ASSOUAD-NAGATA DIMENSION OF NILPOTENT GROUPS WITH ARBITRARY LEFT INVARIANT METRICS

J. HIGES

Abstract. Suppose G is a countable, not necessarily finitely generated, group. We show G admits a proper, left-invariant metric d_G such that the Assouad-Nagata dimension of (G, d_G) is infinite, provided the center of G is not locally finite. As a corollary we solve two problems of A.Dranishnikov.

Contents

1. Introduction

The asymptotic dimension was introduced by Gromov in [13] as a coarse invariant to study the geometric structure of finitely generated groups. We refer to [2] for a survey about this topic. Closely related with the asymptotic dimension is the asymptotic dimension of linear type. It is also called asymptotic Assouad-Nagata dimension in honor of Patrice Assouad who introduced it in [1] from the ideas of Nagata. Such dimension can be considered as the linear version of the asymptotic dimension. In recent years a part of the research activity was focused on this dimension and its relationship with the asymptotic dimension (see for example [16], [9], [10], [3], [4], [6], [5], [17] [12], [15]). One of the main problems of interest consists in studying the differences between the asymptotic dimension and the asymptotic Assouad-Nagata dimension in the context of the geometric group theory. In particular there are two main questions:

(1) Given a finitely generated group G with a word metric d_G. Are the asymptotic dimension and the asymptotic Assouad-Nagata dimension of (G, d_G) equal?
(2) In the case the two dimensions differ, must their difference be infinite?

Date: April 23, 2008.
2000 *Mathematics Subject Classification.* Primary 54F45; Secondary 55M10, 54C65.
Key words and phrases. Assouad-Nagata dimension, asymptotic dimension, nilpotent groups.
The author is supported by Grant AP2004-2494 from the Ministerio de Educación y Ciencia, Spain. He also thanks Jerzy Dydak and N. Brodskyi for helpful comments and support.

It is known that the first question has an affirmative answer for abelian groups, finitely presented groups of asymptotic dimension one, and for hyperbolic groups. But in general the answer is negative. Nowak ([17]) found for every $n \geq 1$ a finitely generated group of asymptotic dimension n but of infinite asymptotic Assouad-Nagata dimension.

As far as the author knows the second question is still open. There is no example of a finitely generated group such that the asympotic dimension is strictly smaller than the asymptotic Assouad-Nagata dimension but both are finite. In [15] the second question was solved in a more general context. It was proved that for every n and m there exists a countable abelian group(non finitely generated) with a proper left invariant metric such that the group is of asymptotic dimension n but of asympotic Assouad-Nagata dimension equal to $n+m$. Proper left invariant metrics are natural generalizations of word metrics. Therefore it is natural to ask the same two problems in the case of finitely generated groups equipped with proper left invariant metrics.

The aim of this note is to study the behaviour of the asymptotic Assouad-Nagata dimension in nilpotent groups with proper left invariant metrics. It is highly likely that both dimensions coincide in nilpotent groups with word metrics (see [7]). For example in [12] it was proved their coincidence for the Heisenberg group. We will show that for every nilpotent group it is possible to find a proper left invariant metric such that both dimensions are different. Dranishnikov in [7] asked about what can be considered a special case:

Problem 1.1. (Dranishnikov [7]) Does $dim_{AN}(\mathbb{Z}, d) = 1$ for every left invariant metric on $\mathbb{Z}$?

In relation to the above problem he asked the following:

Problem 1.2. (Dranishnikov [7]) Does $dim_{AN}(\Gamma \times \mathbb{Z}) = dim_{AN}(\Gamma) + 1$ for any left invariant metric on $\mathbb{Z}$?

Notice that in [9], Dranishnikov and Smith proved that $asdim_{AN}(G \times \mathbb{Z}) = \mathrm{asdim}_{AN}(G) + 1$ for every finitely generated group G but in this case the metrics consideres were the word metrics.

Our main theorem deals with a larger class of groups than nilpotent ones:

Theorem 1.3. *If G is a group such that its center is not locally finite then there exists a proper left invariant metric d_G such that $asdim_{AN}(G, d_G) = \infty$.*

It is clear that if we apply the previous theorem to $G = \mathbb{Z}$, the two questions of Dranishnikov are solved in negative.

