Modal Analysis Using Time-Frequency Transform

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Abstract

Traditional modal analysis methods use either time domain or frequency domain approaches. Because vibration signals are generally non-stationary, time and frequency information is needed simultaneously in many cases. This paper presents an overview of the applications of joint time-frequency methods for modal analysis. Since a joint time-frequency analysis can decouple vibration modes, it has an advantage, especially when information about the excitation is not available. In this paper the modal parameters are estimated using Gabor analysis. Numerical simulations and experiments have also been carried out.

1 INTRODUCTION

Three elementary parameters used in the modal analysis of a dynamic system are its natural frequency, modal damping and magnitude. A considerable amount of attention has been devoted to frequency response function (FRF) analysis [1,2,3]. However, the conventional Fourier analysis approach is restricted to only one domain, because the elementary functions used to decompose the signals exist from negative infinity to positive infinity in the time domain. In other words, Fourier analysis cannot provide information on how the frequency content of a signal changes with time. Therefore, Fourier analysis is only useful for stationary signals. For non-stationary signals, time-frequency representations are needed. Consider a simple case, the decay of free vibration. The decaying vibration signal is non-stationary because its magnitude decreases exponentially. So it can be viewed as a transient phenomenon. Although it is not difficult to calculate the loss factor by the decay rate method for lightly damped systems, the method only works for a single mode at resonance, and the result is very sensitive to noise. Joint time-frequency analysis (JTAF) can be used instead to separate the modal components contained in signals and to reduce noise. The modal parameters of each mode separated thus can be extracted.

Joint time-frequency analysis algorithms fall into two categories: the linear JTFA and the quadratic JTFA. In the linear JTFA, the short-time Fourier transform (STFT) and the Gabor expansion, which can be regarded as the inverse of the STFT, are two algorithms. If we consider the linear JTFA as the evolution of the conventional Fourier transform, the quadratic JTFA is the counterpart of the standard power spectrum. Quadratic algorithms include the Gabor spectrogram, Cohen’s Class and the adaptive spectrogram [4,5,6]. The difference between linear and quadratic JTFA methods is that the linear transform can be inverted to reconstruct the time signal. Thus, the linear transform is suitable for signal processing, such as time-variant filtering. However, the quadratic JTFA describes the energy distribution of the signal in the joint time-frequency domain, which is useful for signal analysis. Since the phase information is lost in the quadratic time-frequency representation, the time histories cannot be reconstructed. In this paper both linear and quadratic JTFA approaches are used for damping calculations.

In some applications, information about the excitation force is not available. Schwarz and Richardson use a curve fitting technique to estimate the modal parameters from ambient response data [7]. Bonato et al. use Cohen’s class to estimate the modal parameters from the non-stationary response to a unknown excitation [8]. However, the algorithms in Cohen’s class, such as the Wigner-Ville Distribution, have cross-term interferences in the time-frequency representation [5]. In this paper we study the modal parameter estimation using the Gabor expansion and the Gabor spectrogram.

2 GABOR ANALYSIS

2.1 Gabor transform

For a given discrete time sequence \( x[k] \), the Gabor transform can be computed from
\[
C_{m,n} = \sum x[k] \gamma^*[k - mT] e^{-j2\pi nk/N},
\]

where \(C_{m,n}\) is a matrix whose entries are called the Gabor coefficients, \(\gamma[k]\) is the analysis window, and \(^*\) denotes the complex conjugate. The parameters \(T\) and \(N\) represent the discrete time sampling interval and the total number of frequency lines [5]. The analysis function \(\gamma[k]\) is localized in the joint time-frequency domain. So the Gabor coefficients will depict the local time-frequency properties of \(x[k]\).

Random noise is evenly distributed over the entire joint time-frequency domain because it is not limited to a short time period or a narrow frequency band. On the contrary, the joint time-frequency representation of a signal is always concentrated in a relatively small region. After identifying the signal component, a mask can be applied to filter the signal components and take the inverse transform in order to obtain the noise-free waveform signal in the time domain.

