Optimization of stochastic fluid model using perturbation analysis: a manufacturing-remanufacturing system with stochastic demand and stochastic returned products

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Abstract—In this paper, a stochastic fluid model is used to study a manufacturing/remanufacturing system composed by two parallel machines, a serviceable inventory, a remanufacturing inventory and customers who demand a stochastic quantity of product. Stochastic fluid model is adopted to describe the system and to take into account machine failure, stochastic demand, stochastic returned products and remanufacturing products. The goal of this paper is to evaluate the optimal serviceable inventory level which allows minimizing the sum of inventory and lost sales costs. Perturbation analysis is applied to the stochastic fluid model to optimize the considered system. The trajectories of the serviceable inventory level are studied and the perturbation analysis estimates are evaluated. The unbiasedness of these estimates is proved and then they are implemented in an optimization algorithm which determines the optimal serviceable inventory in the presence of stochastic returned products and stochastic demand.

Keywords—Manufacturing/remanufacturing system; stochastic fluid model; perturbation analysis; stochastic demand, stochastic returned products.

I. INTRODUCTION

In recent years, due to the pressure of the environmental legislations, the conscious of environment and the economical interesting, many producers have to take their products back after used. Indeed, the process of the returned products after use brings economic burden to the producers, while many of the returned products have economic value and could be remanufactured to satisfy the market demand. However, research on supply chain management has been paying attention on the recovery processes of end of life products for remanufacturing. Many of the studies published on reverse logistics are founded in the literature and which focus on inventory management. Zhou, Naim, Tang, and Towill [1] considered a hybrid system with both manufacturing and remanufacturing. The authors adopted an inventory control strategy in the manufacturing loop which is an automatic pipeline, inventory and order based production control system and in the remanufacturing loop they employ a Kanban policy to represent a typical pull system and to control the remanufacturing process. Indeed, the authors analyzed the dynamic performance of the system which has implications on total costs in terms of inventory holding, capacity utilization and customer service failures. Shi, Zhang, and Sha [2] presented a stochastic model for determining the optimal production and remanufacturing quantities for a product portfolio. The authors assumed that the product demands are independent, for each product new and remanufactured units are perfect substitutes, the returns are of unknown quality and the amount of returned cores is a function of their acquisition price, which is also a decision variable.

Wang, Zhao, and Wang [3] studied a hybrid manufacturing and remanufacturing system for a kind of short life cycle product with stochastic demand and stochastic returned products. The authors analyzed the impacts of the amount of products manufactured and the proportion of the remanufactured part to the returned products on the total cost of the system. They determined the optimum values of the amount of products manufactured and the proportion of the remanufactured part to the returned products which minimize the total cost of the hybrid system. In this paper we use a stochastic fluid model (Cassandras, Wardi, Panayiotou and C. Yao [4], Yao and Cassandras [5], Xie, Hennequin, and Mourani. [6]) to describe the system and to take into account stochastic returned products, remanufacturing products and the stochastic demand.

Stochastic fluid model which is a class of stochastic hybrid is widely used to control, analyze, and improve the performance of manufacturing systems. Indeed, stochastic fluid model is very useful in simulating various kinds of high speed networks, manufacturing systems and more generally, settings where users compete over different sharable resources. The robustness of a stochastic fluid model rests on its ability to aggregate multiple events. In other words, stochastic fluid model paradigm allows the aggregation of multiple events, associated with the movement of individual products over a time period of a constant flow rate [7]. The major reason for choosing this model, that stochastic fluid model allows us to use an interesting gradient method for optimizing the proposed system; this method called perturbation analysis (PA).