The key ingredient of the proof was introduced by the author in [15]. In such paper it was shown that if there exists a sequence of isometric embeddings (up to dilatation) of balls $\{B(0, k_i)\}_{i \in \mathbb{N}}$ of $\mathbb{Z}^m$ into a metric space X where the sequence of radious k_i tends to infinity then the asymptotic Assouad-Nagata dimension of X is greater than m. That result should be viewed as applying the philosophy of Whyte [19] in a rather restricted form. Instead of looking for subsets of a group G that are bi-Lipschitz equivalent to Z^n, we are constructing groups that contain rescaled copies of large balls in Z^n.

Section 3 is devoted to a key ingredient used in Section 4 to present proofs of main results.

2. Preliminaries

Let s be a positive real number. An *s-scale chain* (or s-path) between two points x and y of a metric space (X, d_X) is defined as a finite sequence points $\{x = x_0, x_1, ..., x_m = y\}$ such that $d_X(x_i, x_{i+1}) < s$ for every $i = 0, ..., m-1$. A subset S of a metric space (X, d_X) is said to be *s-scale connected* if there exists an s-scale chain contained in S for every two elements of S.

Definition 2.1. A metric space (X, d_X) is said to be of *asymptotic dimension* at most n (notation $\text{asdim}(X, d) \leq n$) if there is an increasing function $D_X : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $s > 0$ there is a cover $\mathcal{U} = \{\mathcal{U}_0, ..., \mathcal{U}_n\}$ so that the s-scale connected components of each $\mathcal{U}_i$ are $D_X(s)$-bounded i.e. the diameter of such components is bounded by $D_X(s)$.

The function D_X is called an *n-dimensional control function* for X. Depending on the type of D_X one can define the following two invariants:

A metric space (X, d_X) is said to be of *Assouad-Nagata dimension* at most n (notation $\dim_{AN}(X, d) \leq n$) if it has an n-dimensional control function D_X of the form $D_X(s) = C \cdot s$ with $C > 0$ some fixed constant.

A metric space (X, d_X) is said to be of *asymptotic Assouad-Nagata dimension* at most n (notation $\text{asdim}_{AN}(X, d) \leq n$) if it has an n-dimensional control function D_X of the form $D_X(s) = C \cdot s + k$ with $C > 0$ and $k \in \mathbb{R}$ two fixed constants.

It is clear from the definition that for every metric space (X, d_X), $asdim(X, d_X) \leq asdim_{AN}(X, d_X)$.

One important fact about the asymptotic dimension is that it is invariant under coarse equivalences. Given a map $f : (X, d_X) \to (Y, d_Y)$ between two metrics spaces it is said to be a *coarse embedding* if there exist two increasing functions $\rho_+ : \mathbb{R}_+ \to \mathbb{R}_+$ and $\rho_- : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim\limits_{x \to \infty} \rho_-(x) = \infty$ such that:

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)) \text{ for every } x, y \in X.$$

Now a *coarse equivalence* between two metrics spaces (X, d_X) and (Y, d_Y) is defined as a coarse embedding $f : (X, d_X) \to (Y, d_Y)$ for which there exists a constant $K > 0$ such that $d_Y(y, f(X)) \leq K$ for every $y \in Y$. If there exists a coarse equivalence between X and Y both spaces are said to be *coarsely equivalent*.

The metrics spaces in which we are interested are countable groups with proper left invariant metrics.

Definition 2.2. A metric d_G defined in a group G is said to be a *proper left invariant metric* if it satisfies the following conditions:

(1) $d_G(g_1 \cdot g_2, g_1 \cdot g_3) = d_G(g_2, g_3)$ for every $g_1, g_2, g_3 \in G$.
(2) For every $K > 0$ the number of elements g of G such that $d(1_G, g)_G \leq K$ is finite.

The following basic result of Smith will be used to get the main theorem.

