Least-squares-based fitting of paraboloids

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Abstract

A technique for reconstructing a class of quadric surfaces from 3D data is presented. The technique is driven by a linear least-squares-based fitting mechanism. Previously, such fitting was restricted to recovery of central quadrics; here, extension of that basic mechanism to allow recovery of one commonly-occurring class of non-central quadric, the elliptic paraboloids, is described. The extension uses an indirect solution approach that involves introducing a variable to the basic mechanism that is a function of a quadric surface invariant. Results from fitting real and synthetic data are also exhibited.

Keywords: Surface fitting; Quadric surfaces; Paraboloids; Least-squares

1. Introduction

Surface recovery is commonly requisite in automatic object recognition and visual inspection, in particular in surface model-based systems. In such systems, surface recovery is typically a key processing activity [1]. Model construction for computer graphics can also require surface recovery.

Processing schema that enable recovery of a surface-based model from 3D data typically include tasks of local surface property (e.g., curvature) estimation, segmentation, classification, and reconstruction [2]. In the segmentation task, the data is divided into sub-regions of homogeneous surface type, often based on the surface property estimates. In the classification task, sub-region surface type is determined. Reconstruction involves finding the parameters of the best-fitting surface.

Investigation into quadric surface model recovery has been an area of interest for the research community due to the large number of manufactured objects that are composed of quadric surfaces. For example, it has been reported that approximately 85% of manufactured objects can be well-modeled using quadrics [3]. In this paper, focus is on recovery of one class of quadric, the elliptic paraboloids.

Parts with such surfaces occur in some fluid machines and chemical industrial equipment. Many egg-shaped toys and containers can also be well-modeled as pairs of elliptic paraboloids. Some industrial parts composed mainly of basic surface types also contain rounds that bridge simple surface (e.g., plane and cylinder) junctions. The convex rounded surfaces can often be well-modeled as paraboloids.

The elliptic paraboloid recovery technique introduced here extends prior work [4] that allows recovery of central quadrics (also called centered quadrics) from 3D data following a linear least-squares approach. The central quadrics are the quadrics, which when expressed in their standard representation, have a center at the origin. Examples of the central quadrics are ellipsoids and hyperboloids. Quadric surface types which have no center are called non-central (or non-centered) quadrics. The paraboloids are examples of non-central quadrics.

1.1. Quadric recovery

Many methods that address one or more of the components of the quadric surface recovery process have been presented in the literature.
For example, methods to accurately estimate surface properties, especially curvature (e.g., \[5\]), have been studied. Estimation of curvature properties, especially curvature magnitude, in real data is very sensitive to noise [5–7], although Tang and Medioni [6] have described a tensor voting formalism that allows accurate estimation of curvature sign and direction in the presence of noise.

Surface curvature measures have been used to drive both segmentation and classification tasks. For instance, Fan et al. [8] have examined surface curvature measures to find discontinuities that signal potential quadric surface boundaries in range data. Fitzgibbon et al. [9] have segmented and classified objects composed of quadric surfaces from range images based on morphologically-smoothed curvature measures. Hoffman and Jain [10] have used surface curvature measures to perform classification of surface patches in range images into categories of convex, concave, and planar. Bhanu and Nuttall [11] have also used curvature measures to characterize surface types.

1.1.1. Quadric reconstruction

Methods for reconstructing quadric surfaces from 3D data have also been proposed. For example, Hough-based processing that exploits geometric constraints to limit false alarms has been described by Newman et al. [1] for spherical and cylindrical surface reconstruction.

