A Quantized Filtering Scheme for Multi-Sensor Linear State Estimation with Non-Detectability at the Sensors and Fusion Center Feedback

Alex S. Leong, Subhrakanti Dey, and Girish N. Nair

Abstract—In this paper we consider state estimation of a discrete time linear system using multiple sensors, where the sensors quantize their individual innovations, which are then combined at the fusion center to form a global state estimate. It is assumed that detectability does not hold for at least one of the sensors. By allowing the fusion center to broadcast some information back to the sensors, full state estimates can be obtained at the sensors, even without detectability. We prove the stability of the estimation scheme under sufficiently high bit rates, and obtain asymptotic approximations for the error covariance matrix that relates the system parameters and quantization levels used by the different sensors.

I. INTRODUCTION

Linear state estimation using multiple sensors is a commonly performed task in areas such as radar tracking and industrial monitoring. Nowadays, much of the communication systems used in practice are digital in nature. Therefore, analog measurements made by sensors will need to be quantized before transmission to a central processor or fusion center over a bandwidth limited wireless channel.

We consider a discrete time linear system. A number of sensors take measurements, perform some local processing before transmitting a processed signal to a fusion center, which then combines these signals to form a global state estimate. At the sensor level, each sensor will quantize their innovations. This is motivated by the fact that for unstable systems, while the state will become unbounded, the (true) innovations process remains of bounded variance [1]. In this paper, we assume that detectability does not hold for at least one of the sensors, e.g. the sensor cannot fully observe the process. Thus, if the system is unstable and the sensor only uses local information to compute local state estimates, the local innovations process at that sensor will also have unbounded variance. However, by allowing some feedback from the fusion center back to the sensors, this problem can be overcome and the local innovations process will remain of bounded variance.

The work of [2] gave structural results on optimal coding for state estimation with measurements obtained over a finite rate digital link, though the focus is on determining minimum bit rates required for stability rather than performance analysis. For control problems with quantized state feedback, the performance with high rate quantization has been studied in e.g. [3] and [4]. The idea of quantizing innovations for estimation has been considered in [5], [6], [7] with different filtering equations from ours. However [6] and [7] only consider the case of a single sensor, while the multi-sensor setup in [5] does not involve a fusion center but instead requires sensors to broadcast their quantized innovations to all other sensors. In [8], a filter which involves quantizing the true innovations at the sensor is given, but it is shown that for unstable systems the mean squared error always becomes unbounded with this scheme. Particle filtering schemes are also considered in [8], though the performance of such schemes is difficult to analyze theoretically. The case of non-detectability at the individual sensors has been previously studied in [9], [10] in the context of state estimation with data-driven communications.

The paper is organized as follows. We present the system model in Section II. We first give the unquantized decentralized Kalman filter equations with feedback in Section III, in order to motivate our choice of quantized filtering equations which are presented in Section IV. We then prove stability of our filtering scheme in Section V. In Section VI we obtain an asymptotic approximation for the error covariance in terms of the number of quantization levels used by the different sensors, as well as the system parameters.

Numerical comparisons are made in Section VII.

Notations: A matrix $X \geq 0$ if it is positive semidefinite and $X > 0$ if it is positive definite. We use the big-$O$ notation, where for functions $f(.)$ and $g(.)$, we say that $f(x) = O(g(x))$ as $x \to x_0$ if there exists a constant $K$ such that $|f(x)| \leq K |g(x)|$ for all $x$ within some neighbourhood of $x_0$. We will call a matrix $O(1)$ if all its entries are $O(1)$, and call a matrix $O(\epsilon I)$ if all its entries are $O(\epsilon)$.

II. SYSTEM MODEL

Throughout this paper, we will use $k$ to denote the discrete time index, and $i$ the sensor index. We consider a discrete time vector linear system

$$x_{k+1} = Ax_k + w_k,$$  
(1)

where $x_k \in \mathbb{R}^n$ and $w_k$ is i.i.d. zero mean Gaussian with covariance matrix $\Sigma_w > 0$. There are $M$ different sensors each making scalar measurements:

$$y_{i,k} = C_i x_k + v_{i,k}, \quad i = 1, \ldots, M$$  
(2)
where \( y_{i,k} \in \mathbb{R} \), and \( v_{i,k} \) is i.i.d. zero mean Gaussian with variance \( \Sigma_{i,v} > 0 \). We assume that \( \{w_k\} \) and \( \{v_{i,k}\}, \forall i \) are mutually independent, and that the pair \((A, \Sigma_{w}^{1/2})\) is stabilizable. The pair \((A, C_i)\) is detectable, where \( C \triangleq \begin{bmatrix} C_{1}^{T} & \ldots & C_{M}^{T} \end{bmatrix}^{T} \), however there is at least one individual sensor pair \((A, C_i)\) that is not detectable. The case where the individual pairs \((A, C_i)\) are all detectable has been previously studied in [11].

It is assumed that the individual sensors can perform some local processing, with a fusion center then using an appropriate fusion rule to compute a global estimate of the state \( x_k \). See Fig. 1 for a diagram of the system model.

![System model diagram](image)

**Fig. 1. System model**

### III. Decentralized Kalman Filter with Feedback

In [12], it is shown that in the case where there is no quantization, each sensor can run its own individual Kalman filter to obtain local estimates of the full state \( x_k \), which can then be combined at the fusion center to obtain a global state estimate, that is the same as if the fusion center had access to the individual measurements. However, if for some sensor \( i \) the pair \((A, C_i)\) is not detectable and \( A \) is an unstable matrix, then the local error covariance of this sensor becomes unbounded over time. Consequently the local true innovations will also have unbounded variance, making quantization of the local innovations in Section IV infeasible.

