International Journal of Pure and Applied Mathematics
Volume 103 No. 3 2015, 557-565
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v103i3.16

ijpam.eu

NEW OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS

Xhevair Beqiri[1 §], Azir Jusufi[2]

[1,2]Department of Mathematics
State university of Tetova
1200, Tetova, REPUBLIC OF MACEDONIA

Abstract: In this paper we present criteria for oscillation of nonlinear differential equations of second order

$$(a(t)u'(t))' + p(t)f(u(g(t)) = 0, \tag{1}$$

where the coefficient *a(t)* is nonnegative, continuous function and $f(x)$, $g(x)$ are continuous functions which complete certain conditions.

Here we use generalized Riccati technique and the conclusion is also based on building functions where there are involved coefficients of equation (2) and also Philos functions $H(t,s) > 0$.

This criteria is based on the results of Blanka Bakulikova and get argumentum with the example in the end of this work paper.

AMS Subject Classification: 34C10, 34C15
Key Words: oscillation, differential equation, second order, interval, relation etc

1. Introduction

We take into consideration the nonlinear differential equation of second order

$$(a(t)u'(t))' + p(t)f(u(g(t))) = 0, \tag{2}$$

where:

Received: June 20, 2015

§Correspondence author

a) $a \in C(t_0, \infty)$, $a(t) > 0$, $\int_{t_0}^{\infty} a^{-1}(t)dt = \infty$;

b) $p(t) \in C(t_0, \infty), p(t) > 0$;

c) $f(x) \in C((-\infty, \infty))$, $xf(x) > 0$ for $x \neq 0$, $f \in C^1(R_{t_0})$, $R_{t_0} = (-\infty, -t_0) \bigcup (t_0, \infty)$, $t_0 > 0$;

d) $g(t) > 0$, $g(t) \in C^1((t_0, \infty))$,

and

$g(t) < t$, $t_0 \in R^+$, $g'(t) > 0$, $g(t) \to \infty$ when $t \to \infty$, for all t large enough.

We make standing hypothesis (2) possessing solution on (t_0, ∞). Solution of (2), we imply is a function $x(t)$, $t \in [t_x, \infty) \subset (t_0, \infty)$ which has derivate of second order continuously and fulfills (2) on interval $[t_x, \infty)$ where $t_x \geq t_0 \geq 0$. This solution *x(t)* is called oscillatory if it has a sequence of zeros tending to infinity, otherwise it is called nonoscillatory. Equation is oscillatory if all its solution are oscillatory.

The oscillation problem of equation (2) and for less general equations such as the linear differential equation:

$$x''(t) + p(t)x(t) = 0 \tag{3}$$

and nonlinear differential equation

$$x''(t) + p(t)f(x(g(t)) = 0, \tag{4}$$

have been discussed by numerous authors and by different methods (see [1]-[5]).

For equation (3), the condition which guaranties that every solution is oscillatory

$$\lim_{t\to\infty} \sup \frac{1}{t^{\lambda}} \int_{t_0}^{t} (t-s)^{\lambda} q(s)ds = \infty,$$

for $\lambda > 1$ (see Kamenev [7]).

For the equation (4) (see [3]):

$$\int_{t_0}^{\infty} \left[\int_{s_1}^{\infty} p(s)ds \right] ds_1 = \infty$$

and

$$\int_{t_0}^{\infty} \left(tp(t) - \frac{1}{4ktg'(t)} \right) dt = \infty, \quad f'(x) \geq k > 0.$$

For equation (2) if $a(t) = c$, c *is* constant, we have criteria presented in [3] and if $g(t) = t$ we have the criteria presented in [1].

2. Main Results

The following theorem present oscillatory criterion of equation (2).

Theorem 1. *Assume that a) – d) hold true. Let there exists* $k > 0$ *such that* $f'(x) > k$ *for all* $x \in R_{t_0}$. *If*

$$\int_{t_0}^{\infty} [\frac{1}{a(s_1)} \int_{s_1}^{\infty} p(s) ds] ds_1 = \infty \tag{5}$$

and

$$\int_{t_0}^{\infty} (g(t)p(t) - \frac{g'(t)a(t)}{g(t)4k}) dt = \infty \tag{6}$$

then equation (2) *is oscillatory.*

Proof. Assume that *u(t)* is a non-oscillatory solution of (2).
Let *u(t)>0*, then from (2) we have

$$(a(t)u'(t))' = -p(t)f(u(g(t)) < 0$$

and from that the function $a(t)u'(t)$is decreasing, then from a) the function $u'(t)$is decreasing and positive for $t \in (\tau, \infty)$, $\tau \geq t_0$ (see [9]).