Many of the quadric surface reconstruction methods have used numeric processing. For example, Newman et al. [1] have described a regression-based sequence of steps that allows determination of conical surface parameters. Fitzgibbon et al. [9] have performed a two-stage fitting to recover cylindrical, conical, and ellipsoidal surface parameters. The first stage performs general quadric fitting using Taubin’s [12] generalized eigenvector fitting (GEVFIT) procedure. The second stage applies heuristic processing to determine if the fit quadric is a cylinder, cone, or ellipsoid. For surfaces postulated to be cylinders, cones, or ellipsoids, a surface type-specific nonlinear least-squares algorithm is used to recover size, position, and orientation parameters. Hall et al. [4] have presented a least-squares-based algebraic approach for recovering the geometric parameters of central quadric surfaces, such as ellipsoids and hyperboloids. Their approach minimizes the general quadric implicit formulation’s difference from the expected value. While the Hall et al. approach can be tailored to extract the size, position, and orientation parameters of central quadric surfaces in 3D data, it is not applicable to non-central quadrics, such as elliptic paraboloids. Ahn et al. [13] have described a nonlinear (iterative) least-squares-based geometric approach for fitting circles and one class of quadrics (spheres). Their basic approach can also be extended to the fitting of the other conic section curves. Lukács et al. [14] have developed a nonlinear least-squares approach for fitting spheres, cylinders, cones, and tori in 3D data that minimizes the deviation of surface type-specific approximate Euclidean distances between data points and the fit surface. The approach can overcome singularity difficulties of competing methods, such as the GEVFIT [2,14]. Werghi et al. [15] have incorporated inter-surface relationships in a nonlinear least-squares fitting formulation to reconstruct quadric surfaces from segmented 3D data. They demonstrated their approach for scenes composed of spherical, cylindrical, conical, and planar patches. Chivate and Jablkoow [3] have described a least-squares algorithm based on eigenvalues that can be used to recover the parameters of surfaces known to be spheres, cylinders, or planes. Readers interested in the current state of the art in quadric surface recovery from discretized 3D data may wish to consult the recent comprehensive survey of Petitjean [2].

The procedure for elliptic paraboloid fitting presented in this paper expands the state of the art in quadric surface recovery by (1) providing an elliptic paraboloid recovery mechanism and (2) demonstrating that an approach restricted to central quadric recovery can be extended to allow non-central quadric recovery. The procedure is, to our knowledge, also the first linear least-squares approach for reconstruction of a non-central quadric.

1.2. Shape-related defects

The method introduced here can be useful in automated visual inspection of shape. Automated visual inspection has been applied in a wide variety of application areas [16]. Many of the inspection applications require detection of shape-related defects, many of which can be categorized into two gross defect classes, namely the excess and insufficient material defects. Defects of these types are commonly defined in part using deviation thresholds, measured with respect to the reference surface, where the reference surface is a mathematical model of the real underlying surface (i.e., the model that fits the “good” part of the surface—that is, the surface exclusive of its defects). The method introduced aids shape inspection by allowing reference surface recovery.

Automated shape inspection sometimes can be accomplished using single intensity images, although it is generally more feasible if 3D data is available. A number of techniques for inspecting shape using range images have been presented in the literature (e.g., [17,18]). While inspection of objects composed of quadrics is an important problem domain, many techniques have focused on detecting defects only on basic surface types, such as planar, spherical and cylindrical surfaces; the automated shape-based inspection of parts with non-central quadric surfaces has not been well-addressed.

The accuracy of inspection is often impacted by measurement errors. These errors usually result from uncertainties including quantization error in image digitization, illumination error, motion of the object or the camera setup, and parallax [19]. Usually, such error factors can be well modeled using a Gaussian distribution of noise.
1.3. Organization

The rest of this paper is organized as follows. In Section 2 the surface fitting and parameter recovery background related to our technique is described. In Section 3, the new elliptic paraboloid reconstruction method is introduced. In Section 4, we describe a multiple fitting strategy that uses the new reconstruction method to accurately recover a reference surface in the presence of defects. In Section 5, experimental results from testing on synthetic and real data are exhibited. Section 6 concludes the paper.

2. Central quadric parameter recovery

As mentioned in Section 1.1.1, Hall et al. [4] have reported a least-squares-based methodology for estimation of geometric parameters of central quadric surfaces in 3D data. In this paper, we introduce a formulation that allows determination of one type of non-central quadric’s (i.e., the elliptic paraboloid) location, orientation, and shape from 3D data using least-squares-based processing. The basic least-squares central quadric fitting formulation is presented in this section.

The equation for a general quadric in Cartesian space is shown in Eq. (1)

\[ l_1 x^2 + l_2 y^2 + l_3 z^2 + l_4 xy + l_5 yz + l_6 zx + l_7 x + l_8 y + l_9 z + l_{10} = 0. \]

(1)

It is possible to re-write Eq. (1) in a form with nine coefficients since only nine of its 10 coefficients are independent. For example, for the cases that \( l_{10} \neq 0 \), Eq. (1) can be rewritten as

\[ k_1 x^2 + k_2 y^2 + k_3 z^2 + k_4 xy + k_5 yz + k_6 zx + k_7 x + k_8 y + k_9 z + 1 = 0. \]

(2)

Reconstructing a general quadric requires determining the nine independent coefficients \( k_i (i = 1, 2, \ldots, 9) \).