In [13], the approach taken was for sensors to only estimate their observable parts of the state, similar to [9], [10]. An alternative approach which will be studied in this paper is to allow some feedback from the fusion center back to the sensors (e.g. a broadcast to all sensors), which will enable the sensors to have estimates of the full state. In this subsection we present the decentralized Kalman filter equations with fusion center feedback.

In [14] equations for a fully decentralized Kalman filter are presented, where each sensor sends their state and error covariance information to all the other sensors, which allows each sensor to compute the same state estimates as in the centralized case. One can easily adapt their scheme so that after the fusion center has computed the global state estimate, it then broadcasts some information back to the sensors to allow each sensor to also update their estimates with the global estimate. We present the equations below. In Section IV these equations will be further modified for our quantized filtering scheme.

Define the following local quantities:

\[
\hat{x}_{k|k}^{i,f} = E[x_k|y_{0}, \ldots, y_{k-1}, y_{i,k}]
\]

\[
\hat{P}_{k|k}^{i,f} = E[(x_k - \hat{x}_{k|k}^{i,f})(x_k - \hat{x}_{k|k}^{i,f})^{T}|y_{0}, \ldots, y_{k-1}, y_{i,k}]
\]

where \( y_k \triangleq (y_{1,k}^{T}, \ldots, y_{M,k}^{T})^{T} \), and the global quantities:

\[
\hat{x}_{k|k} = E[x_k|y_{0}, \ldots, y_{k-1}]
\]

\[
\hat{P}_{k|k} = E[(x_k - \hat{x}_{k|k})(x_k - \hat{x}_{k|k})^{T}|y_{0}, \ldots, y_{k}]
\]

Note that due to fusion center feedback, individual sensors will also be able to compute the global quantities, see (3).

The sensors run the following equations:

\[
\hat{x}_{k|k-1}^{i,f} = A\hat{x}_{k-1|k-1}^{i,f}
\]

\[
\hat{x}_{k|k}^{i,f} = \hat{x}_{k|k-1}^{i,f} + K_{i,k}^{f}(y_{i,k} - C_{i}\hat{x}_{k|k-1}^{i,f})
\]

\[
K_{i,k}^{f} = P_{k|k-1}^{i,f}C_{i}^{T}(C_{i}P_{k|k-1}^{i,f}C_{i}^{T} + \Sigma_{v})^{-1}
\]

\[
P_{k|k}^{i,f} = AP_{k-1|k-1}^{i,f}A^{T} + \Sigma_{w}
\]

\[
P_{k|k} = \sum K_{i,k}^{f}P_{k|k}^{i,f}
\]

Each sensor transmits \( y_{i,k} - C_{i}\hat{x}_{k|k-1}^{i,f} \) to the fusion center, which can then reconstruct \( \hat{x}_{k|k}^{i,f} \) and \( \hat{P}_{k|k}^{i,f} \) provided the fusion center has knowledge of all the sensor parameters \( C_{i} \) and \( \Sigma_{v} \) for \( i = 1, \ldots, M \). Such parameters can for instance be provided by the sensors to the fusion center beforehand in an offline manner.

The fusion center then runs the following equations:

\[
\hat{x}_{k|k-1}^{f} = A\hat{x}_{k-1|k-1}^{f}
\]

\[
\hat{x}_{k|k}^{f} = P_{k|k-1}^{f} \left( \sum_{i=1}^{M} K_{i,k}^{f}(\hat{x}_{k|k}^{i,f} - \hat{x}_{k|k}^{i,f}) \right) + \hat{P}_{k|k-1}^{f}
\]

\[
P_{k|k}^{f} = AP_{k-1|k-1}^{f}A^{T} + \Sigma_{w}
\]

\[
P_{k|k} = \sum K_{i,k}^{f}P_{k|k}^{i,f}
\]

(4)

where \( C \triangleq \begin{bmatrix} C_{1}^{T} & \ldots & C_{M}^{T} \end{bmatrix}^{T} \) and \( \Sigma_{v} \) is a diagonal matrix given by \( \Sigma_{v} \triangleq \text{diag}(\Sigma_{1,v}, \ldots, \Sigma_{M,v}) \). The fusion center then broadcasts the (generally vector) quantity

\[
(P_{k|k}^{f} - I)\hat{x}_{k|k}^{f} + \sum_{i=1}^{M} K_{i,k}^{f}(\hat{x}_{i|k-1} - \hat{x}_{k|k}^{i,f})
\]

back to all the sensors. One can show that this quantity corresponds to \( K_{k}^{f} = \hat{P}_{k|k-1}^{f}(C_{k}^{f}P_{k|k-1}^{f}C_{k}^{fT} + \Sigma_{v})^{-1} \), and has bounded covariance. The individual sensors can then reconstruct \( \hat{x}_{k|k}^{i,f} \) using...
the equation
\[
\hat{x}_{k|k} = \hat{x}_{k|k-1} + (P_{k|k}^{-1} - I)\hat{x}_{k|k-1}
\]
and \( P_{k|k} \) using the equation
\[
P_{k|k} = P_{k|k-1} - P_{k|k-1} C^T (C P_{k|k-1} C^T + \Sigma_v)^{-1} C P_{k|k-1}
\]
provided the sensors know the parameters \( C \) and \( \Sigma_v \), which depend on all the other sensors. Such knowledge can be broadcast by the fusion center to the sensors beforehand in an offline manner.