The objective of this paper is to develop a PA method for optimizing a manufacturing/remanufacturing system with stochastic demand and stochastic returned products. Perturbation analysis, (Wardi, Giua, and Seatzu [8], Yao and Cassandras [9]) is an approach for sensitivity analysis and which is a technique allowing obtaining sample path derivatives of a random variable with respect to some parameters of interest. The most important advantage of PA method is that the simulation based on PA allows reducing the simulation time comparing to a classical simulation method. This advantage is explained by the fact that the optimization algorithm based on PA computes at every step the gradient estimates which corresponds to the new value of a parameter of interest. Yu and Cassandras [10] applied perturbation analysis method to a stochastic flow model and then derived gradient estimates of throughput and buffer overflow metrics with respect to production control parameters, then they used them as approximate gradient estimates in simple iterative schemes for adjusting thresholds in order to optimize an objective function that trades off throughput and buffer overflow costs.
Before using these gradient estimates in the optimization algorithm the authors proved their unbiasedness. Indeed, the unbiasedness is the main condition for making the application of PA useful in practice, since it enables the use of the sample PA derivative in control and optimization methods that employ stochastic gradient-based techniques. Then, these estimates could be used in stochastic approximation algorithm. In this paper, the PA estimates are determined and then used in an optimization algorithm for finding the optimal serviceable inventory level.

However, the first contribution of this paper is to consider a stochastic flow model which describe a manufacturing/remanufacturing system and take into account of stochastic returned products and stochastic demand, then to derive gradient estimates for performance metrics related to lost sales and inventory levels with respect to threshold parameter (serviceable inventory level). The second contribution is to make use of the PA gradient estimates derived for determining the optimal serviceable inventory level as an optimization problem.

The paper is organized as follows. In section 2, the stochastic fluid model with the problem formulation is presented. The PA approach is applied to the stochastic fluid model in section 3. In section 4, the PA estimates are determined, the unbiasedness is proved and numerical results are presented. Finally, the last section concludes the paper and gives some perspectives to our work.

II. STOCHASTIC FLUID MODEL FOR MANUFACTURING/REMANUFACTURING SYSTEM

We consider a manufacturing/remanufacturing system (Fig.1) composed by two parallel machines which are subject to random failures and repairs denoted $M_1$ and $M_2$ for manufacturing and remanufacturing, respectively. We assume that both machines are producing the same type of product. In this system we consider the production activity in forward direction and reverse logistics (i.e. activity of the remanufacturing of the used products). Also we consider customers who demand a stochastic quantity of product per unit time denoted $d(t)$. This demand is satisfied from a serviceable inventory $S$ which can be filled up by the machines $M_1$ and $M_2$. Another inventory $R$ is available for the stock keeping of the returned products ahead of the remanufacturing process. These returned products will be then remanufactured by the machine $M_2$ and then stoked in the serviceable inventory $S$ with the manufactured products. We denote $Z(t)$ the return rate of the products (i.e. the number of the remanufacturable products per unit time at time $t$) and which is stochastic and proportional to the average customers demand denoted $D$.

The possible events at every time $t$ are: machine failure (MF), repair (MR), serviceable inventory full (SS), serviceable inventory empty (SV), demand event (DE) returned products event (RP). Furthermore, for application of PA and to avoid important discontinuities, we assume that only one event is considered at a time, thus we propose priorities between events if different events occur at the same time. The priority is assigned in a decreasing order as follows: inventory event (SS or SV), machine event (MF or MR) and demand event (DE) or returned products event (RP).

When the machine is up, the production rate of $M_1$ denoted by $u_1(t)$, could take a value between 0 and its maximum rate $U_1$, i.e., $0 \leq u_1(t) \leq U_1$. When the machine is down $u_1(t)$=0. The times to repair and times to failure are exponentially distributed with rate $\lambda_1$ and $\lambda_2$ respectively. We have the same state for the machine $M_2$, with $u_2(t)$ is the production rate, $U_2$ is the maximal production rate, $\mu$ and $\lambda$ are times to repair and times to failure of the machine $M_2$. The failure/repair process is an independent random process. It does not depend on the system parameters.

The following assumptions are considered:
- $U_1$ permits to satisfy the maximum of the demands, (i.e. $U_1 \geq \text{Max}(d(t))$). This assumption allows avoiding having always the serviceable inventory empty.
- $U_2$ is upper to the maximum of the return rate, i.e. $U_2 > \text{Max}(\text{Z}(t))$. This assumption allows avoiding having always the remanufacturing inventory very full.
- If the demand is unsatisfied, the demand is lost with a corresponding cost (lost sales cost). The remanufactured products can be considered meeting the same quality level as the new products so that both type of products can be distributed like new.