After computing the Gabor coefficients by Eq. (1), a time-variant filter, which is actually a two-dimensional binary mask function \(M_{m,n}\), is used to modify the Gabor coefficient as

\[
\hat{C}_{m,n} = M_{m,n} C_{m,n}.
\]
The component of interest can then be extracted. As long as some requirements are satisfied, the component in the time domain can be reconstructed as

$$\hat{x}[k] = \sum_{M=0}^{M-1} \sum_{n=0}^{N-1} \hat{C}_{m,n} h[k - mT] e^{j\frac{2\pi mk}{N}},$$

(3)

where $h[k]$ is called the synthesis function [5, 9]. Qian had shown that if the functions $h[k]$ and $γ[k]$ are identical, the Gabor coefficients of the reconstructed signal $\hat{x}[k]$ will be optimally close to $\hat{C}_{m,n}$, in the sense of least square error. This process is called orthogonal-like Gabor transformation [5]. In this procedure Eq. (1) is called the Gabor transform (or analysis). And the inverse transform Eq. (3) is called the Gabor expansion (or synthesis).

Figure 1 shows the procedure in Eqs. (1) to (3). In Fig. 1, (a) is a signal obtained from the free vibration of an aluminum cantilever beam, and (b) is the original Gabor coefficients calculated using Eq. (1). Three modal responses at 34.5 Hz, 214.4 Hz and 597.5 Hz can be seen clearly. The color intensity in the Gabor coefficient plot represents the displacement magnitude which is displayed in decibels. The noise is distributed in the entire time-frequency domain. By using three time-variant filters, or actually three mask matrices, the three modal responses can be decoupled as shown in Figs. 2 (c), (d) and (e). A 1024-point optimal Gaussian window is used to serve as the analysis and synthesis functions in this calculation. The three decoupled modes thus can be reconstructed using Eq (3). Figure 2 illustrates the reconstructed waveforms and their spectra. It is seen that the property of the signal is improved significantly and the noise is dramatically reduced. Since each reconstructed waveform becomes a single-mode signal, the natural frequency, magnitude, phase and damping ratio can be extracted easily.

![Fig. 2. (a) the three reconstructed modes, (b) comparison of the spectra of the original and reconstructed signals.](image)

2.2 Gabor spectrogram

As mentioned before, the Gabor transform is linear JTFA, but the Gabor spectrogram is quadratic. Based on the Gabor transform, the Gabor spectrogram is defined as

$$GS_D[i, k] = \sum_{m,n | |m-n|\leq D} C_{m,n} C_{m',n'} WVD_{h,i,h'}[i, k],$$

(4)

where $WVD_{h,i,h'}[i, k]$ is the cross Wigner-Ville distribution of the frequency-modulated Gaussian functions. The order of the Gabor spectrogram, $D$, controls the degree of smoothing [6]. Figure 3 illustrates the Gabor spectrogram of the same signal shown in Fig. 1 (a). The energy distribution of the three modes is clearly seen in the spectrogram.
From this the natural frequencies, damping ratios and magnitude relationships between these modes can be extracted. However, since the phase information is lost, time histories cannot be reconstructed from the spectrogram.

Fig. 3. The Gabor spectrogram of a free vibration signal obtained from a cantilever beam.

### 3 A GABOR ANALYSIS-BASED MODAL TESTING

#### 3.1 Damping Calculation

Basically there are four measures of damping: loss factor, quality factor, damping ratio and imaginary part of the complex modulus. However, these four measures are related. The loss factor $\eta$ and the damping ratio $\zeta$ are those most commonly used in measurements of damping. There are many references in which reviews of methods to measure damping are presented [10-16].

The free response of an underdamped single-degree-of-freedom (DOF) system due to an impact excitation is given by

$$y(t) = Ae^{-\zeta \omega_n t} \cos(\omega_d t - \varphi) ,$$  

(5)

where $\zeta$ is the damping ratio, $\omega_n$ is the undamped natural angular frequency and the damped natural angular frequency $\omega_d$ is

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} .$$  

(6)

For a small value of damping coefficient $\zeta$, $\omega_d \approx \omega_n$. The damping ratio can thus be calculated by obtaining the envelope. The traditional decay rate method in which the ratio of successive peak amplitudes is measured, is very sensitive to noise. Another approach to obtain the envelope is to construct the analytic signal by using the Hilbert transform.