Theorem 2.3. *(Smith [18]) Two proper left invariant metrics defined in a countable group are coarsely equivalent.*

One way of constructing proper left invariant metrics in a countable group is via proper norms.

Definition 2.4. A map $\| \cdot \|_G : G \to \mathbb{R}_+$ is called to be a *proper norm* if it satisfies the following conditions:

(1) $\|g\|_G = 0$ if and only if $g = 1_G$.
(2) $\|g\|_G = \|g^{-1}\|_G$ for every $g \in G$.
(3) $\|g \cdot h\|_G \leq \|g\|_G + \|h\|_G$ for every $g, h \in G$.
(4) For every $K > 0$ the number of elements of G such that $\|g\|_G \leq K$ is finite.

It is clear that there is a one-to-one correspondence between proper norms and proper left invariant metrics.

Now we are interested in methods to get proper norms with some special properties. In general this task is not easy. In this paper we will use the method of weights described by Smith in [18]. Let S be a symmetric system of generators(possibly infinite) of a countable group G and let $\omega : L \to \mathbb{R}_+$ be a function(*weight function* or *system of weights*) that satisfies:

(1) $\omega(s) = 0$ if and only if $s = 1_G$
(2) $\omega(s) = w(s^{-1})$.
(3) $\omega^{-1}[0, N]$ is finite for every N .

Then the function $\|\cdot\|_w : G \to \mathbb{R}_+$ defined by:

$$\|g\|_w = \min\{\sum_{i=1}^{n} w(s_i) | x = \Pi_{i=1}^{n} s_i,\ \ s_i \in S\}$$

is a proper norm. Such norm will be called the proper norm *generated by the system of weights* ω and the associated left invariant metric will be the *left invariant metric generated by the system of weights* ω.

Remark 2.5. (1) If we define $w(g) = 1$ for all the elements $g \in S$ of a finite generating system $S \subset G$ (G a finitely generated group) we will obtain the usual word metric.
(2) Notice that if we have a proper norm $\|\cdot\|_G$ in a countable group G and we take the system of weiths defined by $\omega(g) = \|g\|_G$ then the proper norm $\|\cdot\|_\omega$ generated by this system of weights coincides with $\|\cdot\|_G$.
(3) We can construct easily integer valued proper left invariant metrics by getting weight functions with integer range.

3. Lower bounds for Assouad-Nagata dimension

The aim of this section is to give a sufficient condition in a metric space (X, d_X) that implies $asdim_{AN}(X, d_X) \geq n$ for some n. To get that condition we will use the notion of asymptotic cone.

Let (X, d_X) be a metric space. Given a non-principal ultrafilter ω of $\mathbb{N}$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of X, the *ω-limit* of $\{x_n\}_{n\in\mathbb{N}}$ (notation: $\lim_\omega x_n = y$) is an element y of X such that for every neighborhood U_y of y the set $F_{U_y} = \{n | x_n \in U_y\}$ belongs to ω. It can be proved easily that the ω-limit of a sequence always exists in a compact space.

Assume ω is a non principal ultrafilter of $\mathbb{N}$. Let $d = \{d_n\}_{n\in\mathbb{N}}$ be an ω-divergent sequence of positive real numbers and let $c = \{c_n\}_{n\in\mathbb{N}}$ be any sequence of elements of X. Now we can construct the *asymptotic cone* (notation: $Cone_\omega(X, c, d)$) of X as follows:

Firstly define the set of all sequences $\{x_n\}_{n\in\mathbb{N}}$ of elements of X such that $\lim_\omega \frac{d_X(x_n,c_n)}{d_n}$ is bounded. In such set take the pseudo metric given by:

$$D(\{x_n\}_{n\in\mathbb{N}},\{y_n\}_{n\in\mathbb{N}}) = \lim_\omega \frac{d_X(x_n,y_n)}{d_n}.$$

By identifying sequences whose distances is 0 we get the metric space $Cone_\omega(X,c,d)$.

Asymptotic cones were firstly introduced by Gromov in [13]. There has been a lot of research relating properties of groups with topological properties of its asymptotic cones. For example a finitely generated group is virtually nilpotent if and only if all its asymptotic cones are locally finite [14] or a group is hyperbolic if and only if all of its asymptotic cones are $\mathbb{R}$-trees ([13] and [11]).