We define

$$W(t) = \frac{a(t)u'(t)}{f(u(g(t))}, \quad t \in (t_0, \infty). \tag{7}$$

Differentiating $W(t)$ and using (2), we receive

$$\frac{dW(t)}{dt} = (\frac{a(t)u'(t)}{f(u(g(t))})'$$
$$= -p(t)g(t) + \frac{a(t)u'(t)g(t)}{f(u(g(t))} - \frac{W(t)f'(u(g(t))u'(g(t)g'(t)}{f(u(g(t))}.$$

From $g(t) < t$, since *u'(t)* is decreasing function, we come to see that:

$$u'(g(t)) \geq u'(t).$$

Consequently

$$\frac{dW(t)}{dt} \leq -p(t)g(t) + \frac{W(t)g'(t)}{g(t)} - \frac{W^2(t)f'(u(g(t))g'(t)}{g(t)a(t)}.$$

Following the results above and $f'(u(g(t)) > k$, we obtain

$$\frac{dW(t)}{dt} \leq -p(t)g(t) + \frac{W(t)g'(t)}{g(t)} - \frac{W^2(t)kg'(t)}{g(t)a(t)}. \tag{8}$$

Therefore

$$\frac{dW(t)}{dt} \leq -p(t)g(t) - \frac{kg'(t)}{a(t)g(t)}\{[W^2(t) - \frac{a(t)}{2k}]^2 - \frac{a^2(t)}{4k^2}\},$$

$$\frac{dW(t)}{dt} \leq -p(t)g(t) + \frac{a(t)g'(t)}{4kg(t)}. \tag{9}$$

Now we shall show that (5) implies $u(t) \to \infty$ as $t \to \infty$. On the contrary of the assumption that $u(t)$ is bounded from the above, that is $u(t) \in \langle \alpha, \beta \rangle$, where $\alpha > 0$.Using properties of $g(t)$, we assume that $u(g(t)) \in \langle \alpha, \beta \rangle$. While *u'(t)* is positive and decreasing, we have $\lim_{t\to\infty} u'(t)$ exists and it is finite. Integrating equation (2) from t to ∞, we obtain

$$u'(\infty)a(\infty) - u'(t)a(t) = -\int_t^\infty p(s)f(u(g(s))ds,$$

using property of *u'(t)*, we get :

$$u'(t)a(t) \geq \int_t^\infty p(s)f(u(g(s))ds,$$

$$u'(t) \geq \frac{1}{a(t)}\int_t^\infty p(s)f(u(g(s))ds.$$

Let $f_0 = \min_{u\in\langle\alpha,\beta\rangle} f(u)$, for $f' > 0$, we have $f_0 > 0$ because $u(t) > 0$.

Then

$$u'(t) \geq \frac{1}{a(t)}f_0\int_t^\infty p(s)ds.$$

Integrating this inequality from t_0 to t, we obtain

$$\beta \geq u(t) \geq f_0\int_{t_0}^t\left(\frac{1}{a(s_1)}\int_{s_1}^\infty p(s)ds\right)ds_1.$$

When $t \to \infty$ the last inequality comes into contradiction with (5). Therefore we conclude $u(t) \to \infty$ as $t \to \infty$. Thus $u(g(t)) \in R_{t_0}$, for all t large enough.

Now it is easy to see that the condition $f'(u(g(t)) \geq k$ implies (8).

Integrating this inequality from t_1 to t, we obtain

$$W(t) \leq W(t_1) - \int_{t_1}^{t} \left(p(t)g(t) - \frac{a(t)g'(t)}{4kg(t)} \right) dt,$$

from that when $t \to \infty$, we have $W(t) \to -\infty$. This is a contradiction, because *W(t)>0.*

For *u(t)<0* , this case can be treated similarly as the case *u(t)>0* and so it is omitted. The proof is complete. □

For $a(t) = 1$ we obtain the result presented in [3].

From Theorem 1. we can see an easy verifiable oscillation criteria for (2).

Corollary 1. *Assumed that a)-d) and* (5) *hold. Let there exist constant k>0 such that* $f'(x) \geq k$ *for all* $x \in R_{t_0}$ *and*

$$\liminf_{t\to\infty} \left\{ \frac{g^2(t)p(t)}{g'(t)a(t)} \right\} > \frac{1}{4k} \tag{10}$$

then equation (2) *is oscillatory.*

For $a(t) = l > 0$ *we have Corollary 2.5 in [3].*

Proof. A simple calculation shows that (9) implies (6). □

Corollary 2. *Let a) – d) and* (5) *hold true. If*

$$\int_{t_0}^{\infty} (tp(t) - \frac{a(t)}{4kt})dt = \infty \tag{11}$$

then equation

$$(a(t)u'(t))' + p(t)f(u(t)) = 0$$

is oscillatory.