As \( k \to \infty \), the global error covariance matrix \( P_{k|k} \) converges to the steady state value \( P_{\infty|\infty} \) that satisfies the algebraic Riccati equation
\[
P_{\infty|\infty} = AP_{\infty|\infty} A^T + \Sigma_w - AP_{\infty|\infty} C^T (C P_{\infty|\infty} C^T + \Sigma_v)^{-1} C P_{\infty|\infty} A^T
\]
(5)

IV. QUAINTIZED FILTERING SCHEME WITH FEEDBACK

In this paper we consider a suboptimal quantized filtering scheme which are a modified version of the unquantized decentralized Kalman filtering equations given in (3)-(4). We will make the simplifying assumption that the quantization error in the broadcast from the fusion center to the sensors is negligible when compared to the quantization errors in the sensor transmissions, since the fusion center has more resources and can transmit with higher power and better quality. Considering quantization errors in the fusion center feedback will require modifications to our scheme, and will result in a significantly more complicated analysis.

The individual sensors run the following equations, for \( i = 1, \ldots, M \):
\[
\hat{x}_{k|k-1} = A \hat{x}_{k|k-1} - B_i \hat{x}_{i,k}
\]
\[
\tilde{x}_{i,k} = \hat{x}_{i,k} + K_{i,k} q_{i,k}
\]
\[
K_{i,k} = P_{i,k|k-1} C_{i,k}^T (C_i P_{i,k|k-1} C_{i,k}^T + \Sigma_{i,v})^{-1}
\]
(6)
\[
P_{i,k|k-1} = AP_{i,k|k-1} A^T + \Sigma_{w}
\]
\[
P_{i,k} = P_{i,k|k-1} - K_{i,k} C_i P_{i,k|k-1}
\]

The fusion center runs the following equations
\[
\hat{x}_{k|k-1} = A \hat{x}_{k|k-1}
\]
\[
\tilde{x}_{i,k} = P_{i,k} \left( \hat{x}_{k|k-1} - \sum_{i=1}^{M} \left( P_{i,k|k-1} \hat{x}_{i,k} - P_{i,k|k-1} \hat{x}_{k|k-1} \right) \right)
\]
(7)

In (6)-(7), \( \hat{x}_{i,k} \) are the local state estimates and \( P_{i,k} \) the approximation to the local error covariances. Similarly \( \hat{x}_{k|k-1} \), \( \tilde{x}_{i,k} \), \( P_{k|k-1} \), \( P_{i,k} \) are the corresponding global quantities. The term \( l_{i,k} q_{i,k} \) is the quantization of \( y_{i,k} - C_i \tilde{x}_{i,k} \), that is sent by sensor \( i \) to the remote fusion center, and \( \Sigma_{i,k} = \text{diag}(\Sigma_{i,n,k}, \ldots, \Sigma_{i,M,k}) \) is a diagonal matrix with terms \( \Sigma_{i,n,k} \) to account for the quantization noise variances of each sensor. See Section IV-A on how the quantizers \( q_{i,k}(\cdot) \) are chosen and the corresponding expressions for \( \Sigma_{i,n,k} \). The terms \( l_{i,k} \) are the scaling factors of each sensor \( i \), which allows one to adaptively change the quantizer range to account for possible quantizer overload, similar to e.g. [15], [2], and is needed in order to prove the stability of the quantized filtering scheme for noises with infinite support [2]. See Section IV-B for details on how \( l_{i,k} \) are chosen.

The sensors send the quantized values \( l_{i,k} \frac{y_{i,k} - C_i \tilde{x}_{i,k}}{l_{i,k}} \) to the remote fusion center, which can reconstruct \( \tilde{x}_{i,k} \) and \( P_{i,k} \) given knowledge of the sensor parameters \( C_i \), \( \Sigma_{i,v} \) and the scaling factor update rules for \( l_{i,k} \).

The fusion center then broadcasts the vector quantity
\[
P_{k|k} P_{k|k-1}^{-1} (\hat{x}_{k|k-1} - I) \tilde{x}_{k|k-1}
\]
(8)
back to the sensors, which can reconstruct \( \hat{x}_{k|k} \) and \( P_{k|k} \) given knowledge of the parameters \( C \) and \( \Sigma_v \). Again, such knowledge can be provided via a broadcast to the sensors in an offline manner.

We will often use the shorthand \( P_{k} \triangleq P_{k|k-1} \).

A. Choice of quantizer

In this paper the quantizers will be assumed to be fixed rate quantizers, but with time-varying quantizer ranges. In particular, the performance of our quantized filtering scheme using (scalar) uniform quantizers will be analyzed.\(^2\)

Let \( N_i \) denote the number of quantization levels for the quantizer used by sensor \( i \). The rate of the quantizer used by sensor \( i \) is then denoted by \( R_i = \log_2(N_i) \). Thus the case of high rate quantization will refer to either large \( R_i \) or large \( N_i \) interchangeably.

Under high rate quantization, we assume that the quantity \( y_{i,k} - C_i \tilde{x}_{i,k} \) is approximately \( N(0, C_i P_{i|k} C_i^T + \Sigma_{i,v}) \), since the quantization noise is dominated by the Gaussian process and measurement noise. Some studies on the accuracy of the Gaussian approximation of the quantization error, for the case of logarithmic quantizers, can be found in [7].

