

Inter-charge forces in relativistic classical electrodynamics: electromagnetic induction in different reference frames

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Abstract

The force due to electromagnetic induction on a test charge is calculated in different reference frames. The Faraday-Lenz Law and different formulae for the fields of a uniformly moving charge are used. The classical Heaviside formula for the electric field of a moving charge predicts that, for the particular spatial configuration considered, the inductive force vanishes in the frame in which the magnet is in motion and the test charge at rest. In contrast, consistent results, in different frames, are given by the recently derived formulae of relativistic classical electrodynamics.

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In the introduction of his 1905 paper on special relativity [1] Einstein discussed the phenomenon of electromagnetic induction, discovered by Faraday, viewed either from a frame in which the magnet is motion, or from one in which it is at rest. In this letter a careful re-analysis of this problem is performed in terms of the force on a test charge of magnitude q in the vicinity of a magnet. The force on the charge, due to electromagnetic induction, is calculated in both the inertial frame, S , in which the magnet is a rest and the test charge is in motion as well as the frame, S' , in which the magnet is in motion and the test charge is at rest.

Three different methods are used to perform the calculation:

- (i) Application of the Faraday-Lenz Law.
- (ii) Application of the Lorentz Force Law, using standard formulae of Classical Electromagnetism [2, 3] (CEM) for the electric and magnetic fields of a uniformly moving charge.
- (iii) The formulae of Relativistic Classical Electrodynamics (RCED) [4, 5], a covariant formalism developed recently by the present author, are used to calculate directly inter-charge forces.

The corresponding formulae are:

Faraday-Lenz Law

$$\vec{F} = q\vec{E}, \quad -\frac{1}{c}\frac{d\phi}{dt} = \int \vec{E} \cdot d\vec{s} \quad (1)$$

CEM Formulae

$$\vec{E}(CEM) = \frac{Q\vec{r}}{r^3\gamma_u^2(1 - \beta_u^2 \sin^2 \psi)^{\frac{3}{2}}} = \frac{Q(\hat{i} \cos \psi + \hat{j} \sin \psi)}{r^2\gamma_u^2(1 - \beta_u^2 \sin^2 \psi)^{\frac{3}{2}}} \quad (2)$$

$$\vec{B}(CEM) = \frac{Q\vec{\beta}_u \times \vec{r}}{r^3\gamma_u^2(1 - \beta_u^2 \sin^2 \psi)^{\frac{3}{2}}} = \frac{Q\beta_u \hat{k} \sin \psi}{r^2\gamma_u^2(1 - \beta_u^2 \sin^2 \psi)^{\frac{3}{2}}} \quad (3)$$

RCED Formulae^a

$$\vec{E}(RCED) = \frac{Q\gamma_u[\vec{r} - \vec{\beta}_u(\vec{r} \cdot \vec{\beta}_u)]}{r^3} = \frac{Q}{r^2} \left(\frac{\hat{i} \cos \psi}{\gamma_u} + \gamma_u \hat{j} \sin \psi \right) \quad (4)$$

$$\vec{B}(RCED) = \frac{Q\gamma_u\vec{\beta}_u \times \vec{r}}{r^3} = \frac{Q\gamma_u\beta_u \hat{k} \sin \psi}{r^2} \quad (5)$$

For the CEM and RCED calculations the force on the test charge is given by the Lorentz Force Law:

$$\vec{F} = q(\vec{E} + \vec{\beta} \times \vec{B}) \quad (6)$$

where $\beta \equiv v/c$, v is the speed of the test charge, and c is the speed of light in vacuum. In Eqns(2-5) the ‘source’ charge of magnitude Q moves with uniform velocity $u \equiv \beta_u c$ along the x-axis, $\gamma_u \equiv 1/\sqrt{1 - \beta_u^2}$, $\cos \psi = (\vec{v} \cdot \vec{r})/|\vec{v} \cdot \vec{r}|$, \vec{r} is the spatial vector connecting the source and test charges, and \hat{i} , \hat{j} and \hat{k} are unit vectors parallel to the x-, y- and z-axes.