For a given real signal $y(t)$, its analytic signal $y_a(t)$ is

$$y_a(t) = y(t) + jH(y(t)) ,$$  

(7)

where the subscript $a$ stands for **analytic**, and the Hilbert transform of $y(t)$ is defined as

$$H(y(t)) = -\frac{1}{\pi} \int \frac{y(t')}{t-t'} dt' .$$  

(8)

Using Parseval’s formula, we can show that the Fourier transform of $H(y(t))$ is

$$H(\omega) = -j \text{sgn}(\omega)Y(\omega) ,$$  

(9)

where $\text{sgn}(\omega)$ is a sign function, and $Y(\omega)$ is the Fourier transform of $y(t)$. So the Hilbert transform can be easily realized by taking the fast Fourier transform (FFT) of $y(t)$. Then the magnitude of the vector $y_a(t)$ is the envelope of the signal $y(t)$. The damping ratio $\zeta$ associated with each mode can be evaluated by exponential curve fitting from

$$\zeta = -\frac{\Pi_0}{2mf} ,$$  

(10)
where $\pi_s$ is the power of the best exponential fit, and $f$ is the natural frequency extracted from the reconstructed signal.

Figure 4 (a) illustrates a simulated free vibration signal obtained using

$$y(t) = e^{-0.012\pi^2 200t} \sin(2\pi \cdot 200t) + 0.85e^{-0.008\pi^2 350t} \sin(2\pi \cdot 350t) + 0.6 + n(t), \quad (11)$$

There are two damped sinusoids, 200 Hz and 350 Hz. Their damping ratios are chosen to be 0.01 and 0.008, respectively. The noise level $n(t)$ is 0.1. The sampling rate is 1000 Hz and the data length is 300 points. Figure 4 (b) shows the Gabor coefficients. By using the Gabor analysis-based time-variant filters, the two simulated modes can be separated and reconstructed, as shown in Figs. 4 (c) and (d). Figures 4 (e) and (f) show the envelopes selected from the decay parts and the corresponding best exponential fits. The results and the mean squared errors (MSE) are listed in Table 1. The numbers in brackets are the relative errors.
<table>
<thead>
<tr>
<th>Mode</th>
<th>Damping ratio</th>
<th>Calculated damping ratio</th>
<th>MSE of exponential curve fitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>first</td>
<td>0.01</td>
<td>0.009998 (0.020%)</td>
<td>$2.46 \times 10^{-4}$</td>
</tr>
<tr>
<td>second</td>
<td>0.008</td>
<td>0.00797 (0.375%)</td>
<td>$1.14 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 1. Damping ratios of two-mode decay signal calculated using Gabor expansion.

If one is only interested in the damping, and reconstruction is not necessary, an alternative approach, the Gabor spectrogram can also be utilized. Figure 5 (a) illustrates the same signal simulated by (11). Figure 5 (b) is the Gabor spectrogram calculated using Eq. (4). The two modes, their frequencies, and the difference between their magnitudes and damping can be distinguished in this figure. By setting the frequency zoom, we can easily decouple the modes as shown in Figs. 5 (c) and (d).

If the ridges are extracted from the two 3-D plots, then the exponential decay curves are recovered again. Reference [17] describes several algorithms for ridge detection. An advanced wavelet application, the so-called ridgelet, has been developed in recent years [18]. In this paper, it is quite simple to recover the decaying vibration signals because the ridges are concentrated at fixed frequencies.

![Fig. 5. (a) The original signal generated by Eq. (12), (b) Gabor spectrogram, (c) and (d) the first and the second modes separated by zooming Gabor spectrogram.](image)

Unlike the exponential envelope of the signal which can be reconstructed using the Gabor expansion, the modulation term in Eq. (5) is squared because the Gabor spectrogram calculated using Eq. (4) is quadratic. So the term should be divided once more by two compared with Eq. (10). Then the damping ratio is

$$\zeta = -\frac{\Pi_{es}}{4\pi f}, \quad (12)$$

where $\Pi_{es}$ represents the exponential power of the ridge in the Gabor spectrogram. The results are listed in Table 2. The numbers in brackets are the relative errors. On comparing Tables 1 and 2, it can be seen that the error of the curve fitting obtained with the Gabor spectrogram method is smaller than that obtained with the Gabor expansion method.
Mode | Damping ratio | Calculated damping ratio | MSE of exponential curve fitting
--- | --- | --- | ---
first | 0.01 | 0.0101 (1.00%) | $3.00 \times 10^{-9}$
second | 0.008 | 0.00797 (0.375%) | $6.45 \times 10^{-10}$

Table 2. Damping results calculated using the Gabor spectrogram method.

3.2 Natural frequency

The natural frequency of each reconstructed signal can be calculated easily using the FFT. Figure 2 (d) shows the spectra of the three reconstructed modes. The natural frequencies correspond to the three peaks in the spectra.