Suppose now that \( \tilde{q}_{i,k}(\cdot) \) is a quantizer of \( N_i \) levels designed for quantization of \( N(0,1) \) random variables. Then in (6) we can rewrite
\[
\tilde{q}_{i,k}(y_{i,k} - C_i \tilde{x}_{i,k}) = \sigma_{i,k} l_{i,k} \tilde{q}_{i,k}(y_{i,k} - C_i \tilde{x}_{i,k})
\]
where \( \sigma_{i,k} \triangleq C_i P_{i|k} C_i^T + \Sigma_{i,v} \).

For uniform quantization of Gaussian random variables, the asymptotically optimal step sizes of the quantizer for

\(^2\)Other quantizers can also be analyzed using similar techniques, such as the scalar Lloyd-Max “optimal” quantizer and lattice vector quantizers, but are omitted for brevity.
large $N_i$ has been derived in [16]. Under high rate quantization, the step size $\Delta_{N_i}$ is asymptotically
\[ \Delta_{N_i} \sim \frac{4\sqrt{\ln N_i}}{N_i} \sigma \]
where $\sigma^2$ is the variance of the Gaussian random variable that is to be quantized. Using similar notation to [2], the uniform quantizer of [16], for variance $\sigma^2 = 1$, can then be expressed as follows: Partition the real line into $N$ intervals $(-\infty, -2(N_i-2)/\sqrt{N_i},) \cup \cdots \cup (-2(N_i-2)/\sqrt{N_i}, -2(N_i-2)/\sqrt{N_i}) \cup \cdots \cup (\Delta_{N_i}), \cdots, (2(N_i-2)/\sqrt{N_i}, \infty)$. Label these intervals $I(1), I(2), \ldots, I(N_i)$ respectively. The quantized value of $x$ is then
\[ q_i(x) = \begin{cases} \text{midpoint of } I(\omega), & x \in I(\omega), \omega \in \{2, \ldots, N_i - 1\} \\ -2(N_i-1)/\sqrt{N_i}, & x \in I(\omega), \omega = 1 \\ 2(N_i-1)/\sqrt{N_i}, & x \in I(\omega), \omega = N_i \end{cases} \]
where $\omega$ represents the index of the quantizer range that $x$ lies in. The resulting squared error distortion is asymptotically (see [16])
\[ D_{N_i} \sim \frac{4\ln N_i}{3N_i^2} \triangleq \delta_{N_i} \]
The term $\Sigma_{i,n,k}$ in (6) is then defined as
\[ \Sigma_{i,n,k} \triangleq \delta_{N_i}(C_i P_k C_i^T + \Sigma_{i,v}) \]
where $\delta_{N_i} = \frac{4\ln N_i}{3N_i^2}$. We can therefore write
\[ P_{k+1} = AP_k A^T + \Sigma_w - AP_k C^T (C P_k C^T + \Sigma_{i,v} + \Sigma_{n,k})^{-1} C P_k A^T \]
(9)
where
\[ \Sigma_{n,k} = \text{diag}(\delta_{N_i}(C_i P_k C_i^T + \Sigma_{i,v}), \ldots, \delta_{N_M}(C_M P_k C_M^T + \Sigma_{M,v})) \]
To conclude this subsection, we will derive a sufficient condition on the number of quantizer levels $N_i, i = 1, \ldots, M$ needed to ensure that the recursions $P_k$ in (9) converge to a steady state value $P_\infty$ as $k \to \infty$. First, pick a $\delta$ satisfying
\[ \frac{1}{1 + \delta} > 1 - \prod_{j} |\lambda_j^u(A)|^2 \]
(10)
where $\lambda_j^u(A)$ are the unstable eigenvalues of $A$. Now define the following operators:
\[ f(X) = AXA^T + \Sigma_w - AXC^T (CXC^T + \Sigma_v + \text{diag}(\delta_{N_1}(C_1XC_1^T + \Sigma_{1,v}), \ldots, \delta_{N_M}(C_MXC_M^T + \Sigma_{M,v})))^{-1} CXC^T \]
\[ g(X) = AXA^T + \Sigma_w - \frac{1}{1+\delta} AXC^T (CXC^T + \Sigma_v + \text{diag}(\delta_{N_1}(C_1XC_1^T + \Sigma_{1,v}), \ldots, \delta_{N_M}(C_MXC_M^T + \Sigma_{M,v})))^{-1} CXC^T \]
From properties of the modified algebraic Riccati equation (see [17] and the references therein), condition (10) is sufficient to ensure that the equation $g(X) = X$ has a solution. Similar to [18], now pick a $Q_0 \geq \Sigma_w$, and let $U$ be the solution to
\[ U = U A^T + Q_0 - \frac{1}{1 + \delta} AUC^T (CU^T + \Sigma_w)^{-1}CUA^T \]
(11)
Then one can easily verify that $g(U) \leq U$.
We want to show that $f(X) = X$ has a unique solution, i.e. $f(.)$ has a unique fixed point, for sufficiently large $N_i, i = 1, \ldots, M$. We have the following result.

Lemma 4.1: Let $\delta$ satisfy (10), and let $U$ be the solution to (11), where $Q_0 \geq \Sigma_w$. If $\delta_{N_1}, \ldots, \delta_{N_M}$ are such that
\[ AU^T + \Sigma_w - AUC^T (CU^T + \Sigma_v + \text{diag}(\delta_{N_1}(C_1UC_1^T + \Sigma_{1,v}), \ldots, \delta_{N_M}(C_MUC_M^T + \Sigma_{M,v})))^{-1} CUA^T \]
\[ \leq AU^T + \Sigma_w - \frac{1}{1 + \delta} AUC^T (CU^T + \Sigma_v + \text{diag}(\delta_{N_1}(C_1UC_1^T + \Sigma_{1,v}), \ldots, \delta_{N_M}(C_MUC_M^T + \Sigma_{M,v})))^{-1} CUA^T \]
(12)
holds, then the equation $f(X) = X$ has a unique solution.
Proof: See appendix.