In [12] it was shown the following relationship between the topological dimension of an asymptotic cone and the asymptotic Assouad-Nagata dimension of the space:

Theorem 3.1. *[Dydak, Higes [12]] $dim(Cone_\omega(X,c,d) \leq dim_{AN}(Cone_\omega(X,c,d) \leq asdim_{AN}(X,d_X)$ for any metric space (X,d_X) and every asymptotic cone $Cone_\omega(X,c,d)$.*

We recall now the following:

Definition 3.2. A function $f:(X,d_X)\to(Y,d_Y)$ between metric spaces is said to be a dilatation if there exists a constant $C\geq 1$ such that $d_Y(f(x),f(y)) = C\cdot d_X(x,y)$ for every $x,y\in X$.

The number C will be called the *dilatation constant*.

In the following proposition dilatations will be from balls of $\mathbb{Z}^n$ with the l_1-metric to general metric spaces.

Proposition 3.3. *Let (X,d_X) be a metric space and let $\{k_m\}_{m\in\mathbb{N}}$ be an increasing sequence of natural numbers. If for some $n\in\mathbb{N}$ there is a sequence of dilatations $\{f_m\}_{m=1}^\infty$ of the form $f_m: B^n(0,k_m)\to(X,d_X)$ with $B^n(0,k_m)\subset\mathbb{Z}^n$ the ball of radious k_m then there exists an asymptotic cone $Cone_\omega(X,c,d)$ of (X,d_X) such that $[-1,1]^n\subset Cone_\omega(X,c,d)$.*

Proof. Suppose given (X,d_X) and $\{k_m\}_{m\in\mathbb{N}}$ as in the hypothesis. Let us prove firstly the case $n=1$. Assume that $\{C_m\}_{m\in\mathbb{N}}$ is the sequence of dilatation constants of $\{f_m\}_{m\in\mathbb{N}}$. Take ω some ultrafilter of $\mathbb{N}$ and define $c=\{f_m(0)\}_{m=1}^\infty$ and $d=\{d_m\}_{m=1}^\infty$ with $d_m = C_m\cdot k_m$. We will prove that $Cone_\omega(X,c,d)$ contains $[-1,1]$. For each $t\in[-1,1]$ let A_m^t be the subset $\{-k_m,...,k_m\}$ such that $x\in A_m^t$ if and only if the distance between $\frac{C_m\cdot x}{d_m}$ and t is minimum. Notice that this implies that the distance between $C_m\cdot x$ and $d_m\cdot t$ is less thatn C_m. Take now the sequence $\{r_m^t\}_{m=1}^\infty$ where r_m^t is the infimum of A_m^t.

Define the map $g:[-1,1]\to Cone_\omega(X,c,d)$ by $g(t)=x^t$ if the sequence $\{f_m(r_m^t)\}_{m=1}^\infty$ is in the class x^t. As:

$$\lim_\omega \frac{d(f_m(0),f_m(r_m^t))}{d_m} = \lim_\omega \frac{C_m\cdot|r_m^t|}{d_m} \leq \lim_\omega \frac{C_m\cdot k_m}{d_m} = 1$$

the map is well defined. Let us prove it is an isometry. From the definition of r_m^t we get that if $t_1<t_2$ then $r_m^{t_1}\leq r_m^{t_2}$ what implies $\lim_\omega \frac{d(f_m(r_m^{t_1}),f_m(r_m^{t_2}))}{d_m} = \lim_\omega \frac{C_m(r_m^{t_2}-r_m^{t_1})}{d_m}$. So the unique thing we need to show is that $\lim_\omega \frac{C_m\cdot r_m^t}{d_m} = t$ for every t. Notice that we have $\lim_\omega \frac{C_m}{d_m}=0$ as $\lim_\omega k_m=\infty$ but $\lim_\omega \frac{C_m\cdot k_m}{d_m}=1$. This

implies that given $\epsilon > 0$ there exists $G_\epsilon \in \omega$ such that $\frac{C_m}{d_m} < \epsilon$ for every $m \in G_\epsilon$. Therefore by the choice of r^t_m if $m \in G_\epsilon$ we have $|C_m \cdot r^t_m - d_m \cdot t| < C_m$ and then $|\frac{C_m \cdot r^t_m}{d_m} - t| < \frac{C_m}{d_m} \leq \epsilon$.