Proof. It is easy to see that (6) may be reduced to (11), if $g(t) = t$. □

We say that a function $H = H(t, s)$ belongs to *function class* X, denote by $H \in X$, if $H \in C(D, R)$, where $D = \{(t, s), -\infty < s \leq t < \infty\}$, which completes $H(t, s) > 0$, for $t > s$, $H(t, t) = 0$ and has continuous partial derivatives on D such that $\frac{\partial H(t,s)}{\partial s} = -h_2(t, s)\sqrt{H(t, s)}$, $\frac{\partial H(t,s)}{\partial t} = h_1(t, s)\sqrt{H(t, s)}$.

Theorem 2. *Supposed that a)-d) and* $f'(x) \geq k > 0$ *holds for any* $t > t_0$. *If there exists* $(a, b) \subset [t_0, \infty)$, $c \in (a, b)$ *such that*

$$\frac{1}{H(c,a)}\int_a^c \{H(s,a)p(s)g(s) - \frac{g(s)a(s)}{4kg'(s)}\phi_1^2(s,a)\}ds$$
$$+\frac{1}{H(b,c)}\int_b^c \{H(b,c)p(s)g(s) - \frac{g(s)a(s)}{4kg'(s)}\phi_2^2(b,s)\}ds > 0 \quad (12)$$

where

$$\phi_1(s,t) = h_1(s,t) + \frac{g'(s)}{g(s)}\sqrt{H(s,t)},$$
$$\phi_1(t,s) = h_2(t,s) - \frac{g'(s)}{g(s)}\sqrt{H(t,s)},$$

then equation (2) *is oscillatory.*

Proof. Suppose the contrary: $x(t)$ is a nonoscillatory solution of equation (2), say $x(t) \neq 0$ on $[t_0, \infty)$ for some sufficient large $t > t_0$. From (8) if that multiplying by H(s,t) and integrating it over (t,c) for $t \in [a,c)$ it yields (for $s \in (t,c]$)

$$\int_t^c H(s,t)p(s)g(s)ds \leq \int_t^c H(s,t)W'(s)ds$$
$$-\int_t^c \frac{W^2(s)g'(s)H(s,t)}{g(s)a(s)}ds + \int_t^c \frac{W(s)g'(s)H(s,t)}{g(s)a(s)}ds$$
$$= -H(t,s)W(c) - \int_t^c \{\frac{W^2(s)kg'(s)}{g(s)a(s)}H(s,t) + \phi_1(s,t)\sqrt{H(s,t)}W(s)\}ds$$
$$= -H(t,s)W(c) - \int_t^c [W(s)\sqrt{\frac{kg'(s)H(s,t)}{g(s)a(s)}} + \frac{1}{2}\phi_1(s,t)\sqrt{\frac{g(s)a(s)}{kg'(s)}}]^2 ds$$
$$+\frac{1}{4}\int_t^c \phi_1^2(s,t)\frac{g(s)a(s)}{kg'(s)}ds$$
$$\leq -H(t,s)W(c) + \frac{1}{4}\int_t^c \phi_1^2(s,t)\frac{g(s)a(s)}{kg'(s)}ds \quad (13)$$

Letting in (13) $t \to a^+$ and dividing it by $H(b,c)$, we obtain

$$\frac{1}{H(b,a)}\int_a^c H(s,a)p(s)g(s)ds$$
$$\leq -W(c) + \frac{1}{4H(b,a)}\int_a^c \phi_1^2(s,a)\frac{g(s)a(s)}{kg'(s)}ds. \quad (14)$$

On other hand, if we multiply (8) by H(t,s) and integrating it over $[c,t)$ for $t \in [c,b)$ we have (for $s \in [c,t)$):

$$\int_c^t H(t,s)p(s)g(s)ds \leq \int_c^t H(t,s)W'(s)ds - \int_c^t \frac{W^2(s)g'(s)H(t,s)}{g(s)a(s)}ds$$

$$+\int_c^t \frac{W(s)g'(s)H(t,s)}{g(s)a(s)}ds$$
$$=-H(t,c)W(c)-\int_c^t\{\frac{W^2(s)kg'(s)}{g(s)a(s)}H(t,s)+\phi_2(s,t)\sqrt{H(t,s)}W(s)\}ds$$
$$\leq H(t,c)W(c)-\frac{1}{4}\int_c^t \phi_2^2(t,s)\frac{g(s)a(s)}{kg'(s)}ds \tag{15}$$