Next, we describe how the nine quadric coefficients of Eq. (2) can be recovered using a linear least-squares approach modeled after that of Hall et al. [4]. We assume the least-squares fitting is performed on an input of \( m \) 3D sample points \((x_i, y_i, z_i)\). To enable robust fitting, an over-constrained system of \( m \) linear equations (i.e., \( m > 9 \)) is considered

\[ A_0 \mathbf{X}_0 = \mathbf{b}, \]

(3)

where

\[
A_{0m \times n} = \\
\begin{bmatrix}
x_1^2 & y_1^2 & z_1^2 & x_1 y_1 & y_1 z_1 & z_1 x_1 & x_1 & y_1 & z_1 \\
x_2^2 & y_2^2 & z_2^2 & x_2 y_2 & y_2 z_2 & z_2 x_2 & x_2 & y_2 & z_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x_m^2 & y_m^2 & z_m^2 & x_m y_m & y_m z_m & z_m x_m & x_m & y_m & z_m
\end{bmatrix}
\]

(4)

is the coefficient matrix of the linear equation group and \( n = 9 \). Additionally

\[ \mathbf{X}_0 = [k_1 \ k_2 \ k_3 \ k_4 \ k_5 \ k_6 \ k_7 \ k_8 \ k_9]^T \]

(5)

is the vector of unknown variables. The right-side term is

\[ \mathbf{b}_{m \times 1} = [1 \ 1 \ \ldots \ 1]^T. \]

(6)

The least-squares solution \( \mathbf{X}_0 \) to this system of equations is the solution that minimizes the residual error

\[ e_r = \| A_0 \mathbf{X}_0 - \mathbf{b} \|_2 \]

(7)

and this solution is optimal in the least-squares sense [20]. The solution \( \mathbf{X}_0 \) can be computed using the normal equation

\[ \mathbf{X}_0 = (A_0^T A_0)^{-1} A_0^T \mathbf{b}. \]

(8)

If the sample data points are not concentrated at a single point, line or plane (which are the reduced dimension cases), then the rows and columns in matrix \( A_0 \) will not be linearly co-related, and therefore the matrix \( A_0^T A_0 \) will be non-singular. When \( A_0^T A_0 \) is non-singular, there is guaranteed to exist a least-squares solution vector \( \mathbf{X}_0 \).

For object recognition or inspection applications, determining the coefficients of the Eq. (2) does not complete the surface recovery process since geometrically meaningful parameters required in matching and inspection have not been obtained. To complete the recovery process, parameters such as the surface type, shape, position, and orientation must be determined. Some additional computations are necessary to extract such information from the coefficients of Eq. (2).

First, surface type can be determined using the surface’s invariants \( \Delta \) and \( D \), which are invariant with respect to translation and rotation [21,22]. For an arbitrary general quadric defined in the form of Eq. (1), the invariant \( \Delta \) is [21,22]

\[ \Delta = \begin{bmatrix}
l_1 & l_4/2 & l_6/2 & l_{10}/2 \\
l_4/2 & l_2 & l_5/2 & l_{10}/2 \\
l_5/2 & l_6/2 & l_9/2 & l_{10}/2 \\
l_{10}/2 & l_9/2 & l_{10}/2 & l_3
\end{bmatrix}. \]

(9)

and the invariant \( D \) is [21,22]

\[ D = \begin{bmatrix}
l_1 & l_4/2 & l_6/2 & l_{10}/2 \\
l_4/2 & l_2 & l_5/2 & l_{10}/2 \\
l_5/2 & l_6/2 & l_9/2 & l_{10}/2 \\
l_{10}/2 & l_9/2 & l_{10}/2 & l_3
\end{bmatrix}. \]

(10)

The relationship between quadric surface type and the invariants is summarized in Table 1.