The serviceable inventory level denoted by $s(t)$ and the remanufacturing inventory level denoted by $r(t)$. We assume the case of infinite capacity for $S$ and $R$, where the system dynamics are given by the following equations:

\[
\frac{dr(t)}{dt} = Z(t) - u_1(t) \tag{3}
\]

\[
\frac{ds(t)}{dt} = u_1(t) + u_2(t) - d(t) \tag{4}
\]
Remark 1: In the first step of the approach which determines the optimal inventory level, the capacity of serviceable inventory $S$ is supposed infinite, and then the optimal inventory level will be determined according to the hedging point.

The hedging point policy has been proved to be the optimal for a one-product manufacturing system (Akella and Kumm [11]). Indeed, the hedging point policy ensures that the part does not exceed a given number of products, denoted by $h$. According to this policy, the production rate is equal to the demand rate when $S$ becomes full (i.e. $s(t)=h$). We assume in this case that the machines $M_1$ and $M_2$ share the production. In other word, each machine produces half of the production (i.e. $u_1(t)=d(t)/2$ and $u_2(t)=d(t)/2$). The control policy is defined as follows:

For the machine $M_1$:

$$u_1(t)=\begin{cases} U_j & \text{if } \alpha(t)=1 \text{ and } s(t)<h \\ d(t)/2 & \text{if } \alpha(t)=1 \text{ and } s(t)=h \\ 0 & \text{if } \alpha(t)=0 \text{ or } s(t)>h \end{cases}$$

(5)

For the machine $M_2$:

$$u_2(t)=\begin{cases} U_j & \text{if } \beta(t)=1 \text{ and } r(t)>0 \text{ and } s(t)<h \\ d(t)/2 & \text{if } \beta(t)=1 \text{ and } r(t)>0 \text{ and } s(t)=h \\ Z(t) & \text{if } \beta(t)=1 \text{ and } r(t)=0 \text{ and } s(t)\leq h \\ 0 & \text{if } \beta(t)=0 \text{ or } s(t)>h \end{cases}$$

(6)

We assume that the return rate $Z(t)$ is proportional to the average demand rate $D$. We denote by $w(t)$ ($0<w(t)<1$) the percentage of sales which are returned for remanufacturing at time $t$ and which is stochastic. Indeed, the customer returns rate may be as high as 15% of sales in the coming years, and in sectors such as catalogue sales and e-commerce it could reach as much as 35% (Rubio and Corominas [12]). Then we have:

$$Z(t) = w(t). D$$

(7)

We denote $d^*(t)$ the number of unsatisfied demands (lost) per unit time, and which depends on $d(t)$ and $s(t)$. Indeed, when the customer orders a demand $d(t)$ and the serviceable inventory is empty, the demand will not be satisfied at all and will be lost. However, when $s(t)$ is positive $d^*(t)$ is null and is equal to the demand if the serviceable inventory is empty ($s(t)=0$). The number of unsatisfied demands per unit time is defined as follows:

$$d^*(t)=\begin{cases} 0 & \text{if } s(t)>0 \\ d(t) & \text{if } s(t)=0 \end{cases}$$

(8)

The number of unsatisfied demands at time $t$ denoted by $Y(t)$ is given by:

$$\frac{dY(t)}{dt} = d^*(t) = d(t) \text{ if } s(t)=0$$

$$Y(t) = 0 \text{ if } s(t)>0$$

(9)

Remark 2: We defined $Y(t)$, because we will need it for writing the cost function.

We denote by $C(t)$ the cost function at time $t$ and which is composed by the inventory cost and the lost sale cost. $C(t)$ is given by:

$$C(t) = cs.s(t) + cr.r(t) + cs\cdot Y(t)$$

(10)

Where:

- $cs$ and $cr$ : are the units inventory cost respectively for $S$ and $R$;
- $cs\cdot Y(t)$ : unit lost sale cost.

The expected average cost, denoted by $G(h)$ depending on $h$ is given by :

$$G(h) = \lim_{T \to \infty} \frac{1}{T} E \left[ \int_0^T C(t) dt \right]$$

(11)

$\forall t \in [0,T]$ with $T$ the total simulation time.

We will study the behavior of the sample path trajectories of $s(t)$, $r(t)$ and $Y(t)$ in the next section. This study will allow us to find the PA estimates and prove that these estimates are unbiased.