Letting $t\to b^-$ in (15) and dividing it by $H(b,c)$ we get

$$\frac{1}{H(b,c)}\int_c^b H(b,s)p(s)g(s)ds$$
$$\leq W(c)-\frac{1}{4H(b,c)}\int_c^b \phi_2^2(b,s)\frac{g(s)a(s)}{kg'(s)}ds. \tag{16}$$

Adding equations (14) to (16), we obtain the following inequation

$$\frac{1}{H(c,a)}\int_a^c\{p(s)g(s)H(s,a)-\frac{g(s)a(s)}{4kg'(s)}\phi_1^2(s,a)\}ds$$
$$+\frac{1}{H(b,c)}\int_c^b\{p(s)g(s)H(b,s)-\frac{g(s)a(s)}{4kg'(s)}\phi_2^2(b,s)\}ds\leq 0. \tag{17}$$

The last inequation is a contradiction to the condition (12), therefore, every solution of equation (2) is oscillatory. The proof is complete. □

If $H(t,s)=H(t-s)$, then $h_1(t-s)=h_2(t-s)\overset{def}{=}h(t-s)$.

Theorem 3. *Suppose that a)-d) hold and $f'(x)\geq k>0$. If for any $t>t_0$ there exist numbers a, c ; $t<a<c$, such that for some $H\in X$ holds*

$$\int_a^c\{[p(s)g(s)-p(2c-s)g(2c-s)]H(s-a)-\frac{g(s)a(s)}{4kg'(s)}(h(s-a)+\frac{g(s)}{g'(s)}\sqrt{H(s-a)})^2$$
$$-\frac{g(2c-s)a(2c-s)}{4kg'(2c-s)}(h(s-a)+\frac{g(2c-s)}{g'(2c-s)}\sqrt{H(s-a)})^2\}ds>0, \tag{18}$$

then equation (2) is oscillatory.

Proof. If $c\in(a,b)$ and $2c=a+b$, then we have $H(b-c)=H(c-a)=H(\frac{b-a}{2})$.

Using

$$\int_c^b w(s)ds=\int_a^c w(2c-s)ds$$

or

$$\int_{\frac{a+b}{2}}^{b} f(t)dt = \int_{a}^{\frac{a+b}{2}} f(a+b-s)ds$$

in (12) we have (19). The proof is complete. □

Example. Consider the differential equations

$$(\frac{x'(t)}{e^t})' + e^t(x(\frac{t}{2})(1+x^2(\frac{t}{2})) = 0, \tag{19}$$

where $a(t) = e^{-t}$, $p(t) = e^t$ and $f(x) = x(1+x^2)$, $f'(x) = 1+3x^2 > 1$.

We have

$$\int_{t_0}^{\infty} \left[\frac{1}{a(s_1)} \int_{s_1}^{\infty} p(s)ds\right] ds_1 = \int_{t_0}^{\infty} \left[e^t \int_{t}^{\infty} e^s ds\right] dt = \infty,$$

and

$$\int_{t_0}^{\infty} \left(g(t)p(t) - \frac{g'(t)a(t)}{g(t)4k}\right) dt = \int_{t_0}^{\infty} \left(\frac{t}{2}e^t - \frac{\frac{1}{2}e^{-t}}{4k\frac{t}{2}}\right) dt = \infty,$$

that follows from Theorem 1.

The equation (19) is oscillatory.

Acknowledgments

The authors wishes to thank the referees for all his/her suggestions and useful remarks.

References

[1] Wang Tong Li, Interval oscillation criteria for nonlinear second order differential equations, *Indian J. Pure Appl. Math.*, **32** (2001), 1003-1014.

[2] A. Lomotidze, Oscillation and nonoscillation criteria for second order linear differential equations, *Georgian Mathematical Jurnal*, **4**, No. 2 (1997), 129-138.

[3] Blanka Baculikova, Oscillation criteria for second order nonlinear differential equations, *Archivum Matematicum*, Brno, **42** (2006), 141-149.

[4] Xh. Beqiri, Interval oscillation of nonlinear diferential equations second order, *The Heritage*, **4** (2011), 122-128.

[5] Qinghua Feng, Interval criteria for second order delay differential equations, In: *Proceding of the World Congres on Enginnering, 2009*, **II**, (2009), July 1-3, 2009, London, UK.

[6] I.T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)\,|u|^n\, signu = 0$, *Mat. Sb.*, **65** (1964), 172-187 (In Russian).

[7] I.V. Kamenev, An integral criterion for oscillation of linear differential equations of second order, *Mat. Zametki*, **23** (1978), 249-251.

[8] Aydin Tiryaki, Oscillation criteria for a certain second order nonlinear differential equations with deviating arguments, **61** (2009), 1-11.

[9] Ioannis P. Stavroulakis, Oscillation criteria for functional differential equations, *Electronic Journal of Differential Equations* (2005), 171-180.