Solution of the system of ordinary differential equations by Adomian decomposition method

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Abstract

In this article we use Adomian decomposition method, which is a well-known method for solving functional equations now-a-days, to solve systems of differential equations of the first order and an ordinary differential equation of any order by converting it into a system of differential of the order one. Theoretical considerations are being discussed, and convergence of the method for thses systems is addressed. Some examples are presented to show the ability of the method for linear and non-linear systems of differential equations.

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1. Introduction

A system of ordinary differential equations of the first order can be considered as:
\[
\begin{cases}
y_1' = f_1(x, y_1, \ldots, y_n) \\
y_2' = f_2(x, y_1, \ldots, y_n) \\
\vdots \\
y_n' = f_n(x, y_1, \ldots, y_n)
\end{cases}
\]  
(1)

where each equation represents the first derivative of one of the unknown functions as a mapping depending on the independent variable \(x\), and \(n\) unknown functions \(f_1, \ldots, f_n\).

Since every ordinary differential equation of order \(n\) can be written as a system consisting of \(n\) ordinary differential equation of order one, we restrict our study to a system of differential equations of the first order.

2. Using Adomian decomposition method to solve (1)

We can present the system (1), by using the \(i\)th equation as:

\[
L y_i = f_i(x, y_1, \ldots, y_n) \quad i = 1, 2, \ldots, n
\]  
(2)

where \(L\) is the linear operator \(\frac{d}{dx}\) with the inverse \(L^{-1} = \int_0^x \cdot \, dx\). Applying the inverse operator on (2) we get the following canonical form, which is suitable for applying Adomian decomposition method.

\[
y_i = y_i(0) + \int_0^x f_i(x, y_1, \ldots, y_n) \, dx \quad i = 1, 2, \ldots, n
\]  
(3)

As usual in Adomian decomposition method the solution of Eq. (3) is considered to be as the sum of a series:

\[
y_i = \sum_{j=0}^{\infty} f_{i,j}
\]  
(4)

And the integrand in the Eq. (3), as the sum of the following series:

\[
f_i(x, y_1, \ldots, y_n) = \sum_{j=0}^{\infty} A_{i,j}(f_{i,0}, f_{i,1}, \ldots, f_{i,j})
\]  
(5)

where \(A_{i,j}(f_{i,0}, f_{i,1}, \ldots, f_{i,n})\) are called Adomian polynomials. Substituting (4) and (5) into (3) we get:

\[
\sum_{j=0}^{\infty} f_{i,j} = y_i(0) + \int_0^x \sum_{j=0}^{n} A_{i,j}(f_{i,0}, \ldots, f_{i,j}) \, dx
\]

\[
= y_i(0) + \sum_{j=0}^{\infty} \int_0^x A_{i,j}(f_{i,0}, \ldots, f_{i,j}) \, dx
\]  
(6)
From which we define:

\[ f_{i,0} = y_i(0) \]
\[ f_{i,n+1} = \int_0^x A_{i,n}(f_{i,0}, \ldots, f_{i,n}) \, dx \quad n = 0, 1, 2, \ldots \] (7)

3. Convergence of the method

Since after the first step of the mentioned procedure we derive the Eq. (3), which is a system of Volterra integral equations of the second kind, for the convergence of the method, we refer the reader to [1] in which the problem of convergence has been discussed briefly.

4. Numerical examples

In this part we present three examples. The first and the second examples are considered to illustrate the method for linear and non-linear systems of ordinary differential equations of order one. While in third example we solve a differential equation of order three by transforming it into a system of differential equations of the first order.

Example 1. Consider the following system of differential equations, with initial values \( y_1(0) = 1, y_2(0) = 0, \) and \( y_3(0) = 2. \)

\[
\begin{align*}
    y_1' &= y_3 - \cos(x) \\
    y_2' &= y_3 - e^x \\
    y_3' &= y_1 - y_2
\end{align*}
\] (8)

Using the inverse operator \( L^{-1} = \int_0^x (\cdot) \, dx \) we get:

\[
\begin{align*}
    y_1 &= 1 - \int_0^x \cos(x) \, dx + \int_0^x y_3 \, dx \\
    y_2 &= -\int_0^x e^x \, dx + \int_0^x y_3 \, dx \\
    y_3 &= 2 + \int_0^x (y_{1,n} - y_{2,n}) \, dx
\end{align*}
\]

Using the alternate algorithm for computing the Adomian polynomials [2,3], the Adomian procedure would be as the following:
\[ y_{1,0} = 1 - \sin(x) \quad y_{1,n+1} = \int_0^x y_{3,n} \, dx \]
\[ y_{2,0} = 1 - e^x \quad y_{2,n+1} = \int_0^x y_{3,n} \, dx \quad n = 0, 1, 2, \ldots \]
\[ y_{3,0} = 2 \quad y_{3,n+1} = \int_0^x (y_{1,n} - y_{2,n}) \, dx \]

After two steps we get the exact solutions \( y_1 = e^x \), \( y_2 = \sin(x) \), \( y_3 = e^x + \cos(x) \).