Now let us do the general case. Let $(s_1, ..., s_n) \in [-1,1]^n$. By the previous case we get that for every $j = 1, ..., n$ there exists a sequence $\{r^{s_j}_m\}_{m\in\mathbb{N}}$ with $r^{s_j}_m \in \{-k_m, ..., k_m\}$ such that $\lim_\omega \frac{C_m \cdot r^t_m}{d_m} = s_j$. In a similar way as before we construct a map $g : [-1,1]^n \to Cone_\omega(X, c, d)$ by defining $g(s_1, ..., s_m)$ as the class that contains the sequence $\{f_m(r^{s_1}_m, ..., r^{s_n}_m)\}^\infty_{m=1}$. To finish the proof it will be enough to check that for every $s, t \in [-1,1]^n$ with $s = (s_1, ..., s_n)$ and $t = (t_1, ..., t_n)$, the following equality holds:

$$\lim_\omega \frac{d_X(f_m(r^{s_1}_m, ..., r^{s_n}_m), f_m(r^{t_1}_m, ..., r^{t_n}_m))}{d_m} = \sum_{i=1}^n |s_i - t_i|$$

As f_m is a dilatation of constant C_m we can write:

$$\lim_\omega \frac{d_X(f_m(r^{s_1}_m, ..., r^{s_n}_m), f_m(r^{t_1}_m, ..., r^{t_n}_m))}{d_m} = \sum_{i=1}^n \lim_\omega \frac{C_m \cdot |r^{s_i}_m - r^{t_i}_m|}{d_m}$$

And again by the case $n = 1$ we can deduce that the last term satisfies the equality:

$$\sum_{i=1}^n \lim_\omega \frac{C_m \cdot |r^{s_i}_m - r^{t_i}_m|}{d_m} = \sum_{i=1}^n |s_i - t_i|$$

. □

From the previous proposition we can get the following result that is one of the ingredients of the main theorem.

Corollary 3.4. *If for metric space (X, d_X) and form some $n \in \mathbb{N}$ there exists a sequence of dilatations $f_m : B^n(0, k_m) \to (X, d_X)$ with $\lim_{m\to\infty} k_m = \infty$ and $B^n(0, k_m) \subset \mathbb{Z}^n$ the ball of radious k_m then $asdim_{AN}(X, d_X) \geq n$.*

Proof. By proposition 3.3 we get that there exists an asymptotic cone of X such that $[-1,1]^n \subset Cone_\omega(X, c, d)$. Applying theorem 3.1 we obtain immediately:

$$n \leq dim(Cone_\omega(X, c, d)) \leq dim_{AN}(Cone_\omega(X, c, d)) \leq asdim_{AN}(X, d_X)$$

□

4. Main results

The idea of the proof of the main theorem consists in creating a metric inductively. In each step we will construct a new metric that satisfies two conditions. First condtion says that the new metric doer not change a sufficiently large ball of the old metric. Second condtion implies there is a dilatation from some sufficiently large ball of $\mathbb{Z}^n$ into the group with the new metric. In fact that dilatation will be the restriction of some homomorphism $f : \mathbb{Z}^n \to G$. Then we will apply corollary 3.4. The following lemma could be considered as the induction step.

Lemma 4.1. *Let G be a finitely generated group such that its center is not locally finite. Let d_G be a proper left invariant metric. In such condtions for every $k, m, R \in \mathbb{N}$ there exists a proper left invariant metric d_ω that satisfies the following conditions:*

(1) $\|g\|_\omega \leq \|g\|_G$.
(2) $\|g\|_G = \|g\|_\omega$ *if* $\|g\|_\omega \leq R$.
(3) *There is an homomorphism* $f : \mathbb{Z}^m \to G$ *such that the restriction* $f|_{B(0,k)}$ *of* f *to the ball radious* k *is a dilatation in* (G, d_ω).