Once quadric type has been determined, the remaining parameters of surface shape and pose need to be found. The main contribution of the Hall et al. [4] paper is its description of the process of extracting the remaining parameters for the central quadrics. Next, we describe this process through an illustration for one central quadric, the single-sheet hyperboloids.
Table 1
Quadric surface classification via invariant values

<table>
<thead>
<tr>
<th>$D \neq 0$</th>
<th>$D = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Central quadric surface</td>
<td>Non-central quadric surface</td>
</tr>
<tr>
<td>$\Delta &gt; 0$</td>
<td>Single-sheet hyperboloid</td>
</tr>
<tr>
<td>$\Delta &lt; 0$</td>
<td>Ellipsoid or dual-sheet hyperboloid</td>
</tr>
<tr>
<td>$\Delta = 0$</td>
<td>Cone</td>
</tr>
</tbody>
</table>

For single-sheet hyperboloids

$$A = \begin{bmatrix} 1/a^2 & 1/b^2 & -1/c^2 \end{bmatrix},$$  \hspace{1cm} (14)$$

$$g = [0 \ 0 \ 0]^T$$  \hspace{1cm} (15)$$

and

$$d = 1.$$  \hspace{1cm} (16)$$

After translation and rotation, the Eq. (12) becomes

$$x^T A \hat{x} + g \cdot \hat{x} = d,$$  \hspace{1cm} (17)$$

where

$$x = [x - x_0 \ y - y_0 \ z - z_0]^T,$$  \hspace{1cm} (18)$$

$$A_1 = R^T A R,$$  \hspace{1cm} (19)$$

$$g_1 = g^T R,$$  \hspace{1cm} (20)$$

and $(x_0, y_0, z_0)$ is the translation vector that describes the quadric’s position with respect to the standard position and

$$R = R(\theta_x, \theta_y, \theta_z) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$  \hspace{1cm} (21)$$

is a 3 $\times$ 3 standard orthogonal matrix representing a rotation transformation, with $\theta_x, \theta_y, \theta_z$ the three rotation angles that describe the quadric’s orientation with respect to the standard orientation (e.g., roll-pitch-yaw angles).

Since $g_1$ is the product of $g^T$ and $R$, it is a vector that we will denote as follows:

$$g_1 = [w_1 \ w_2 \ w_3].$$  \hspace{1cm} (22)$$

For hyperboloids, $w_1 = w_2 = w_3 = 0$, since $g$ is a zero vector.

The matrix $A_1$ is symmetric, hence there are only six independent elements in it, as shown in Eq. (23):

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$  \hspace{1cm} (23)$$

To simplify matters, the $a_{ij}$ coefficients will be used in the remainder of this development. According to Eqs. (15), (18)–(23), Eq. (17) can be expanded as

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}zx$$

$$- (2a_{11}x_0 + 2a_{12}y_0 + 2a_{13}z_0 + w_1)x$$

$$- (2a_{12}x_0 + 2a_{22}y_0 + 2a_{23}z_0 + w_2)y$$

$$- (2a_{13}x_0 + 2a_{23}y_0 + 2a_{33}z_0 + w_3)z$$

$$+ a_{11}x_0^2 + a_{22}y_0^2 + a_{33}z_0^2 + 2a_{12}x_0y_0$$

$$+ 2a_{23}y_0z_0 + 2a_{13}x_0z_0$$

$$+ w_1x_0 + w_2y_0 + w_3z_0 = d.$$  \hspace{1cm} (24)$$

The standard equation of a single-sheet hyperboloid (i.e., a single-sheet hyperboloid in standard position and orientation) is

$$f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$  \hspace{1cm} (11)$$

where $a$, $b$, and $c$ are the three basic geometric parameters representing the shape of the hyperboloid (i.e., the half lengths of the real major axis, real minor axis, and the imaginary axis, respectively). A diagram of a single-sheet hyperboloid is shown in Fig. 1.

Eq. (11) for the hyperboloid can be rewritten in terms of matrices and vectors as shown in Eq. (12)

$$\hat{x}^T A \hat{x} + g \cdot \hat{x} = d,$$  \hspace{1cm} (12)$$

where

$$\hat{x} = [x \ y \ z]^T.$$  \hspace{1cm} (13)$$

Fig. 1. Single-sheet hyperboloid.
where \( w_1 = w_2 = w_3 = 0 \) for hyperboloids. After comparing Eq. (24) with Eq. (2), the coefficients in Eq. (2) can be represented as follows:

\[
\begin{align*}
  k_1 &= a_{11}/u, \\
  k_2 &= a_{22}/u, \\
  k_3 &= a_{33}/u, \\
  k_4 &= 2a_{12}/u, \\
  k_5 &= 2a_{23}/u, \\
  k_6 &= 2a_{13}/u, \\
  k_7 &= (-2a_{11}x_0 - 2a_{12}y_0 - 2a_{13}z_0 - w_1)/u, \\
  k_8 &= (-2a_{12}x_0 - 2a_{22}y_0 - 2a_{23}z_0 - w_2)/u, \\
  k_9 &= (-2a_{13}x_0 - 2a_{23}y_0 - 2a_{33}z_0 - w_3)/u,
\end{align*}
\]