### III. Perturbation Analysis

The PA method is applied to the stochastic fluid model in this section for investigating the trajectories of $s(t)$, $r(t)$ and $Y(t)$. Further to this study we will determine in next section, the gradient estimates of the cost function with respect to the optimal serviceable inventory level. Indeed, the PA method consists on observing and analyzing two sample paths, one is the nominal sample path ($s(t)$), and the other is the perturbed sample path ($s'(t)$) (see Fig. 2). We assume that the optimal serviceable inventory level is increased by a perturbation, denoted by $\gamma$. In this work, we assume that $\gamma > 0$ and we study the resulting changes in the cost function using geometric arguments (similar results could be obtained for $\gamma < 0$). The optimal serviceable inventory level of the perturbed sample path ($s'(t)$) is $h+\gamma$.

The following assumptions are considered:

- The perturbation of $h$ is $\gamma$.
- The maximal production and unit costs are the same for both sample paths.
- The same distribution of random variables ($time to failure$, $time to repair$) is used for both sample paths.

In the following (Fig. 2), we give an example of simple paths for $s(t)$ and $s'(t)$.

The following notations are used:

- $s(t)$: The serviceable inventory ($S$) level for the perturbed path;
- $r(t)$: The remanufacturing inventory ($R$) level for the perturbed path;
- $u_1(t)$: The production rate of the machine $M_1$ at time $t$ for the perturbed path;
- $u_2(t)$: The production rate of the machine $M_2$ at time $t$ for the perturbed path;
Theorem 2 shows that the remanufacturing inventory level of the perturbed path is equal to nominal path minus the lag ψ(t) when \( t \in [t_0, t_{\mu}] \) and the nominal path and of the perturbed path are equal when \( t \in [t_{\mu}', t_{\psi(t)}] \).

**Theorem 2:** If \( \gamma > 0 \), \( r'(0) = r(0) \) we have:

- \( t \in [t_0, t_{\mu}] \), then \( r'(t) = r(t) - \psi(t) \).
- \( t \in [t_{\mu}', t_{\psi(t)}] \), then \( r'(t) = r(t) \).

**Proof:**

The perturbation on \( R \) is caused by the perturbation of \( S \) when \( t \in [t_0, t_{\mu}] \). If the machine \( \beta(t) = 1 \) and at time \( t \), the machine becomes empty. We assume that we have \( r'(t) = r(t) \). The lag which is between the perturbed and nominal paths at time \( t' \) and \( r(t) > 0 \). According to the equation 3, we have:

\[
r(t) = r(t) + (Z(t) - u_z(t))(t-t')
\]

According to the equation 6, when \( t \in [t_{\mu}', t_{\psi(t)}] \), we have:

\[
r'(t) = r'(t) + (Z(t) - u_z(t))(t-t')
\]

We assume that \( \psi(t) = U_z(t-t') \) then we have:

\[
r'(t) = r(t) - \psi(t)
\]

Therefore, this lag which is between the perturbed and nominal path will disappear when \( R \) becomes empty on the perturbed and nominal path (i.e., \( t = t_{\mu} \)). Then when \( t \in [t_{\mu}', t_{\psi(t)}] \), we have:

\[
r'(t) = r(t) - \psi(t)
\]

Q.E.D.

**Theorem 3:** If \( \gamma > 0 \), \( s'(0) = s(0) \) we have:

- \( t \in [t_0, t_{\mu}] \), then \( Y'(t) = Y(t) - \gamma \).
- \( t \in [t_{\mu}', t_{\psi(t)}] \), then \( Y'(t) = Y(t) \).

**Proof:**

When \( t \in [0,T] \), the demands are satisfied for both types of trajectories, thus we have \( Y'(t) = Y(t) \). When \( t \in [t_0, t_{\mu}] \) we have, the number of satisfied products is equal to the number of products in the serviceable inventory and then we
have \( s'(t) - s(t) = \psi(t) \). Contrariwise, the difference between the number of unsatisfied products for both trajectories is equal to the opposite for the case of satisfied products, thus \( Y'(t) - Y(t) = -\psi(t) \).

Q.E.D.

We will determine the PA estimates and will prove their unbiasedness in following section.