3.3 Mode shape

For an N-degree-of-freedom (DOF) damped system, the general equations of motion written in matrix form are

$$[M]\ddot{y} + [C]y + [K]y = \{f(t)\},$$

where $[M]$, $[C]$ and $[K]$ are the mass, damping and stiffness matrices, and $\{f(t)\}$ is the force vector. For a passive system, the $N \times N$ matrices $[M]$, $[C]$ and $[K]$ are symmetric and positive definite. Then the mode shapes are identical to the mode shapes for the undamped system [19].

For an undamped system, the natural frequencies are the eigenvalues of the matrix $[M]^{-1}[K]$, and the mode shape corresponding to one natural frequency is the eigenvalue which satisfies

$$[M]^{-1}[K]\{y_i\} = \omega_i^2\{y_i\},$$

where $\omega_i$ is the $i$-th natural frequency, and the mode shape $\{y_i\}$ is an $N$-dimensional column vector

$$\{y_i\} = [y_{i1}, y_{i2}, \ldots, y_{iN}]^T.$$

Here the subscripts 1 through $N$ indicate the grid points which are evenly distributed on the $N$-DOFs structure.

The absolute values of these elements are the magnitudes of the vibration at the $N$ grid points. The signs indicate the phase differences. The same sign for two elements means that the vibration at these two points is in phase. Different signs mean that the two points are vibrating $180^\circ$ out-of-phase. Although the actual values of the vector elements are arbitrary, the ratios between them are unique. Therefore, even without the information of excitation, the mode shapes can be obtained by simply measuring the magnitudes and phase angles of the responses at all the grid points. We can choose one of the grid points as the reference point and compare the magnitudes and phase angles measured at other points to those measured at the reference point for all the modes of interest.

After the modes in a vibration signal measured at a point are decoupled and reconstructed, the magnitude and the phase angle for each mode can be obtained using the FFT. Fourier transformation is a complex process, resulting in both magnitude and phase information.

Let $p(t) = Ae^{-j\omega_d t}$ and $q(t) = \cos(\omega_d t - \phi)$. Then Eq. (5) becomes $y(t) = p(t) \cdot q(t)$. The single-sided spectrum of the pure cosine function $q(t)$ is $Q(\omega) = 2\pi A e^{-j\omega_d} \delta(\omega_d)$. The Fourier transform of $p(t)$ is

$$P(\omega) = \int_0^\infty Ae^{-j\omega t} e^{-j\omega t} dt = A \int_0^\infty e^{-j(\omega_n + j\omega)t} dt = -A \frac{A}{j\omega + j\omega_n} e^{-j(\omega_n + j\omega)t} \bigg|_0^\infty$$

$$= \frac{A}{j\omega + j\omega_n}.$$ (16)

Using the convolution property of the Fourier transform,

$$Y(\omega) = P(\omega) \ast Q(\omega) = \frac{2\pi A e^{-j\omega_d}}{j\omega_d + j\omega_n} = \frac{2\pi A e^{-j\omega_d}}{(\omega_n)^2 + (\omega_d)^2} \cdot j \cdot \frac{\omega_n - j\omega_d}{(\omega_d)^2}$$

$$= \frac{2\pi A}{(\omega_d)^2} [j(\omega_n \cos \phi - \omega_d \sin \phi) - j(\omega_n \cos \phi + \omega_d \sin \phi)]$$

$$= |Y(\omega)| \cdot e^{j\phi}.$$ (17)
It is easy to show that the calculated magnitude $|Y(\omega)|$ and phase angle $\Phi$ of a single-mode free vibration at its natural frequency $\omega_n$ is

$$|Y(\omega)| = \frac{2\pi A}{\omega_n^2}, \text{ and } \Phi = 90 - \varphi.$$  

(18)

Then the mode shape can be obtained by the magnitude ratios and phase differences which are

$$\frac{|Y_{i1}|}{|Y_{im}|} \equiv \frac{A_{i1}}{A_{im}} = \frac{|Y_{i1}(\omega)|}{|Y_{im}(\omega)|}, \quad m = 2, \cdots N,$$  

(19)

and

$$\varphi_{i1} - \varphi_{im} = \Phi_{i1} - \Phi_{im}, \quad m = 2, \cdots N.$$  

(20)

4 EXPERIMENTS

4.1 Experimental setup

A three-DOF model of an aluminum cantilever beam was studied experimentally as shown in Fig. 6. For a cantilever beam, the first three natural frequencies are

$$f_{1,2,3} = (1.194^2, 2.988^2, 5.5^2) \pi \cdot \kappa \cdot c_l / 8L^2,$$  

(21)

where $L$ is the length of the cantilever beam, $\kappa$ is the radius of gyration and $c_l$ is the longitudinal wave speed. The thickness of the cantilever beam is 6.43 mm. So $\kappa = h / \sqrt{12} = 1.86 \times 10^{-3}$ m. And the wave speed in aluminum is 5055 m/s. The theoretical natural frequencies are listed in Table 3.