In order to reduce the problem to its essentials, the ‘magnet’ is constituted of just two equal charges of magnitude Q with equal and opposite velocities \vec{u}_+ , \vec{u}_- , $|\vec{u}_+| = |\vec{u}_-| = u$ in the configuration shown in Fig.1a. The charges move parallel to the z-axis and are situated at $(x,y,z) = (0,y,0)$ and $(0,-y,0)$, while the test charge is near to the symmetry point $(x,0,0)$ and moves with velocity $\hat{i}v$ in the rest frame of the ‘magnet’ constituted by the two source charges. Adding further moving charges, equidistant from the test charge, uniformly on a ring of radius y , to give a ‘one turn solenoid’ complicates the evaluation of the fields and forces, but adds nothing to the essential dynamics of the problem. Since magnets are usually electrically neutral, the correspondence with a magnet constituted by an electron circulating in an atom or a one-turn solenoid would be made more exact by placing charges $-Q$, at rest in S , adjacent to the moving charges. Since however such charges produce no magnetic field in S , and an electric field at the test charge confined to the x-y plane in both S and S' , the following calculations of electromagnetic induction, where both electric and magnetic forces are parallel to the z-axis, is unchanged by the presence of such ‘neutralising’ charges. They are therefore not considered in the following.

In order to apply the Faraday-Lenz Law an imaginary rectangular current loop ABCD is drawn through the test charge in a plane perpendicular to the x-axis as shown in Fig.1a. If $a = AB \ll b = BC$, then, because of the symmetrical position of the loop, magnetic flux will, to a good approximation, cross only the short sides AB and DC as the loop attached to the test charge moves through the field. In consequence, the line integral in (1) reduces to $2E_z a$. Since (see Fig.1a) $\psi = \pi/2$, (3) or (5) give $\vec{B}(CEM) = \vec{B}(RCED)$ and the magnetic flux, ϕ threading the loop ABCD is:

$$\phi(CEM) = \phi(RCED) = ab[(B_+)_{x} + (B_-)_{x}] = \frac{2abQ\gamma_u\beta_u \cos \theta}{r^2} = \frac{2abQ\gamma_u\beta_u y}{r^3} \quad (7)$$

^aIn RCEM the forces between charges are calculated directly without the introduction of any field concept [4]. For comparison purposes the terms in the force formula corresponding to the usual definitions of electric and magnetic forces in Eqn(6) are expressed here in terms of corresponding electric and magnetic fields $\vec{E}(RCED)$ and $\vec{B}(RCED)$.

where \vec{B}_+ and \vec{B}_- are the magnetic fields due to the charges with velocity \vec{u}_+ and \vec{u}_- respectively. Differentiating (7) w.r.t. x , noting that $v = dx/dt$, and using (1) gives:

$$-\frac{1}{c} \frac{d\phi}{dt} = 2aE_z = \frac{6abQ\gamma_u\beta_u\beta xy}{r^5} \quad (8)$$

So that the force on the test charge is:

$$F_z(FL) = qE_z = \frac{3bqQ\gamma_u\beta_u\beta \cos \theta \sin \theta}{r^3} \quad (9)$$

The force on the test charge in S is now calculated using the Lorentz Force Law (6). Since both the CEM and RCED electric fields at the test charge lie in the x-y plane, only the magnetic force contributes in the z-direction. This force is given by the y-component of $\vec{B}_+ + \vec{B}_-$ at the point $(x, b/2, 0)$. From the geometry of the x-y plane, shown in Fig.1b, and (3) or (5) with $\psi = \pi/2$:

$$B_y(x, \frac{b}{2}, 0) = (B_+)_y + (B_-)_y = Q\gamma_u\beta_u x \left(\frac{1}{r_+^3} - \frac{1}{r_-^3} \right) \quad (10)$$

Assuming then that $b \ll x, y$ it is found that:

$$\frac{1}{r_+^3} - \frac{1}{r_-^3} = \frac{3b \cos \theta}{r^4} + O((b^2/r^5)) \quad (11)$$

so that from (10) and (11):

$$B_y(CEM) = B_y(RCED) = \frac{3bQ\gamma_u\beta_u \cos \theta \sin \theta}{r^3} + O((b^2/r^4)) \quad (12)$$

Hence, using (6):

$$F_z(CEM) = F_z(RCED) = \frac{3bqQ\gamma_u\beta_u\beta \cos \theta \sin \theta}{r^3} + O((b^2/r^4)) \quad (13)$$

in agreement, to first order in b , with the Faraday-Lenz Law result (9).