where

\[
\begin{align*}
  u &= d - (a_{11}x_0^2 + a_{22}y_0^2 + a_{33}z_0^2 + 2a_{12}x_0y_0 \\
 & \quad + 2a_{23}y_0z_0 + 2a_{13}x_0z_0 + w_1x_0 + w_2y_0 + w_3z_0).
\end{align*}
\]

The values of \( k_1, k_2, \ldots, k_9 \) can be determined from the solution vector \( \mathbf{X}_0 \) produced by Eq. (8). For hyperboloids and other central quadric surfaces, the translation values of \( x_0, y_0, z_0 \) can be solved from Eqs. (25)–(33) since \( w_1 = w_2 = w_3 = 0 \). Then, the value for \( u \) and for the six elements in \( A_1 \) (as defined in Eq. (23)) can be solved by Eqs. (25)–(30) and by Eq. (34).

For single-sheet hyperboloids, the three eigenvalues of \( A_1 \) are exactly the three diagonal elements in Eq. (14) (i.e., the inverse squares of the three basic geometric parameters of the surface equation [4]). Thus, the negative eigenvalue determines \( c \), the smaller positive eigenvalue corresponds to \( a \), and the other eigenvalue corresponds to \( b \). The three corresponding eigenvectors make up the orthogonal matrix \( \mathbf{R} \). The rotation angles, \( \theta_x, \theta_y, \theta_z \), can be computed from \( \mathbf{R} \) according to its definition.

### 3. Paraboloid reconstruction

In this section, we describe the new formulation that allows the parameter recovery process to be extended to allow extraction of one type of non-central quadric, the elliptic paraboloids.

The standard equation of an elliptic paraboloid is

\[
f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0,
\]

where \( a \) and \( b \) are the shape parameters of the paraboloid (i.e., the half lengths of its cross-sectional ellipse at \( z = 1 \)).

A diagram of an elliptic paraboloid is shown in Fig. 2.

The standard equation of an elliptic paraboloid can be rewritten in terms of matrices and vectors, to yield a form similar to Eq. (12), except for the elliptic paraboloids the values of \( \mathbf{A}, \mathbf{g}, \) and \( d \) are

\[
\mathbf{A} = \begin{bmatrix} 1/a^2 & 0 & 0 \\ 0 & 1/b^2 & 0 \end{bmatrix},
\]

\[
\mathbf{g} = [0 \ 0 \ -1]^T,
\]

and

\[
d = 0.
\]

After rotation and translation, the standard equation for paraboloids can be written in a form similar to Eq. (17) or (24). For paraboloids, the values \( w_i \) in these expressions are non-zero; \( w_1 = r_{31}, w_2 = r_{32}, \) and \( w_3 = r_{33} \). Since \( w_1, w_2, \) and \( w_3 \) are not 0, the values of \( x_0, y_0, z_0 \) cannot be solved directly using the linear least-squares formulation of Hall et al. [4]. Since these values cannot be found, the values which directly and indirectly depend on them also cannot be determined. Specifically, the value for \( u \) and the elements of \( A_1 \) cannot be found, which prevents determination of the shape and orientation parameters. Next, our indirect approach to find the location, orientation, and shape parameters for elliptic paraboloids is described.

First, we introduce a variable \( v \), where

\[
v = 1/u,
\]

and perform the orthogonal decomposition (diagonalization) on a matrix \( \mathbf{A}_2 \) that is the product \( v \mathbf{A}_1 \)

\[
\mathbf{A}_2 = v \mathbf{A}_1 = \begin{bmatrix} k_1 & k_4/2 & k_6/2 \\ k_4/2 & k_2 & k_5/2 \\ k_6/2 & k_5/2 & k_3 \end{bmatrix}.
\]

