the right answer follows. The current generated by the $F_2(q^2)$ term is directly related to the orbital motion of the quarks [14].

Forming the second moment of x, we have the *mass distribution* of the proton from the quarks [15],

$$\begin{aligned}\rho^m_+(\vec{r}) \ &= \ \int dxx\rho_+(\vec{r},x)\\ &= \ \int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}\left[(A_q(q^2)+\xi^2C_q(q^2))\right.\\ &\qquad \left. -\ \tau(B_q(q^2)-\xi^2C_q(q^2)\right] \ , \quad (12)\end{aligned}$$

where $A_q(q^2)$, $B_q(q^2)$ and $C_q(q^2)$ are the quark part of the proton's energy-momentum form factors [6, 16],

$$\begin{aligned}\langle P'|T^{\mu\nu}|P\rangle \ &= \ \overline{U}(P')\left[A(q^2)\overline{P}^\mu\gamma^\nu+B(q^2)\frac{i\sigma^{\mu\alpha}q_\alpha P^\nu}{2M}\right.\\ &\qquad \left.+C(q^2)(q^\mu q^\nu-g^{\mu\nu}q^2)\right]U(P) \ . \quad (13)\end{aligned}$$

Here the energy-momentum tensor $T^{\mu\nu}$ is symmetric and conserved, and contain both quark and gluon contributions. The normalization of the form factors $A(0)$ and $B(0)$ are well-known: $A(0)=1$ is related to the energy-momentum conservation and $B(0)=0$ is a result of the angular momentum conservation [6, 17].

The energy-momentum form factors can *in principle* be measured in elastic graviton-nucleon scattering. Here we use this fact to motivate the physical interpretation of $\rho^m_+(\vec{r})$ as the proton's mass distribution. Gravitions have spin 2, and an on-shell massless graviton has helicity $\lambda=\pm2$. An off-shell (massive) graviton can have additional helicity states $\lambda=\pm1$ and 0. Consider the graviton-proton scattering in the Breit frame. Assume the helicity of the initial proton is 1/2, then the final proton has helicity $1/2-\lambda$. Since the proton has spin-1/2, only off-shell gravitons of helicity 1 and 0 contribute. A graviton with helicity 1 and 0 acts very much like a photon, and we can mimic the Sachs's electromagnetic form factors by defining the gravitational "electric" (or mass) form factor

$$G^m_E(q^2)=A(q^2)+\frac{t}{4M^2}B(q^2) \ , \quad (14)$$

proportional to the helicity non-flip graviton-nucleon scattering amplitude [18, 19]. If one ignores the deformation due to the probes ($\xi=0$), $\rho^m_+(\vec{r})$ in Eq. (12) is just the Fourier transformation of the above form factor.

The x-moment of the quark current is the *momentum density* in the proton

$$\begin{aligned}j^{zp}(\vec{r}) \ &= \ \frac{M}{2}\int dxxj^z_+(\vec{r},x)\\ &= \ \int \frac{d^3\vec{q}}{(2\pi)^3}e^{-i\vec{q}\cdot\vec{r}}i(\vec{s}\times\vec{q})^z\\ &\qquad \times\frac{1}{4M}\left[A_q(q^2)+B_q(q^2)\right] \ . \quad (15)\end{aligned}$$

The combination in the integrand is the gravitational "magnetic" (or angular momentum) form factor,

$$G^m_M(q^2)=A(q^2)+B(q^2) \ , \quad (16)$$

proportional to the helicity-flip graviton-proton scattering amplitude [18, 19]. Knowing the momentum density as a function of $\vec{r}$, it is simple to calculate the quark contribution to the angular momentum (spin) of the proton according to $\int\vec{r}\times\vec{j}^p$. The answer is $(A_q(0)+B_q(0))/2$ [6]. A contribution from the gluons can be obtained in a similar way.

Let us remark that gravity plays a dominant role at two extreme distance scales: the cosmic scale and the Plank scale. At the atomic and subatomic levels, with the exception of the motion of these particles under strong gravitational fields, gravitation has little effects because of its extremely weak coupling. Thus while mass distribution and moment-of-inertia are important concepts in classical mechanics, they have rarely been discussed for microscopic quantum systems such as the proton. Thus, it is very interesting that the GPD's provide gravitational form factors without actual gravitational scattering [6, 18]. A study of the gravitational Sachs's form factors (and gravitational radius) in chiral perturbation theory can be found in [19]. A related coordinate-space interpretation of the gravitational form factors, in particular, the stress density T^{ij}, can be found in Ref. [20].

If integrating over $\vec{r}$ in the Wigner distributions in Eq. (5), one obtains the transverse-momentum dependent parton distributions. There is a lot of interesting physics associated with these distributions which has been discussed recently. For instance, in a transversely polarized proton, the quark momentum distribution has an azimuthal angular dependence [5, 10, 21]. The so-called Siver's function can produce a novel single-spin asymmetry in deep-inelastic scattering. We will not pursue this topic here, except emphasizing that they have the same generating functions as the GPD's.

The phase-space distributions in the transverse coordinates have also been used in small-x physics [22], where the parton physics becomes truly classical at high density.

We end the paper by making a few remarks. First, as we have mentioned before, the Wigner distributions are quantum phase-space distributions which contain much more information of a quantum system than conventional observables. Originally introduced as a theoretical construction, it is surprising that these distributions for the proton are directly measurable through high-energy processes. Second, the Wigner distributions are more than just quantum interference amplitudes; they have a clear physical interpretation in the classical limit. This endows the GPD's a new physics feature which has not been recognized before. QCD in the limit of large number of colors might be a case in which the Wigner distributions become truly classical densities. Finally, the extent to which the Wigner distributions in the proton can be regarded as classical densities may be judged once the phenomenological GPD's is available from experiment data.

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