The above calculations are now carried out in the frame S' where the magnet is in motion and the test charge is at rest. Using Eqns(3) and (5) it follows from the geometry of Fig.2 that, the magnetic fluxes threading the loop ABCD in the frame S' are:

$$\phi'(CEM) = \frac{2abQ\gamma_{u'}\beta_{u'}y}{\gamma r^3(1 - \gamma_{u'}^2\beta^2 \sin^2 \theta)^{\frac{3}{2}}} = \frac{\gamma_{u'}}{\gamma\gamma_u(1 - \gamma_{u'}^2\beta^2 \sin^2 \theta)^{\frac{3}{2}}} \phi(CEM) \quad (14)$$

$$\phi'(RCED) = \frac{2abQ\gamma_{u'}\beta_{u'}y}{\gamma r^3} = \frac{\gamma_{u'}}{\gamma\gamma_u} \phi(RCED) \quad (15)$$

where, from the geometry of Fig.2b:

$$\beta_{u'} = \frac{\sqrt{\beta_u^2 + \gamma^2\beta^2}}{\gamma}, \quad \gamma_{u'} = \frac{1}{\sqrt{1 - \beta_{u'}^2}}$$

Note that each component of the spatial separation of the test and source charges remains invariant under Lorentz transformation [4, 6] so that, as shown in Fig.2, there is no distinction between the spatial interval x and x' , y and y' and r and r' , or between the