IV. OPTIMIZATION BASED ON PERTURBATION ANALYSIS

In this section, the previous study will be used for determining the PA estimates. Indeed, these estimates will be implemented in an optimization algorithm and which determines the value of \( h \). The values of PA estimates allow orienting quickly the algorithm to the optimal value of \( h \). Thus, the advantage of the optimization algorithm based on PA compared with an exhaustive optimization algorithm is that it takes smaller computational time (i.e. simulation time).

A. PA Estimates

The average cost of the perturbed path is given by:

\[
G'(h + \gamma) = \lim_{r \to -\infty} \frac{1}{T} E \left[ \int_0^T C'(t) \, dt \right] \tag{12}
\]

With

\[
C'(t) = cs.s'(t) + cr.r'(t) + cs'.Y'(t) \tag{13}
\]

The sampled estimation for the expected average cost of the nominal path is given by:

\[
G_s(h) = \frac{1}{T} E \left[ \int_0^T C(t) \, dt \right] \tag{14}
\]

The sampled estimation for the expected average cost of the perturbed path is given by:

\[
G'_s(h + \gamma) = \frac{1}{T} E \left[ \int_0^T C'(t) \, dt \right] \tag{15}
\]

We determine the PA estimates of the cost function by computing the difference between the perturbed average cost and the nominal average cost and which is given by:

\[
G'(h + \gamma) - G_s(h) = \frac{1}{T} E \left[ \int_0^T (C'(t) - C(t)) \, dt \right]
\]

\[
G'_s(h + \gamma) - G_s(h) = \frac{1}{T} E \left[ \int_0^T \left( cs.s'(t) - s(t) \right) + cr.(r'(t) - r(t)) + cs'.\left( Y'(t) - Y(t) \right) \right] \, dt \tag{16}
\]

We assume that in the interval \([0,T]\) we have \( m \) intervals \( [t_{i-1}, t_i] \), \( n \) intervals \( [t_{i-1}^*, t_{i, t_{i+1}}] \), \( p \) intervals \( t_{m+1}, t_{m+p} \) and \( q \) \( [t_{m+p}, t_{m+p+1}] \) then we have:

- For the trajectories of \( s(t) \) and \( s'(t) \) the interval \([0,T]\) is divided on two sums of periods: \( T_1' \) is the sum of periods when \( t \in \{ t_i, t_i^* \} \) and \( T_i \) is the sum of periods when \( t \in \{ t_{i, t_{i+1}} \} \). Then we have:

\[
\int_0^T cs.s'(t) - s(t) \, dt = \sum_{i=1}^m (t_i - t_{i-1}) \psi(t) \, dt \text{ (theorems 1)}.
\]

Then we have:

\[
\int_0^T cs.s'(t) - s(t) \, dt = \sum_{i=1}^m \psi(t) \, dt = cs.T_1'.\gamma.
\]

- For the trajectories of \( r(t) \) and \( r'(t) \) the interval \([0,T]\) is divided on two sums of periods: \( T_2' \) is the sum of periods when \( t \in \{ t_i, t_i^* \} \) and \( T_3 \) is the sum of periods when \( t \in \{ t_{i, t_{i+1}} \} \). Thus, according to the theorems 2 we have:

\[
\int_0^T cr.r'(t) - r(t) \, dt = -(cr.T_2'.\gamma)
\]

- For the trajectories of \( Y(t) \) and \( Y'(t) \) the interval \([0,T]\) is divided on two sums of periods: \( T_1' \) is the sum of periods when \( t \in \{ t_i, t_i^* \} \) and \( T_3 \) is the sum of periods when \( t \in \{ t_{i, t_{i+1}} \} \). Then, according to the theorems 3 we have:

\[
\int_0^T cs'.\left( Y'(t) - Y(t) \right) \, dt = -(cs'.T_3'.\gamma)
\]

Then, the gradient estimates of the cost function are given by:

\[
\frac{\partial J_s(h)}{\partial h} = \frac{1}{T} E \left[ cs.T_1' - cr.T_2' - cs'.T_3' \right]
\]

We should prove the unbiasedness of estimates in order to verify them reliability.

Remark 3: The fact that the estimates are statistically unbiased, mean that the estimated value equal to the real value.