Proof. Suppose k, m and R given and let a and C be two natural numbers that satisfy:

$$R < C < \frac{a}{2 \cdot k \cdot m^2}.$$

As the center of G is not locally finite there exists an element g in the center of infinite order. The restriction of the metric d_G to the subgroup generated by g defines a proper left invariant metric in $\mathbb{Z} =< g >$. By theorem 2.3 we know that two proper left invariant metrics defined in a group are coarsely equivalent, hence we can find an integer number $h_1 \in \mathbb{Z}$ such that if $|h| \geq |h_1|$ then $\|g^h\|_G \geq a$. Let $p_1 = 1$ and for every $j = 2...m$ we define p_j as a sufficiently large number that satisfies $\sum_{i=1}^{j-1}(2 \cdot k \cdot m)2^{p_i} < 2^{p_j}$. Take now the finite set of integer numbers $\{h_1, ..., h_m\}$ with $h_j = 2^{p_j} \cdot h_1$ for every $j = 2, ..., m$. In this situation we create the norm $\| \cdot \|_\omega$ generated by the following system of weights:

$$\omega(z) = \begin{cases} \|z\|_G \text{ if } z \neq g^{\pm h_i} \text{ for every } i = 1...m \\ C \text{ otherwise} \end{cases}$$

By the choice of C and $\{h_1, ..., h_m\}$ it is clear that the two first conditions of the lemma are satisfied. To prove the third condition we define the homomorphism $f : \mathbb{Z}^m \to G$ as $f(x_1, ..., x_m) = g^h$ with $h = \sum_{i=1}^m x_i \cdot h_i$. Let us show that the restriction $f|_{B(0,k)} : B(0, k) \to G$ to the ball of radious k is a dilatation of constant C. It will be enough to check that:

$$\|g^h\|_\omega = \sum_{i=1}^m |x_i| \cdot C \text{ if } h = \sum_{i=1}^m x_i \cdot h_i \text{ and } |x_i| \leq k$$

The reasoning will be by contradiction. Suppose there exists an element of the form g^h with $h = \sum_{i=1}^m x_i \cdot h_i$ and $|x_i| \leq k$ such that $\|g^h\|_\omega < \sum_{i=1}^m |x_i| \cdot C$. This implies that there exist and $r = \sum_{i=1}^m y_i \cdot h_i$ and an $s \in G$ such that $g^h = g^r \cdot s$ and:

$$\|g^h\|_\omega = \sum_{i=1}^m |y_i| \cdot C + \|s\|_G.$$

Notice that $|y_i| \leq k \cdot m$. There are now two possible cases:

Case $s = 1_G$: In this situation we have $\sum_{i=1}^m |y_i| \cdot C < \sum_{i=1}^m |x_i| \cdot C$ so there exists an i such that $x_i \neq y_i$. Let $j = \max\{i | x_i \neq y_i\}$. From the fact $g^h = g^r$ we can deduce $(x_j - y_j) \cdot h_j = \sum_{i=1}^{j-1}(y_i - x_i) \cdot h_i$ it means $(x_j - y_j) \cdot 2^{p_j} \cdot h_1 = \sum_{i=1}^{j-1}(y_i - x_i) \cdot 2^{p_i} \cdot h_1$ and then:

$$2^{p_j} \leq |x_j - y_j| \cdot 2^{p_j} \leq \sum_{i=1}^{j-1} |y_i - x_i| \cdot 2^{p_i} \leq \sum_{i=1}^{j-1} |y_i| + |x_i| \cdot 2^{p_i} \leq \sum_{i=1}^{j-1} 2^{p_i} \cdot (k \cdot m + k) < 2^{p_j}$$

A contradiction. Therefore the first case is not possible.