The result must be

\[
\mathbf{A}_2 = \mathbf{R}^T (v \mathbf{A}) \mathbf{R}.
\]
The $\mathbf{R}$ in Eq. (41) is the same as the $\mathbf{R}$ in Eq. (19) (since $\mathbf{R}$ is defined as a standard orthogonal matrix). Therefore, the values for $w_1$ (i.e., $r_{31}$), $w_2$ (i.e., $r_{32}$), and $w_3$ (i.e., $r_{33}$) can be determined from eigenanalysis of $A_2$. If Eqs. (25)–(30) and (39) are substituted into Eqs. (31)–(33), then we have

\[2k_1x_0 + k_4y_0 + k_6z_0 = -k_7 - r_{31}v,\]  
\[k_4x_0 + 2k_2y_0 + k_5z_0 = -k_8 - r_{32}v,\]  
\[k_6x_0 + k_5y_0 + 2k_3z_0 = -k_9 - r_{33}v,\]

where $x_0$, $y_0$, $z_0$, and $v$ are unknown. The unknown translational variables $x_0$, $y_0$, and $z_0$ can be represented by expressions of $v$ according to Eqs. (42)–(44)

\[x_0 = g_1(v),\]  
\[y_0 = g_2(v),\]  
\[z_0 = g_3(v).\]

After dividing both sides of Eq. (34) by $u$ and then substituting Eqs. (25)–(30) and (45)–(47) into it, the result is an expression of the form:

\[pv^2 + qv + s = 0,\]  
where

\[q = 0,\]

and $p$ and $s$ are functions of the known coefficients $k_1, k_2, \ldots, k_9$, and $r_{31}, r_{32}$, and $r_{33}$. Furthermore, the value $s$ is

\[s = \begin{vmatrix} k_1 & k_4 & k_6 & k_7 \\ k_4 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & k_5 & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & \frac{2}{2} & k_8 & k_9 \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & -1 \end{vmatrix},\]

which is exactly the invariant $\Delta$ for a general quadric (i.e., that is expressed in the form of Eq. (2)). Since $\Delta$ is invariant under rotational and translational transformation of a quadric, translation of the quadric back to the standard position (i.e., when $x_0 = y_0 = z_0 = 0$) produces an equation whose invariant $\Delta$ is unchanged from the value of $s$, although the invariant $\Delta$ in such a case will be $\hat{s} = s$:

\[\hat{s} = \begin{vmatrix} k_1 & k_4 & k_6 & r_{31}v \\ k_4 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & k_5 & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & \frac{2}{2} & k_8 & k_9 \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{vmatrix},\]

due to Eqs. (31)–(33) and (39) and since $D = 0$. In addition, the value of $p$ is

\[p = -\begin{vmatrix} k_1 & k_4 & k_6 & r_{31} \\ k_4 & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & k_5 & \frac{2}{2} & \frac{2}{2} \\ \frac{2}{2} & \frac{2}{2} & k_8 & k_9 \\ \frac{2}{2} & \frac{2}{2} & \frac{2}{2} & \frac{2}{2} \end{vmatrix} ;\]

$p$ is thus:

\[p = -\frac{s}{v^2}.\]

Since each of $p$ and $s$ can be expressed as a real multiple of the invariant $\Delta$ of a paraboloid surface, and since $\Delta$ must be negative for elliptic paraboloids (Table 1), $s < 0$ and $p > 0$; a real solution for $v$ is ensured. Therefore, the value of $v$ can be found, as shown in Eq. (54):

\[v = \pm \sqrt{-\frac{s}{p}}.\]

It is necessary to determine the sign of $v$. Because of Eq. (41), $v\mathbf{A}$ is a diagonal matrix. Furthermore, according to the definition of matrix $\mathbf{A}$ in Eq. (36), its non-zero eigenvalues (diagonal entries) must be positive. Therefore, the sign of $v$ can be determined by finding $\mathbf{A}_2$’s eigenvalues; if $\mathbf{A}_2$’s eigenvalues are negative, then $v$ must be negative.

Once $v$ has been determined, evaluation of Eqs. (36) and (41)–(44) allows determination of $x_0, y_0, z_0, a, b$. Thus all of the location, orientation, and shape information can be extracted for elliptic paraboloids.

### 3.1. Alternate normalization

For the cases that $l_{10} = 0$ in Eq. (1), the equation can be rewritten in terms of nine independent coefficients by normalizing some other term. The least-squares resolving and geometric parameter recovery processing will then proceed in a similar fashion to what has been described in this section.

### 4. Reference surface reconstruction

Next, we describe use of the least-squares-based fitting for recovering a reference surface in the presence of localized excess or insufficient material defects.