Theorem 4: the gradient estimates of the average cost are unbiased.

The proof of this theorem is similar to the proof of theorem 12 in [13].

In the following, we present numerical results to study the impact of the percentage of the returned products \((w(t))\) on the value of the optimal serviceable inventory level \( h \).

B. Optimization Using PA Estimates

In this part the PA estimates are used in an optimization algorithm to determine the values of \( h \). Indeed, we are interesting to study the impact of the percentage of the returned products \((w(t))\) on the value of the optimal serviceable inventory level \( h \). Such as the cost function depends of return rate of the products, the value of \( h \) which minimizes this cost function will certainly also depend on the percentage of the returned products. Therefore, we vary the values of the percentage of the returned...
products and we determine the value of h by using the PA optimization algorithm.

The following parameters are used for the simulation:
- $U_1 = 8$ products /time unit;
- $U_2 = 5$ products /time unit;
- $d(t)$ is generated by a Uniform distribution (for the demand event) and truncated Normal distribution (for the demand value). Then, the average demand $D=6$ products /time unit (truncated Normal distribution) and for the demand event the boundaries of the Uniform distribution are 1 (minimum) and 3 (maximum).
- The total simulation time is equal to $T=1E+07$ time units;
- The times to failure or repair are given by exponential distribution, the mean time between failures $MTBF$ is equal to 2.8 and the mean time to repair $MTTR$ is equal to 1.1;
- The unit inventory cost $cs$ is equal to 1 monetary unit;
- The unit lost sales cost $cs$- is equal to 50 monetary units;
- The unit inventory cost $cr$ is equal to 1 monetary unit.

The value of percentage of the returned products (w(t)) is generated by a truncated Normal distribution. In the following table, we vary the value of the average of the distribution and the bounds (lower bound and upper bound) and then we determine the optimal serviceable inventory level. Then, simulation results are presented in the following table to show the impact of the percentage of the returned products on h.

**TABLE I. IMPACT OF THE PERCENTAGE OF THE RETURNED PRODUCTS ON THE OPTIMAL SERVICEABLE INVENTORY.**

<table>
<thead>
<tr>
<th>w(t)</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>5%</td>
<td>0%</td>
<td>10%</td>
<td>13.95</td>
</tr>
<tr>
<td>10%</td>
<td>5%</td>
<td>15%</td>
<td>11.36</td>
</tr>
<tr>
<td>15%</td>
<td>10%</td>
<td>20%</td>
<td>9.18</td>
</tr>
<tr>
<td>20%</td>
<td>15%</td>
<td>25%</td>
<td>7.73</td>
</tr>
<tr>
<td>25%</td>
<td>20%</td>
<td>30%</td>
<td>5.15</td>
</tr>
<tr>
<td>30%</td>
<td>25%</td>
<td>35%</td>
<td>3.21</td>
</tr>
</tbody>
</table>

As we see, more the percentage of the returned products (i.e. the number of remanufacturable products) increases more the value of h decreases. Indeed, when the number of returned products increases, the remanufacturing inventory fills up and then the machine $M_2$ can fill more serviceable inventory (i.e. $u(t) > Z(t)$). Then, when the machine $M_2$ fills more S the customers demand is more satisfied. Thus, the unsatisfied demand decreases and normally the lost sales cost decreases; consequently, the optimal serviceable inventory level which minimizes the total cost decreases.

**V. CONCLUSION**

In this paper, we studied a manufacturing/remanufacturing system composed by two parallel machines, a serviceable inventory, a remanufacturing inventory and customers who demand a stochastic quantity of product. Stochastic fluid model is adopted to describe the system and to take into account machine failure, stochastic demand, stochastic returned products and remanufacturing products. The times to failure and times to repair are random variables with exponential distribution. The serviceable inventory level trajectories is studied and analyzed. The perturbation analysis estimates are determined and shown to be unbiased. These estimates are then implemented in an optimization algorithm for determining the optimal serviceable inventory level value. The impact of the percentage of the returned products on the value of the optimal serviceable inventory level h is studied. Indeed, more the percentage of the returned products increases; more the value of h decreases.

For future research, we will take into account the delivery time between the serviceable inventory and the customers. Also, we will consider a more complex model with random delivery time.

**REFERENCES**