By the characterization that a matrix is positive definite if and only if the determinants of all its principal submatrices are positive [19], the fact that determinants are continuous in the matrix entries, and that $\delta_{N_i} \to 0$ as $N_i \to \infty$, condition (12) of Lemma 4.1 can be met for $N_i, i = 1, \ldots, M$ sufficiently large. Consequently other quantities such as $\sigma_{i,k}, K_i, K_k, P_{i,k}, P_{k,i}$ also converge to steady state values as $k \to \infty$, which will be used in the next subsection, see e.g. (15).

B. Choice of scaling factors

We now describe how the scaling factors $l_{i,k}$ in (6) are chosen. Following a similar approach to [20], the scaling factors $l_{i,k}$ are updated recursively as follows:
\[ l_{i,k} = \frac{||C_i|| l_{i,k} + d_{i,v}}{\sigma_{i,k}} \]
\[ l_{i,k+1} = ||A(I - K_k C)|| l_{i,k} + d_{i,w} + \sum_{j=1}^{M} ||A P_{k,j} P_{k,j}^{-1} K_{j,k}|| d_{j,v} \]
\[ + \sum_{j=1}^{M} ||A P_{k,j} P_{k,j}^{-1} K_{j,k}|| (||C_j|| l_{j,k} + d_{j,v} + \kappa_j(\omega_{j,k}) \right) \]
(13)
where $d_{i,v} > 0, d_{i,w} > 0, \forall i$ are arbitrary constants, $K_k \equiv P_{k,k-1} - P_{k,k-1} C^T (P_{k,k-1} C^T + \Sigma_v + \Sigma_{n,k})^{-1}$, and $||.||_p$ is a matrix norm that approximates the spectral radius (norms which can approximate the spectral radius arbitrarily closely are known to exist, see [19]). $\kappa_i(\omega_{i,k})$ is defined as:
\[ \kappa_i(\omega_{i,k}) = \left\{ \begin{array}{ll} \beta(N_i), & \omega_{i,k} \in \{2, \ldots, N_i - 1\} \\
\gamma(N_i), & \omega_{i,k} \in \{1, N_i\} \end{array} \right. \]
(14)
where $\beta(\cdot)$ and $\gamma(\cdot)$ are functions of $N_i$ that need to be chosen appropriately in order to prove stability in Section V, see e.g. Lemma 5.1. We will choose here
\[ \beta(N_i) = \frac{2\sqrt{\ln N_i}}{N_i}, \]
which corresponds to the half length of a quantizer interval, similar to [2]. The function \( \gamma(.) \) will be chosen to be
\[
\gamma(N_j) = \sqrt{\ln N_j}.
\]
In order to update the scaling factors, each sensor needs to know the values \( \kappa_{ij}(\omega_{j,k}) \) of all other sensors, which can be obtained if the fusion center broadcasts these values back to the sensors, in addition to the quantity (8).

The choice of values of \( d_{i,v} \) and \( d_w \) in (13) will affect the performance of the filtering scheme. The intuitive reason is that the locations of the quantizer points for the quantizer \( \tilde{q}_i(.) \) are designed assuming \( l_{i,k} = 1 \), so the performance is expected to perform well when \( l_{i,k} \approx 1 \) most of the time. Under high rate quantization, this can be achieved as follows. For a given \( d_w, d_{i,v} \) are chosen from the solution \((\tilde{l}_{1,min}, \ldots, \tilde{l}_{M,min}, d_{1,v}, \ldots, d_{M,v})\) of the set of 2M linear equations
\[
\begin{align*}
\tilde{l}_{1,min} &= ||A(I-KC)||_F \tilde{l}_{1,min} + d_w + \sum_{j=1}^{M} ||APP_j^{-1}K_j|| d_{j,v} + \sum_{j=1}^{M} ||APP_j^{-1}K_j|| ||C_j|| \tilde{l}_{j,min} + d_{j,v}|| ||B(N_j)||, i = 1, \ldots, M \\
\frac{||C_i||}{||C_{i,min} + d_{i,v} \sigma_i ||} &= 1, \quad i = 1, \ldots, M
\end{align*}
\]
where \( \sigma_i, K_j, K, P_j, P \) are the steady state values of \( \sigma_{i,k}, K_{i,k}, P_{j,k}, P_{k} \) respectively.

Lemma 4.2: Let \( d_{i,v} > 0 \) and \( d_w > 0 \) satisfy (15), and suppose that \( \omega_{i,k} \in \{2, \ldots, N_i - 1\}, i = 1, \ldots, M, \forall k \). If \( N_i, i = 1, \ldots, M \) are large enough that
\[
||A(I-KC)||_F + ||APP_i^{-1}K_i|| ||C_i|| ||B(N_i)|| < 1, \quad i = 1, \ldots, M
\]
then \( l_{i,k} \rightarrow 1, i = 1, \ldots, M \).

Proof: See Appendix.

Thus in the case of high rate quantization, where quantizer saturation is rare so that \( \omega_{i,k} \in \{2, \ldots, N_i - 1\} \) for much of the time, one can keep \( l_{i,k} \) close to 1 with this choice of \( d_{i,v} \) and \( d_w \).