One challenge in reconstructing surfaces that have shape-based defects is that the defects can perturb the recovered geometric parameters for the reference (i.e., underlying) surface. For example, an elliptic paraboloid surface in the standard pose that has a circular cross-section of radius $r$ in the $z = 1$ plane and which has several small excess material defects could be reconstructed as a paraboloid that has a cross-section of radius $r + \varepsilon$ in the $z = 1$ plane, where $\varepsilon$
is a small nonnegative value. Generally, excess material defects are prone to “lift” the fit surface above the actual reference surface while insufficient material defects are prone to “drop” the fit surface below the actual reference surface. In addition, when defects need to be localized on surfaces that are reconstructed above or below their true position, the measured deviations in the defect regions will be too conservative, leading to higher rates of false acceptances. Thus, measured deviations in the defect regions will be too conservative, leading to higher rates of false acceptances. Therefore, the strategy can be easily extended to the scenario where both significant excess and insufficient material defects are present. Our strategy proceeds as follows:

Step 1: A surface is fit to the 3D data points using the linear least-squares-based procedure. The fit surface serves as an estimate of the underlying (reference) surface.

Step 2: The “side” of the surface most likely to contain defect(s) is hypothesized. This side is the one that contains the fewest number of points, as illustrated in Fig. 3.

Step 3: For each point, approximate orthogonal point-to-surface distances are computed, for example using the procedure of Cao et al. [20], as described in Section 4.1. The Point \( P_A \) with greatest point-to-surface distance \( D_A \) is marked.

Step 4: If \( D_A \) exceeds a rejection threshold \( T \) or if \( D_A < T_a \), where \( T_a \) is an acceptance threshold \( (T_a < T) \), the fit surface is acceptable as a reference surface and processing ends. (In the former case, the part is defective, and it would also be rejected.)

Step 5: A search is started at \( P_A \) and carried out recursively to visit all points near to it. The search terminates at each point that is within distance \( T_a \) of the surface. All points found that exceed the acceptance threshold are marked with the label \( B \).

Step 6. All points labelled \( B \) are removed and replaced with their approximate projections on the reference surface. Then, another iteration is performed (commencing from Step 1), unless a termination criterion, such as completing some predefined maximum number of iterations or observing little change between successive iterations, is reached.

4.1. Approximate orthogonal distances

An acceptance threshold \( T \) is normally given as an orthogonal distance to the real surface. Determining the precise distance from a point to the reference surface requires solution of a sixth order equation, however, and there is no general radical solution for equations of fifth or higher order. An alternative is to compute an approximation that is precise enough.

For elliptic paraboloid (or single-sheet hyperboloid) cases, the reference surface \( Q \) can be expressed by an implicit equation

\[
f(x, y, z) = 0.
\]  

(55)

An arbitrary straight line passing through a sample point \( P_B(x_B, y_B, z_B) \) can be represented using the parametric form

\[
L(x, y, z) = (x_B + p_Lt, y_B + q_Lt, z_B + r_Lt),
\]  

(56)

where \((p_L, q_L, r_L)\) is the unit-length vector describing the line’s direction and \( t \) is the distance from a point on the line to the reference point \( P_B \). By substituting Eq. (56) into Eq. (55), the intersection, \( Q_L \), of the line and the surface \( Q \) can be solved. The distance \( |P_BQ_L| \) can be considered as a candidate of the distance from \( P_B \) to the surface. Using different combinations of the values of \( p_L, q_L, \) and \( r_L \), a series of such candidates can be computed. If enough combinations of \( p_L, q_L, \) and \( r_L \) are considered, the smallest \( |P_BQ_L| \) value will be precise enough to be taken as a reasonable approximation of the real orthogonal distance.

In Cao et al. [3], the values of \((p_L, q_L, r_L)\) are chosen using an approximate equal subdivision of a viewsphere based on an icosahedron. The 12 vertices of an icosahedron lie on the sphere that circumscribes the icosahedron. Each vertex is exactly 60° from adjacent vertices and the vertices cover the sphere. They are also equally distributed on the surface of the sphere. Therefore, the directions from the center of the sphere to the vertices can be taken as an equal distribution of viewing directions. However, 12 directions are not enough for the approximate orthogonal distance computation. Therefore, in our approach the directions to icosahedron vertices as well as the directions to the mid-points of edges between vertices are used to form the collection of
directions \((p_L, q_L, r_L)\). The selection of the mid-points leads to an unequal subdivision of a viewsphere although the subdivision is very close to an equal subdivision; the subdivision is an approximate equal subdivision of a viewsphere. There are 30 edges on an icosahedron, therefore totally 42 combinations of \((p_L, q_L, r_L)\) are considered.