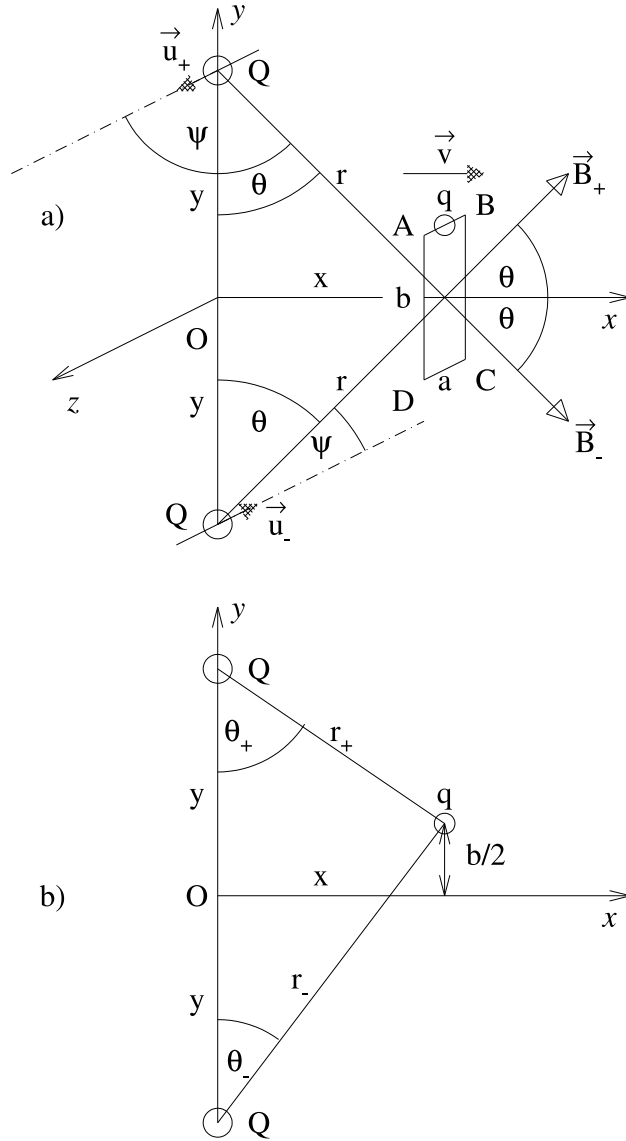


Figure 1: Geometry for calculation of electromagnetic induction in the frame S in which the magnet is at rest and the test charge, of magnitude q , moves with velocity \vec{v} along the $+ve$ x -axis. The ‘magnet’ consists of two charges of magnitude Q moving along the z -axis in opposite directions, each with speed u . The imaginary flux-loop $ABCD$ is attached to the test charge. Various distances and angles are defined. \vec{B}_+ and \vec{B}_- are the magnetic fields at $(x,0,0)$ generated by the charges of velocity \vec{u}_+ and \vec{u}_- . a) shows a perspective view and b) the x - y projection.

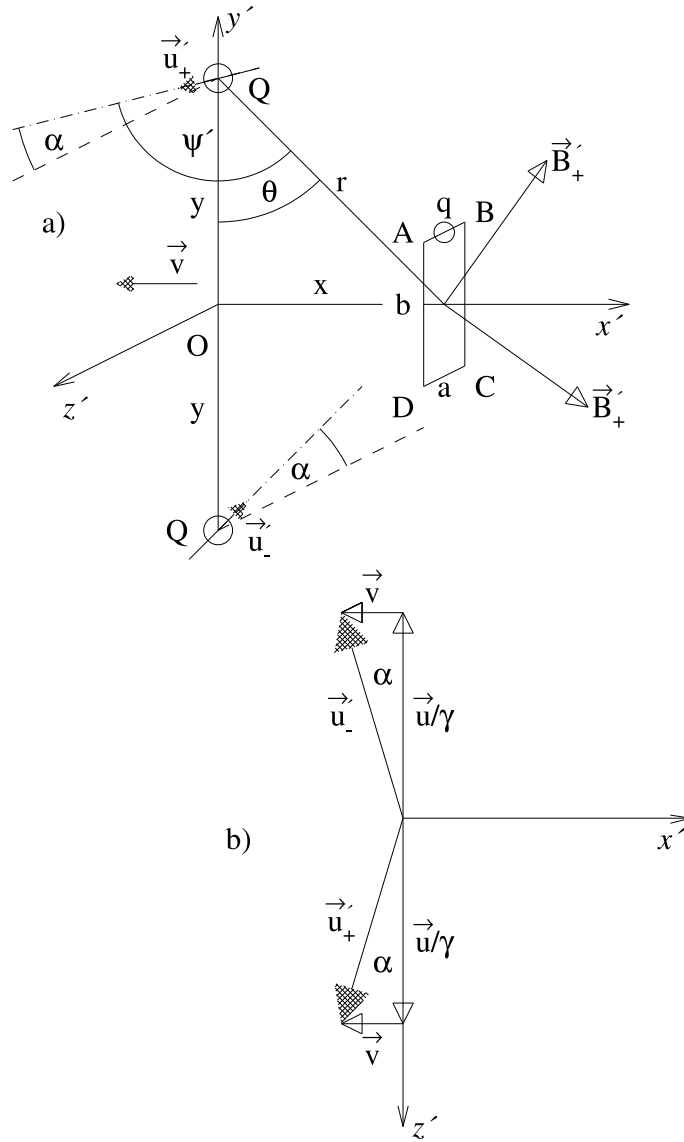


Figure 2: Geometry for calculation of electromagnetic induction in the frame S' in which the test charge is at rest and the magnet moves at velocity \vec{v} along the $-ve$ x -axis. Distances, angles, velocity vectors and magnetic fields are defined in a manner similar to those in Fig.1. a) shows a perspective view and b) the x' - z' projection.

angles θ and θ' . Since the calculation of the rate of change of the flux threading the loop is the same whether the loop is displaced with velocity v along the +ve x-axis as in S (Fig.1) or the source of the magnetic field is displaced with velocity v along the -ve x-axis as in S' (Fig.2) the calculation of the z-component of the electric field using the Faraday-Lenz Law proceeds as above, with the results:

$$F_{z'}(CEM) = \frac{3bqQ\gamma_{u'}\beta_u\beta\cos\theta\sin\theta}{\gamma r^3(1-\gamma_{u'}^2\beta^2\sin^2\theta)^{\frac{3}{2}}} = \frac{\gamma_{u'}}{\gamma\gamma_u(1-\gamma_{u'}^2\beta^2\sin^2\theta)^{\frac{3}{2}}}F_z(CEM) \quad (16)$$

$$F_{z'}(RCED) = \frac{3bqQ\gamma_{u'}\beta_u\beta\cos\theta\sin\theta}{\gamma r^3} = \frac{\gamma_{u'}}{\gamma\gamma_u}F_z(RCED) \quad (17)$$