Case 2: $s \neq 1_G$. In this case we have $g^{(h-r)} = s$ and $h - r \neq 0$ what implies $|h - r| \geq |h_1|$ and hence $\|s\|_G \geq a$. But from the fact $\|g^h\|_\omega = \sum_{i=1}^m |y_i| \cdot C + \|s\|_G <$

$\sum_{i=1}^{m} |x_i| \cdot C$ we can deduce:

$$a \leq \|s\|_G < \sum_{i=1}^{m}(|x_i| - |y_i|) \cdot C \leq \sum_{i=1}^{m}(k + k \cdot m) \cdot C \leq 2 \cdot k \cdot m^2 \cdot C.$$

Therefore $C \geq \frac{a}{2 \cdot k \cdot m^2}$ and this contradicts the choice of a and C. □

Here is the main theorem.

Theorem 4.2. *If G is a group such that its center is not locally finite then there exists a proper left invariant metric d_G such that $asdim_{AN}(G, d_G) = \infty$.*

Proof. We will use corollary 3.4 and the previous lemma. Take any integer valued proper left invariant metric d in G (see remark 2.5) and some increasing sequences $\{k_i\}_{i\in\mathbb{N}}$ and $\{M_i\}_{i\in\mathbb{N}}$ of natural numbers. Let us construct the metric d_G of the theorem by an inductive process.

Step 1: Apply the previous lemma to d with $k = k_1$, $m = 1$ and $R = M_1$. We obtain a proper left invariant metric d_{ω_1} such that the ball $B_{\omega_1}(1_G, R_1)$ is equal to the ball of radious R_1 of d. Also there exists a dilatation $f : B(0, k_1) \to G$ from the ball or radious k_1 of $\mathbb{Z}$ to G.

Induction Step: Suppose now that we have contructed a finite sequence of proper left invariant metrics $L = \{d_{\omega_1}, ..., d_{\omega_n}\}$ and a finite sequence of natural numbers $R_1 < R_2 < ... < R_n$ that satisfy the following conditions:

(1) $\|g\|_{\omega_i} \leq \|g\|_{\omega_{i-1}}$.
(2) $\|g\|_{\omega_i} = \|g\|_{\omega_{i-1}}$ if $\|g\|_{\omega_i} \leq R_i$
(3) There exists an homomorphism $f_i : \mathbb{Z}^i \to G$ such that the restriction $f_i|_{B(0,k_i)}$ is a dilatation in (G, d_{ω_i}) for every $i = 1, ..., n$
(4) $diam(f_i(B(0, k_i)) < R_{i+1}$ for every $i = 1, ..., n-1$

In these conditions define $R_{n+1} = \max\{M_{n+1}, R_n + 1, diam(f_n(B(0, k_n))\}$ and apply the previous lemma to d_{ω_n} with $k = k_{n+1}$, $m = n + 1$ and $R = R_{n+1}$. We have now a new proper left invariant metric $d_{\omega_{n+1}}$. It is clear that the new finite sequence of proper left invariant metrics $\{d_{\omega_{n+1}}\} \cup L$ and the new finite sequence of numbers $R_1 < R_2 < ... < R_n < R_{n+1}$ satisfy the same four conditions.

Repeating this procedure we construct a sequence of integer valued proper left invariant metrics $\{d_{\omega_i}\}_{i=1}^{\infty}$ and an increasing sequence of natural numbers $\{R_i\}_{i=1}^{\infty}$. By the first two properties and the fact $\lim_{i\to\infty} R_i = \infty$ we deduce that for every $g \in G$ the sequence $\{\|g\|_{\omega_i}\}_{i=1}^{\infty}$ is asymptotically constant. Define now the function $\|\cdot\|_G : G \to \mathbb{N}$ by $\|g\|_G = \lim_{i\to\infty} \|g\|_{\omega_i}$. Again by the first two properties we can check $\|\cdot\|_G$ is a proper norm. So it defines a proper left invariant metric d_G. Using the third and fourth properties we have that for every $i \in \mathbb{N}$ there exists an homomorphism $f_i : \mathbb{Z}^i \to G$ such that the restriction to the ball $B(0, k_i)$ is a dilatation in (G, d_G). As in each step we are increasing the dimension of the balls, we get that for every $m \in \mathbb{N}$ the metric space (G, d_G) satisfies the conditions of corollary 3.4 so we get $asdim_{AN}(G, d_G) \geq m$. Therefore $asdim_{AN}(G, d_G) = \infty$. □

Recall that in [8] Dranishnikov and Smith showed that the asymptotic dimension of finitely generated nilpotent groups is equal to the hirsch length of the group. Hence the asymptotic dimension of a nilpotent group is always finite for every proper left invariant metric. Next trivial corollary shows that the unique nilpotent groups that satisfy the same property for the asymptotic Assouad-Nagata dimension are finite.