V. STABILITY OF QUANTIZED FILTERING SCHEME

Define the estimation error
\[
f_k = x_k - \hat{x}_{k|k-1}
\]
The objective in this subsection is to prove Theorem 5.2, which says that \( \mathbb{E}[||f_k||^2] \) is always bounded, when using our choice of \( l_{i,k} \) in (13) and sufficiently high bit rates for all sensors (or sufficiently large \( N_i, \forall i \)).

Similar to [2], [20], the approach used to prove this is as follows: instead of showing directly that \( \mathbb{E}[||f_k||^2] \) is bounded, we show instead that upper bounds to \( \mathbb{E}[||f_k||^2] \), given by \( ||f_k||_x, ||l_{i,k}||^2 ||f_k||_x ||f_k||_x ||f_k||_x \), are unbounded, where \( ||f_k||_x, ||l_{i,k}||^2 ||f_k||_x ||f_k||_x ||f_k||_x \), are defined as
\[
||X, L|| \triangleq \sqrt{\mathbb{E}[L^2 + ||X||^2 + ||L||^2]}
\]
for some random vector \( X \) and random variable \( L \geq 0 \), and some \( \epsilon > 0 \). The fact that \( ||X, L||^2 \) is an upper bound to \( \mathbb{E}[||X||^2] \) is proved in [2], and further pseudo-norm properties of \( ||\cdot, \cdot|| \), namely
\[
||dX, dL||_e = d||X, L||_e, \forall d > 0
\]
\[
||X_1 + X_2, L_1 + L_2||_e \leq ||X_1, L_1||_e + ||X_2, L_2||_e
\]
are proved in [20]. We first have the following result, which is similar to Lemma 5.2 of [2], but for the uniform quantizer of [16].

Lemma 5.1: Let \( X \in \mathbb{R}, L > 0 \) be random variables with \( \mathbb{E}[||X||^2 + ||L||^2 < \infty \) for some \( \epsilon > 0 \). Suppose \( \gamma(N) = \sqrt{\ln N} \) in (14). Then for \( N \geq 4 \) and the uniform quantizer of [16],
\[
||X - Lq(\gamma(N))||_e^2 \leq \max\{2\beta(N)^2, 3 \frac{\mathbb{E}[||X||^2]}{\ln N^{3/2}} \} ||X, L||^2
\]
Proof: The proof is omitted. See [13].

In the case where all moments of \( X \) exist (e.g. if \( X \) is a Gaussian random variable), Lemma 5.1 will hold for any \( \epsilon > 0 \). In particular, for a given \( N \), one can choose \( \epsilon(N) \) such that \( \mathbb{E}[||X||^2] \leq 2\beta(N)^2 \), so one then has for suitably chosen \( \epsilon(N) \) that \( ||X - Lq(\gamma(N))||_e^2 \leq 2(2\beta(N)^2||X, L||^2) \).

We now have the following stability result.

Theorem 5.2: Let \( N_i, i = 1, \ldots, M \) be such that the matrix \( \Xi \) is stable, where the entries of \( \Xi \) are given by
\[
\Xi_{ij} = \begin{cases} ||A(I-KC)||_F + ||APP_i^{-1}K_j|| ||\tilde{B}(N_i)|| ||C_j||, & i = j \\
||APP_i^{-1}K_j|| ||\tilde{B}(N_i)|| ||C_j||, & i \neq j \\
\end{cases}
\]
is stable. Then \( \mathbb{E}[||f_k||^2] \) is bounded \( \forall k \).

Proof: See Appendix.

VI. ASYMPOTIC ANALYSIS

The quantity \( P_k \) in (7) can be regarded as an approximation to the true error covariance. In this subsection we will determine the asymptotic behaviour of \( P_{\infty} \) for large \( N_i \), \( i = 1, \ldots, M \) (corresponding to the situation of high rate quantization at all sensors), where \( P_{\infty} \) is the limit of \( P_k \) as \( k \rightarrow \infty \), that satisfies the equation
\[
P_{\infty} = AP_{\infty} A^T + \Sigma_w - AP_{\infty} C^T (CP_{\infty} C^T + \Sigma_w + \Sigma_v) \Sigma_v C P_{\infty} A^T
\]
(18)

Similar to [11], assume that \( P_{\infty} \) takes the form
\[
P_{\infty} = \Phi_0 + \sum_{i=1}^{M} \delta_{N_i} \Phi_{1,i} + \sum_{i,j} O(\delta_{N_i} \delta_{N_j}) I
\]
where \( \Phi_0, \Phi_{1,i}, i = 1, \ldots, M \) are matrices not dependent on \( N_i \). Using similar asymptotic analysis techniques to [11] (the details are omitted for brevity), we can determine \( \Phi_0 \) by solving the algebraic Riccati equation
\[
\Phi_0 = A \Phi_0 A^T - \Sigma_w - A \Phi_0 C^T (C \Phi_0 C^T + \Sigma_v)^{-1} C \Phi_0 A^T
\]
and \( \Phi_{1,i} \) by solving the discrete Lyapunov equations
\[
\Phi_{1,i} = (A - A \Phi_0 C^T (C \Phi_0 C^T + \Sigma_v)^{-1} C) \Phi_{1,i}
\]
\[
+ (A - A \Phi_0 C^T (C \Phi_0 C^T + \Sigma_v)^{-1} C)^T
\]
\[
+ A \Phi_0 C^T (C \Phi_0 C^T + \Sigma_v)^{-1} F_i (C \Phi_0 C^T + \Sigma_v)^{-1} C \Phi_0 A^T
\]
where $F_i$ is a diagonal matrix with $i$-th diagonal entry equal to $C_i P_k^i C_i^T + \Sigma_{i,v}$.