**Example 2.** In this example we solve the following non-linear system of differential equations, with exact solutions \( y_1 = e^{2x} \), \( y_2(x) = e^x \), and \( y_3(x) = xe^x \).

\[
\begin{align*}
  y_1' &= 2y_2^2 \\
  y_2' &= e^{-x}y_1 \\
  y_3' &= y_2 + y_3 
\end{align*}
\]  \hspace{1cm} (9)

Applying the inverse operator \( L^{-1} = \int_0^x (.) \, dx \), we get:

\[
\begin{align*}
  y_1 &= 1 + 2 \int_0^x y_2^2 \, dx \\
  y_2 &= 1 + \int_0^x e^{-x}y_1 \, dx \\
  y_3 &= \int_0^x (y_2 + y_3) \, dx 
\end{align*}
\]  \hspace{1cm} (10)

Computing the Adomian polynomials by the algorithm presented in [2], Adomian method leads to the following scheme:

\[
\begin{align*}
  y_{1,0} &= 1 \quad y_{1,n+1} = 2 \int_0^x \left( \sum_{k=0}^{n} y_{2,k}y_{2,n-k} \right) \, dx \\
  y_{2,0} &= 1 \quad y_{2,n+1} = \int_0^x e^{-x}y_{1,n} \, dx \\
  y_{3,0} &= 0 \quad y_{2,n+1} = \int_0^x (y_{2,n} + y_{3,n}) \, dx 
\end{align*}
\]
Approximations to the solutions with five terms are as follows:

\[ y_1(x) \approx 4(26x + 31)e^{-x} + (-4x^2 - 16x + 15)e^{-2x} - 3.11111e^{-3x} + 58.6666x - 134.888 \]

\[ y_2(x) \approx -4(8x - 7)e^{-x} - 8(2x + 5)e^{-2x} + 0.111111(12x + 13)e^{-3x} + 11.5555 \]

\[ y_3(x) \approx (12x + 3)e^{-x} + (2x + 7.5)e^{-2x} - 0.111111e^{-3x} + 0.00833333x^5 + 0.0833333x^4 + 0.5x^3 + x^2 + 4.66666x - 10.3888 \]

Some numerical values of these solutions are presented in Table 1.

**Example 3.** Consider the following non-linear ordinary differential equation of order 3, with the initial conditions \( y(0) = 0, \ y'(0) = 1 \) and \( y''(0) = 2 \), and the exact solution \( y(x) = xe^x \).

\[ y'' = \frac{1}{x}y + y' \quad (11) \]

Considering three functions, \( y_1(x) = y(x), \ y_2(x) = y'(x), \) and \( y_3(x) = y''(x) \), we can convert (11) into the following non-linear system of three differential equation of order one.

\[ y_1' = y_2 \]
\[ y_2' = y_3 \]
\[ y_3' = \frac{1}{x}y_1 + y_3 \quad (12) \]

As in the previous examples if we apply the inverse operator, and using the alternate algorithm for computing Adomian polynomials \([2]\), we would have the following scheme:
\begin{align*}
y_{1,0} &= 0 \quad y_{1,n+1} = \int_0^x y_{2,n} \, dx \\
y_{2,0} &= 1 \quad y_{2,n+1} = \int_0^x y_{3,n} \, dx \quad n = 0, 1, \ldots \tag{13} \\
y_{3,0} &= 2 \quad y_{3,n+1} = \int_0^x \left( \frac{1}{x} y_{1,n} + y_{3,n} \right) \, dx
\end{align*}

Let \( y^p = y_{1,0} + y_{1,1} + \cdots + y_{1,p} \) is a notation for an approximation to the solution with \( p + 1 \) term. From (12) some computed approximations are as follows:

\begin{align*}
y^3 &= x \left( 1 + x + \frac{x^2}{3} \right) \\
y^4 &= x \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{12} \right) \\
y^5 &= x \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + x^4 \right) \\
y^6 &= x \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{7x^4}{180} + \frac{x^5}{360} \right) \\
y^7 &= x \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{31x^5}{4320} + \frac{x^7}{2520} \right) \\
y^8 &= x \left( 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{167x^7}{151,200} + \frac{x^8}{20,160} \right) \\
\vdots
\end{align*}

It seems to be reasonable to conclude the exact solution, \( y(x) = xe^x \).

5. Conclusion

Adomian decomposition method has been known to be a powerful device for solving many functional equations as algebraic equations, ordinary and partial differential equations, integral equations and so on. Here we used this method for solving systems of differential equations. It is demonstrated that this method has the ability of solving systems of both linear and non-linear differential equations. In Example 1, the system was a linear system and we derived the exact solutions. There is another linear example in which we obtained an exact solution. For non-linear systems, we usually derive a very good approximations to the solutions, as in Example 2, and some times the exact solutions can be found, as in Example 3. Extension of the method for solving
systems of partial differential equations offers an excellent oppurtunity for future research.

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References