When the force calculations are performed in the frame S', by use of the Faraday-Lenz Law, consistent results are therefore, in general, no longer obtained. Only in the limit $\theta \rightarrow 0$ when the inductive force vanishes do the CEM and RCED predictions become equal.

Since the vectors \vec{r}_+ , \vec{r}_- lie in the x'-y' plane, and the electric field in the CEM formula (2) is radial, it follows that the electric field at the test charge in S' also lies in this plane. Thus Eqn(2) predicts *no force, parallel to the z' axis, acts on the test charge in S'*. That is, that there is no effect of electromagnetic induction in this frame, in contradiction both the requirements of special relativity and the prediction of the Faraday-Lenz Law using either CEM or RCED fields.

Finally the calculation is performed in the frame S' using the RCED electric field (4). From the geometries of Fig.1b and Fig.2b:

$$\vec{r}_+ = r_+(\hat{i}\sin\theta_+ - \hat{j}\cos\theta_+) \quad (18)$$

$$\vec{r}_- = r_-(\hat{i}\sin\theta_+ + \hat{j}\cos\theta_-) \quad (19)$$

$$\vec{\beta}'_+ = \beta_{u'}(-\hat{i}\sin\alpha + \hat{k}\cos\alpha) \quad (20)$$

$$\vec{\beta}'_- = \beta_{u'}(-\hat{i}\sin\alpha - \hat{k}\cos\alpha) \quad (21)$$

so that:

$$\vec{r}_+ \cdot \vec{\beta}'_+ = -\beta_{u'}r_+\sin\theta_+\sin\alpha = -\beta_{u'}x\sin\alpha \quad (22)$$

$$\vec{r}_- \cdot \vec{\beta}'_- = -\beta_{u'}r_-\sin\theta_+\sin\alpha = -\beta_{u'}x\sin\alpha \quad (23)$$

Eqns(18), (19), (22), (23) and (4) then give:

$$\begin{aligned} E_{z'}(RCED) &= (E_+)_z + (E_-)_z = Q\gamma_{u'}\beta_{u'}^2x\sin\alpha\cos\alpha\left(\frac{1}{r_+^3} - \frac{1}{r_-^3}\right) \\ &= \frac{3bQ\gamma_{u'}\beta_{u'}^2\cos\theta\sin\theta\sin\alpha\cos\alpha}{r^3} + O((b^2/r^4)) \end{aligned} \quad (24)$$

where Eqn(11) has been used. Hence:

$$\begin{aligned} F_{z'}(RCED) &= qE_{z'}(RCED) = \frac{3bqQ\gamma_{u'}\beta_{u'}^2\cos\theta\sin\theta\sin\alpha\cos\alpha}{r^3} + O((b^2/r^4)) \\ &= \frac{3bqQ\gamma_{u'}\beta_u\beta\cos\theta\sin\theta}{\gamma r^3} + O((b^2/r^4)) \\ &= \frac{\gamma_{u'}}{\gamma\gamma_u}F_z(RCED) \end{aligned} \quad (25)$$

where the relations following from the geometry of Fig.2b:

$$\sin \alpha = \frac{\beta}{\beta_{u'}}, \quad \cos \alpha = \frac{\beta_u}{\gamma \beta_{u'}}$$

have been used. This result agrees with that, (17), obtained by use of the Faraday-Lenz Law.