VII. NUMERICAL RESULTS

Consider a two sensor situation, with parameters

$$A = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}, \quad \Sigma_w = I, \quad C_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix},$$

$\Sigma_{1,v} = 1, \quad C_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}, \quad \Sigma_{2,v} = 0.5$. One can easily verify that the sensor pairs $(A, C_i), i = 1, 2$ are not detectable.

In Fig. 2 we plot the results from Monte Carlo simulations of the true error covariance $\text{tr}(x_k - \hat{x}_{k|k-1})(x_k - \hat{x}_{k|k-1})^T$, together with $\text{tr}(P_\infty)$ and the asymptotic expression (19) for $\text{tr}(P_\infty)$, for different values of $N_1 = N_2 = N$. We see that the asymptotic approximations to $P_\infty$ become more accurate as $N$ increases.

![Fig. 2. Error covariance and asymptotic expression](image)

VIII. CONCLUSION

In this paper we have considered a multi-sensor quantized filter, where individual sensors do not have detectability. By allowing some fusion center feedback, full state estimates can still be obtained at the sensors. We have given conditions on the number of quantization levels needed to guarantee filter stability, and studied asymptotic analysis of its error performance in the case of high rate quantization.

**APPENDIX**

A. Proof of Lemma 4.1

The arguments in the proof of Theorem 3.3 of [21] can be adapted to prove that $f(.)$ has a unique fixed point. The main difference is that condition (v) of the proof, showing that there exists a $V$ such that $f(V) \leq V$, will require a different argument. We will verify that the $U$ in the statement of the Lemma will satisfy this condition. By (12), we have

$$f(U) = AU^T + \Sigma_w - AU^T \left( CUC^T + \Sigma_v + \text{diag}(\delta N_1(CUC^T + \Sigma_v), \ldots, \delta N_M(CMUC_M^T + \Sigma_M, v)) \right)^{-1} \Sigma_v + \text{diag}(\delta N_1(CUC^T + \Sigma_v), \ldots, \delta N_M(CMUC_M^T + \Sigma_M, v))^{-1} \Sigma_w$$

+ diag($\delta N_1(CUC^T + \Sigma_v), \ldots, \delta N_M(CMUC_M^T + \Sigma_M, v))^{-1} \Sigma_v + \Sigma_w \leq AU^T + \Sigma_w - \frac{1}{1 + \delta} AU^T (CUC^T + \Sigma_v)^{-1} \Sigma_v + \Sigma_w \leq AU^T + Q_0 - \frac{1}{1 + \delta} AU^T (CUC^T + \Sigma_v)^{-1} \Sigma_v + \Sigma_w = U$$

by (11). Hence $f(U) \leq U$.

Similar arguments in the rest of the proof of Theorem 3.3 of [21] can then be used to conclude that $f(.)$ has a unique fixed point.

B. Proof of Lemma 4.2

Since $\omega_i, k \in \{2, \ldots, N_i - 1\}$, $\forall k$, we have $\kappa(\omega_i, k) = \beta(N_i), i = 1, \ldots, M, \forall k$. Hence (13) becomes

$$\hat{l}_{i,k+1} = \left( ||A(I - K_i C)||_p + \sum_{j=1}^{M} ||AP_{i|k} \hat{P}_{i,j|k} K_{i,k}|| \right) ||C_i|| \beta(N_i)$$

$$\times \hat{l}_{i,k} + d_w + \sum_{j=1}^{M} ||AP_{i|k} \hat{P}_{i,j|k} K_{j,k}|| d_{j,v}$$

$$\times \sum_{j \neq i} ||AP_{i|k} \hat{P}_{j,k|k} K_{j,k}|| (||C_j|| \delta_{j,k} + d_{j,v}) \beta(N_j)$$

Now if

$$\lim_{k \to \infty} \left( ||A(I - K_i C)||_p + \sum_{j=1}^{M} ||AP_{i|k} \hat{P}_{i,j|k} K_{i,k}|| \right) ||C_i|| \beta(N_i)$$

$$= ||A(I - KC)||_p + \sum_{j=1}^{M} ||AP_{i|k} \hat{P}_{i,j|k}|| ||C_i|| \beta(N_i) < 1,$$

then $\hat{l}_{i,k}$ converges to $\tilde{l}_i$ satisfying the equation

$$\tilde{l}_i = \left( ||A(I - K_i C)||_p + \sum_{j=1}^{M} ||AP_{i|k} \hat{P}_{i,j|k} K_{i,k}|| \right) \tilde{l}_i$$

$$+ d_w + \sum_{j=1}^{M} ||AP_{i|k} \hat{P}_{i,j|k} K_{i,k}|| d_{j,v}$$

$$+ \sum_{j \neq i} ||AP_{i|k} \hat{P}_{i,j|k}|| (||C_j|| \delta_{j,k} + d_{j,v}) \beta(N_j)$$

Since the $\tilde{l}_i$’s satisfy the same equation (15) as $\tilde{l}_{i,min}$, we have $\tilde{l}_i = \tilde{l}_{i,min}, \forall i$. Hence

$$l_{i,k} = \frac{||C_i|| \tilde{l}_{i,k} + d_{i,v}}{\sigma_{i,k}} \Rightarrow \frac{||C_i|| \tilde{l}_{i,min} + d_{i,v}}{\sigma_{i,k}} = 1.$$