A heavy steel block and two clamps were used to fix the beam at the left end. The free end, point #1 was selected as the reference point. Two Polytech laser vibrometers were employed to measure the beam responses. Since the reference point and one of the other points must be measured simultaneously, one laser vibrometer was fixed at point 1, and the other one was used to measure the responses at the other two points. A National Instruments PCMCIA 6036E card was used for data acquisition and analysis programs were developed in LabVIEW. The sampling rate was 1500 Hz. Totally 20 measurements were carried out at points #2 and #3, ten measurements at each point. Each data file also contains the free response acquired at the reference point.

4.2 Natural frequency and mode shape

Figures 1 and 2 show one of the typical measurements. Based on the Gabor transform procedure in Eqs. (1) to (3), the modes which were overlapping in one free vibration signal were separated and reconstructed. Then the natural frequency, magnitude and phase angle associated with each mode were extracted using the FFT.
Consequently, the magnitude ratios and phase differences at different grid points are determined using Eqs. (19) and (20). Table 3 lists the theoretical and calculated values. The measured natural frequencies are less than the theoretical values due to the non-ideal boundary conditions. However, the magnitude ratios and phase differences are quite accurate. Figure 7 compares the theoretical and measured mode shapes for the three modes.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fundamental</th>
<th>Second</th>
<th>Third</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (Hz)</td>
<td>35.788</td>
<td>224.398</td>
<td>628.265</td>
</tr>
<tr>
<td>$A_{11}/A_{12}$</td>
<td>1.792</td>
<td>2.175</td>
<td>1.365</td>
</tr>
<tr>
<td>$A_{11}/A_{13}$</td>
<td>5.968</td>
<td>1.577</td>
<td>1.385</td>
</tr>
<tr>
<td>$\phi_{11} - \phi_{12}$</td>
<td>0°</td>
<td>180°</td>
<td>180°</td>
</tr>
<tr>
<td>$\phi_{11} - \phi_{13}$</td>
<td>0°</td>
<td>0.09°</td>
<td>0°</td>
</tr>
</tbody>
</table>

Table 3. Error analysis of the new modal testing method.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Fundamental</th>
<th>Second</th>
<th>Third</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{21}/A_{22}$</td>
<td>2.175</td>
<td>2.390</td>
<td>1.383</td>
</tr>
<tr>
<td>$A_{21}/A_{23}$</td>
<td>1.577</td>
<td>1.583</td>
<td>1.366</td>
</tr>
<tr>
<td>$\phi_{21} - \phi_{22}$</td>
<td>180°</td>
<td>178.92°</td>
<td>182.49°</td>
</tr>
<tr>
<td>$\phi_{21} - \phi_{23}$</td>
<td>180°</td>
<td>181.27°</td>
<td>182.49°</td>
</tr>
</tbody>
</table>

Table 4. Damping ratios calculated using the Gabor expansion and the Gabor spectrogram.
5 CONCLUSIONS

For an N-DOF system, in general a vibration signal contains the dynamic deflection of N modes if these modes are all properly excited. The Gabor transform and expansion can be used to decouple and reconstruct these modes to effectively make them into single-mode signals. Then the natural frequency, modal damping, vibration magnitude and phase can be extracted for each mode. The mode shape can also be obtained by comparing the magnitudes and phase angles at different grid points. A quadratic joint time-frequency method, the Gabor spectrogram, is an alternative approach which can be used to calculate the natural frequency and modal damping of each mode excited. However, the Gabor spectrogram cannot be used to determine the mode shape since the phase information is lost in such a quadratic process. The new Gabor analysis-based modal testing method is especially useful for non-stationary vibration signals, because the joint time-frequency properties of a non-stationary signal can be represented by these methods. In this paper the modal parameters for a simulated free vibration signal were studied. The method produced quite accurate results. The measurements were made on a cantilever beam. Without any information on the excitation, the modal parameters can be obtained very well using the Gabor expansion approach.

REFERENCES