The factor relating the inductive forces on the test charge in S and S' is:

$$\frac{\gamma_{u'}}{\gamma \gamma_u} = 1 - \frac{1}{2}[\beta^2(\beta^2 - \beta_u^2) + \beta_u^4] + O(\beta^2 \beta_u^4, \beta_u^2 \beta^4) \quad (26)$$

For $\beta \gg \beta_u$:

$$\frac{\gamma_{u'}}{\gamma \gamma_u} = 1 + \beta^4 + O(\beta^6) \quad (27)$$

while for $\beta_u \gg \beta$:

$$\frac{\gamma_{u'}}{\gamma \gamma_u} = 1 - \beta_u^4 + O(\beta_u^6) \quad (28)$$

so in these cases the forces in S and S' differ only by corrections of order the fourth power in the ratio of charge velocities to the speed of light. In summary, in the frame S, where the magnet is at rest, so that the magnetic fields are 'static', and the test charge is in motion, all three methods of calculation yield the same result (9) or (13) to the considered calculational accuracy. However when the Faraday-Lenz Law is used to perform the calculation in the frame S' where the magnet is in motion and the test charge is at rest, the CEM result (16) is found to differ from the RCED one (17) by terms of $O(\beta^2)$. The CEM electric field formula (2) predicts the complete absence of electromagnetic induction in the frame S', in contradiction with the Faraday-Lenz Law predictions in this frame, and with special relativity. The incompatibility of this formula, first derived by Heaviside [7], more than a decade before the advent of special relativity, with the requirements of the latter has been previously demonstrated by comparing calculations of Rutherford scattering in different inertial frames [5] as well as by Jackson's 'Torque Paradox' [8], which is resolved [5] by the use of the RCED force formula that is the combination of (4),(5) and (6).

All four results (9),(13),(17) and (26) of the calculations of the force, using the RCED formulae, give consistent results. The forces in the frames S and S' are found to differ only by corrections of order the fourth power in the ratio of velocities to the speed of light. The forces are different due to the relativistic time dilatation effect which results in different accelerations in different inertial frames. That forces are different in different inertial frames is already evident from inspection of Eqns(4) and (5) by comparing the fields in the frame where the source charge is at rest ($\beta_u = 0, \gamma_u = 1$) with those shown for an arbitrary value of β_u .

Since the original version of this letter was written some two years ago, convincing experimental evidence has been obtained [9, 10] for the non-retarded nature of electro-dynamical force fields, as in RCED. The relation of these results to the basis of RCED in QED has also been discussed [11].

References

- [1] A.Einstein, Annalen der Physik **17**, 891 (1905).
English translation by W.Perrett and G.B.Jeffery in 'The Principle of Relativity'

- (Dover, New York, 1952) P37, or in ‘Einstein’s Miraculous Year’ (Princeton University Press, Princeton, New Jersey, 1998) P123.
- [2] J.D.Jackson, ‘Classical Electrodynamics’, 3rd Ed (Wiley, New York, 1998) P558,560.
 - [3] L.D.Landau and E.M.Lifshitz, ‘The Classical Theory of Fields’, 2nd Edition (Pergamon Press, Oxford, 1962) Section 38, P103.
 - [4] J.H.Field, Phys. Scr. **74** 702 (2006), <http://xxx.lanl.gov/abs/physics/0501130>.
 - [5] J.H.Field, ‘Forces between electric charges in motion: Rutherford scattering, circular Keplerian orbits, action-at-a-distance and Newton’s third law in relativistic classical electrodynamics’, <http://xxx.lanl.gov/abs/physics/0507150>.
 - [6] J.H.Field, Phys. Scr. **73** 639 (2006), <http://xxx.lanl.gov/abs/physics/0409103>.
 - [7] O.Heaviside, The Electrician, **22** 1477 (1888), Philos. Mag. **27** 2324 (1889).
 - [8] J.D.Jackson, Am. J. Phys. **72** 1484 (2004).
 - [9] A.L.Kholmetskii *et al.* J. Appl. Phys. **101** 023532 (2007).
 - [10] A.L.Kholmetskii. O.V.Missevitch and R.Smirnov-Rueda, J. Appl. Phys. **102** 013529 (2007).
 - [11] J.H.Field, ‘Quantum electrodynamics and experiment demonstrate the non-retarded nature of electrodynamic force fields’, <http://xxx.lanl.gov/abs/0706.1661>.