Corollary 4.3. *Let G be a finitely generated nilpotent group. G is non finite if and only if there exists a proper left invariant metric d_ω defined in G such that $asdim_{AN}(G, d_\omega) = \infty$.*

Proof. For one implication just use that the asymptotic Assouad-Nagata dimension is zero for all bounded spaces. The other implication is a particular case of the main theorem. □

References

[1] P. Assouad, *Sur la distance de Nagata*, C.R. Acad. Sci. Paris Sr. I Math. **294** (1982), no. 1, 31–34.

[2] G.Bell, A.Dranishnikov *Asymptotic dimension*, preprint math.GT/0703766

[3] N.Brodskiy, J.Dydak, J.Higes, A.Mitra, *Dimension zero at all scales*, Topology ans its Appl. **154** (2007) no.14, 2729-2740.

[4] N.Brodskiy, J.Dydak, J.Higes, A.Mitra, *Nagata-Assouad dimension via Lipschitz extension*, Israel Journal of Mathematics (to appear), math.MG/0601226.

[5] N.Brodskiy, J.Dydak, M.Levin, A.Mitra, *Hurewicz theorem for Assouad-Nagata dimension*, Journal of the London Math.Soc. (to appear), math.MG/0605416.

[6] N.Brodskiy, J.Dydak, U.Lang, *Assouad-Nagata dimension of wreath products of groups*, preprint math.MG/0611331v2.

[7] A. Dranishnikov, *Open problems in asymptotic dimension theory*, preprint, in http://aimbri12.securesites.net/pggt

[8] A.Dranishnikov and J. Smith *Asymptotic dimension of discrete groups*, Fund Math. **189** (2006), no.1, 27–34.

[9] A.Dranishnikov and J.Smith *On asymptotic Assouad-Nagata dimension*, Topology and its Appl. **154** (2007), no. 4, 934–952

[10] A. Dranishnikov and M.Zarichnyi, *Universal spaces for asymptotic dimension*, Topology and its Appl. **140** (2004), no.2-3, 203–225.

[11] C. Drutu, *Quasi-isometry invariants and asymptotic cones*, Int. J. Algebra Comput **12**(1 and 2)(2002),99–135.

[12] J.Dydak, J.Higes, *Asymptotic cones and Assouad-Nagata dimension*, to appear in PAMS.

[13] M. Gromov, *Asymptotic invariants for infinite groups*, in Geometric Group Theory, vol. 2, 1–295, G.Niblo and M.Roller, eds., Cambridge University Press, 1993.

[14] M. Gromov, *Groups of polynomial growth and expanding maps*, in Inst. Hautes Études Sci. Publ. Math. (1981), no.53, 53–73.

[15] J. Higes,*Assouad-Nagata dimension of locally finite groups and asymptotic cones*, preprint math.MG0711.1512.

[16] U. Lang, T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, IMRN International Mathematics Research Notices (2005), no.58, 3625–3655.

[17] P.W. Nowak, *On exactness and isoperimetric profiles of discrete groups*, Journal of functional analysis. **243** (2007), no.1, 323-344.

[18] J. Smith, *On Asymptotic dimension of Countable Abelian Groups*, Topology Appl. **153** (2006), no. 12, 2047–2054

[19] K. Whyte, *Amenability, Bilipschitz Equivalence, and the Von Neumann Conjecture*, Duke Journal of Mathematics.(1999), 93–112.

Departamento de Geometría y Topología, Facultad de CC.Matemáticas. Universidad Complutense de Madrid. Madrid, 28040 Spain

E-mail address: josemhiges@yahoo.es