A Course in Metric Geometry

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Preface

This book is not a research monograph or a reference book (although research interests of the authors influenced it a lot)—this is a textbook. Its structure is similar to that of a graduate course. A graduate course usually begins with a course description, and so do we.

Course description. The objective of this book is twofold. First of all, we wanted to give a detailed exposition of basic notions and techniques in the theory of length spaces, a theory which experienced a very fast development in the past few decades and penetrated into many other mathematical disciplines (such as Group Theory, Dynamical Systems, and Partial Differential Equations). However, we have a wider goal of giving an elementary introduction into a broad variety of the most geometrical topics in geometry—the ones related to the notion of distance. This is the reason why we included metric introductions to Riemannian and hyperbolic geometries. This book tends to work with "easy-to-touch" mathematical objects by means of "easy-to-visualize" methods. There is a remarkable book [**Gro3**], which gives a vast panorama of "geometrical mathematics from a metric viewpoint". Unfortunately, Gromov's book seems hardly accessible to graduate students and non-experts in geometry. One of the objectives of this book is to bridge the gap between students and researchers interested in metric geometry, and modern mathematical literature.

Prerequisite. It is minimal. We set a challenging goal of making the core part of the book accessible to first-year graduate students. Our expectations of the reader's background gradually grow as we move further in the book. We tried to introduce and illustrate most of new concepts and methods by using their simplest case and avoiding technicalities that take attention away from the gist of the matter. For instance, our introduction to Riemannian geometry begins with metrics on planar regions, and we even avoid the notion of a manifold. Of course, manifolds do show up in more advanced sections. Some exercises and remarks assume more mathematical background than the rest of our exposition; they are optional, and a reader unfamiliar with some notions can just ignore them. For instance, solid background in differential geometry of curves and surfaces in $\mathbb{R}^3$ is not a mandatory prerequisite for this book. However, we would hope that the reader possesses some knowledge of differential geometry, and from time to time we draw analogies from or suggest exercises based on it. We also make a special emphasis on motivations and visualizations. A reader not interested in them will be able to skip certain sections. The first chapter is a clinic in metric topology; we recommend that the reader with a reasonable idea of metric spaces just skip it and use it for reference: it may be boring to read it. The last chapters are more advanced and dry than the first four.

Figures. There are several figures in the book, which are added just to make it look nicer. If we included all necessary figures, there would be at least five of them for each page.

• It is a must that the reader systematically studying this book makes a figure for every proposition, theorem, and construction!

Exercises. Exercises form a vital part of our exposition. This does not mean that the reader should solve all the exercises; it is very individual. The difficulty of exercises varies from trivial to rather tricky, and their importance goes all the way up from funny examples to statements that are extensively used later in the book. This is often indicated in the text. It is a very helpful strategy to perceive *every* proposition and theorem as an exercise. You should try to prove each on your own, possibly after having a brief glance at our argument to get a hint. Just reading our proof is the last resort.

Optional material. Our exposition can be conditionally subdivided into two parts: core material and optional sections. Some sections and chapters are preceded by a brief plan, which can be used as a guide through them. It is usually a good idea to begin with a first reading, skipping all optional sections (and even the less important parts of the core ones). Of course, this approach often requires going back and looking for important notions that were accidentally missed. A first reading can give a general picture of the theory, helping to separate its core and give a good idea of its logic. Then the reader goes through the book again, transforming theoretical knowledge into the genuine one by filling it with all the details, digressions, examples and experience that makes knowledge practical.

About metric geometry. Whereas the borderlines between mathematical disciplines are very conditional, geometry historically began from very "down-to-earth" notions (even literally). However, for most of the last century it was a common belief that "geometry of manifolds" basically boiled down to "analysis on manifolds". Geometric methods heavily relied on differential machinery, as it can be guessed even from the name "Differential geometry". It is now understood that a tremendous part of geometry essentially belongs to metric geometry, and the differential apparatus can be used just to define some class of objects and extract the starting data to feed into the synthetic methods. This certainly cannot be applied to all geometric notions. Even the curvature tensor remains an obscure monster, and the geometric meaning of only some of its simplest appearances (such as the sectional curvature) are more or less understood. Many modern results involving more advanced structures still sound quite analytical. On the other hand, expelling analytical machinery from a certain sphere of definitions and arguments brought several major benefits. First of all, it enhanced mathematical understanding of classical objects (such as smooth Riemannian manifolds) both ideologically, and by concrete results. From a methodological viewpoint, it is important to understand what assumptions a particular result relies on; for instance, in this respect it is more satisfying to know that geometrical properties of positively curved manifolds are based on a certain inequality on distances between quadruples of points rather than on some properties of the curvature tensor. This is very similar to two ways of thinking about convex functions. One can say that a function is convex if its second derivative is nonnegative (notice that the definition already assumes that the function is smooth, leaving out such functions as $f(x) = |x|$). An alternative definition says that a function is convex if its epigraph (the set $\{(x, y) : y \geq f(x)\}$) is; the latter definition is equivalent to Jensen's inequality $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$ for all nonnegative α, β with $\alpha + \beta = 1$, and it is robust and does not rely on the notion of a limit. From this viewpoint, the condition $f'' \geq 0$ can be regarded as a convenient criterion for a smooth function to be convex.

As a more specific illustration of an advantage of this way of thinking, imagine that one wants to estimate a certain quantity over all metrics on a sphere. It is so tempting to study a metric for which the quantity attains its maximum, but alas this metric may fail exist within smooth metrics, or even metrics that induce the same topology. It turns out that it still may exist if we widen our search to a class of more general length spaces. Furthermore, mathematical topics whose study used to lie outside the range of noticeable applications of geometrical technique now turned out to be traditional objects of methods originally rooted in differential geometry. Combinatorial group theory can serve as a model example of this situation. By now the scope of the theory of length spaces has grown quite far from its cradle (which was a theory of convex surfaces), including most of classical Riemannian geometry and many areas beyond it. At the same time, geometry of length spaces perhaps remains one of the most "hands-on" mathematical techniques. This combination of reasons urged us to write this "beginners' course in geometry from a length structure viewpoint".

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"Metric geometry" is an approach to geometry based on the notion of length on a topological space. This approach experienced a very fast development in the last few decades and penetrated into many other mathematical disciplines, such as group theory, dynamical systems, and partial differential equations.

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