C. Proof of Theorem 5.2

From (6)-(7) we have

$$\hat{x}_{k+1|k} = AP_{k|k} \left( P_{k|k-1}^{-1} \hat{x}_{k|k-1} + \sum_{i=1}^{M} \left( P_{i|k}^{-1} \hat{x}_{i|k-1} + K_{i,k} \sigma_{i,k} \hat{y}_{i,k} \left( \frac{y_{i,k} - C_i \hat{x}_{i|k-1}}{\hat{l}_{i,k} \sigma_{i,k}} \right) - P_{k|k-1}^{-1} \hat{x}_{k|k-1} \right) \right)$$

$$= A \hat{x}_{k|k-1} + AP_{k|k} \left( \sum_{i=1}^{M} \left( P_{i|k}^{-1} \sigma_{i,k} \hat{y}_{i,k} \left( \frac{y_{i,k} - C_i \hat{x}_{i|k-1}}{\hat{l}_{i,k} \sigma_{i,k}} \right) \right) \right)$$
by using the relation $P_{k|k} \left[ P_{k|k-1}^{-1} + \sum_{i=1}^{M} \tilde{P}_{i,k|k}^{-1} - P_{k|k-1}^{-1} \right] = I$, which can be derived using [14]. Then

$$f_{k+1} = x_{k+1} - \hat{x}_{k+1|k}$$

$$= A_{f} f_{k} + w_{k} - A_{P} P_{k|k} \sum_{i=1}^{M} \tilde{P}_{i,k|k}^{-1}\sigma_{i,k} l_{i,k} K_{i,k} \tilde{q}_{i} \left( \frac{C_{i} f_{k} + v_{i,k}}{\sigma_{i,k} l_{i,k}} \right)$$

$$= (I - K_{k} C) f_{k} + w_{k} - A_{P} P_{k|k} \sum_{i=1}^{M} \tilde{P}_{i,k|k}^{-1} l_{i,k} K_{i,k} \tilde{q}_{i} \left( \frac{C_{i} f_{k} + v_{i,k}}{\sigma_{i,k} l_{i,k}} \right)$$

$$+ A_{P} P_{k|k} \sum_{i=1}^{M} \tilde{P}_{i,k|k}^{-1} \sigma_{i,k} l_{i,k} K_{i,k} \left[ \frac{C_{i} f_{k} + v_{i,k}}{\sigma_{i,k} l_{i,k}} - l_{i,k} \tilde{q}_{i} \left( \frac{C_{i} f_{k} + v_{i,k}}{\sigma_{i,k} l_{i,k}} \right) \right]$$

where we have used the relation $P_{k|k} \sum_{i=1}^{M} \tilde{P}_{i,k|k}^{-1} K_{i,k} C_{i} = K_{k} C$, which can also be derived using [14].

Then, from the definition (13) and the pseudo-norm properties (17) we have

$$\|f_{k+1} - \tilde{f}_{k+1|k}\|_{*} \leq \|A(I - K_{k} C)\|_{p} \|f_{k} - \tilde{f}_{k|k}\|_{*} + \|w_{k}, d_{w}\|_{*}$$

$$+ \sum_{j=1}^{M} \|A_{P} P_{k|k} \tilde{P}_{j,k|k}^{-1} K_{j,k}\| \|v_{j,k}, d_{v,j}\|_{*} + \|A_{P} P_{k|k} \tilde{P}_{j,k|k}^{-1} K_{j,k}\| \sigma_{j,k} \frac{\|C_{j} f_{k} + v_{j,k}\|_{*}}{\sigma_{j,k}}$$

$$\leq \|A(I - K_{k} C)\|_{p} \|f_{k} - \tilde{f}_{k|k}\|_{*} + \|w_{k}, d_{w}\|_{*}$$

$$+ \sum_{j=1}^{M} \|A_{P} P_{k|k} \tilde{P}_{j,k|k}^{-1} K_{j,k}\| \|v_{j,k}, d_{v,j}\|_{*}$$

$$+ \sum_{j=1}^{M} \|A_{P} P_{k|k} \tilde{P}_{j,k|k}^{-1} K_{j,k}\| \sigma_{j,k} \frac{\|C_{j} f_{k} + v_{j,k}\|_{*}}{\sigma_{j,k}}$$

where the second inequality uses Lemma 5.1. Writing the inequalities in matrix form as

$$\begin{bmatrix} \|f_{k+1} - \tilde{f}_{k+1|k}\|_{*} \\ \|f_{k+1} - \tilde{f}_{k|k}\|_{*} \end{bmatrix} \leq \Xi \begin{bmatrix} \|f_{k} - \tilde{f}_{k|k}\|_{*} \\ \|w_{k}, d_{w}\|_{*} \end{bmatrix}$$

$$+ \sum_{j=1}^{M} \|A_{P} P_{k|k} \tilde{P}_{j,k|k}^{-1} K_{j,k}\| \|v_{j,k}, d_{v,j}\|_{*}$$

$$+ \sum_{j=1}^{M} \|A_{P} P_{k|k} \tilde{P}_{j,k|k}^{-1} K_{j,k}\| \sigma_{j,k} \frac{\|C_{j} f_{k} + v_{j,k}\|_{*}}{\sigma_{j,k}}$$

where the inequality is applied component-wise, and noting that $K_{i,k}, K_{k}, P_{j,k|k}, P_{k|k}$ converge to steady state values as $k \to \infty$, then gives the result.

REFERENCES