5. Experimental results

In this section, the performance of fitting on both synthetic and real data is reported. To provide a frame of reference, we present fitting results for one central quadric (hyperboloids) as well as for the paraboloids. For many of the tests, mean error in fit is reported. These errors are computed from the vertical \((z)\) direction RMS error \(E_{rms}\), defined as shown in Eq. (57):

\[
E_{rms} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(\hat{z}_i - z_i)^2},
\]

where \((x_i, y_i, z_i)\) are the \(n\) dataset points and \(\hat{z}_i(x_i, y_i)\) is the \(z\) coordinate of the fit surface at \((x_i, y_i)\). For some tests, mean shape parameter error, \(E_p\), is defined, as shown in Eq. (58) for elliptic paraboloids:

\[
E_p = \frac{1}{2} \left( \left| \frac{\hat{a} - a}{a} \right| + \left| \frac{\hat{b} - b}{b} \right| \right),
\]

where \(\hat{a}\) and \(\hat{b}\) are the fit parameters and \(a\) and \(b\) are the real parameters. Mean shape parameter error expresses the overall relative error in the fit shape parameter values.

Synthetic datasets, each of which contained several thousand 3D data points, were used to determine the general behavior of the fitting process. Each sample point’s coordinate components were perturbed by additive normally distributed random noise with \(\mu = 0\). Sample points were limited to a range in \(z\) of \([-30, 30]\) from the surface’s centroid (or, for paraboloids, its vertex). Sample points were not limited in \(x\) and \(y\); data was generated all around the surface. The time for a single fitting on these data points was low—0.11 s on a Sun Ultra 10/333 workstation.

5.1. Tests on hyperboloids

We first report tests on reconstructing single-sheet hyperboloids using the method of Hall et al. [4].

5.1.1. Defect-free hyperboloids

A series of Monte-Carlo trials were performed on synthetically generated defect-free datasets of hyperboloids in standard pose. Each dataset consisted of a set of between 4000 and 7000 sample points from the hyperboloid surface. Points were perturbed with additive noise at the level of \(\sigma = 0.05\). For each combination of \(a\), \(b\), and \(c\) parameters tested, 500 trial data sets were generated.

Table 2 lists the mean vertical direction RMS errors in the fit surface for each combination of tested parameters. In Table 3 the mean shape parameter error is reported for each combination of tested parameters. In these tests, it was typical for the recovered parameters to be very close to the true parameters. For example, for the hyperboloid with shape parameters \(a=20\), \(b=18\), and \(c=10\), typical recovered shape parameters were \(\hat{a} = 20.01\), \(\hat{b} = 18.01\), and \(\hat{c} = 10.01\). We did observe an increase in fitting error as the relative amount of noise increased. For example, at \(\sigma = 0.20\), the mean \(E_{rms}\) value varied between 0.31 and 0.45 for these cases.

5.1.2. Hyperboloids with defects

A hyperboloid vase with a real excessive material defect of height approximately 15 mm was also digitized for testing the fitting performance of the linear least-squares method. The vase’s surface was dimpled, making the problem more challenging. The vase’s measurable parameters were approximately \(a = b = 37\) mm. A 3D digitizer with mean accuracy 0.89 mm was used to collect the data. A range image constructed from the data collected from the vase is shown in Fig. 4. The defect is the whitish area near the center of the figure. The surface shape parameters recovered using the reference surface reconstruction strategy are shown for the first and 30th iterations in Table 4. The process converged at the 30th iteration, using a criterion of terminating when the change in maximum deviation from the reference surface between successive iterations was less than 0.05. The iterative process was able to recover the reference surface’s
5.2. Tests on paraboloids

Next, we report tests on elliptic paraboloid reconstruction.

5.2.1. Defect-free paraboloids

To test the performance of the new technique, Monte-Carlo trials were performed on synthetically-generated defect-free datasets of elliptic paraboloids in standard orientation with vertex at (10, 10, 10). For each combination of \(a\) and \(b\) parameters tested, 500 trial data sets were generated. Table 5 lists the mean vertical direction RMS errors in the fit surface for each combination of tested parameters for data with additive noise level \(\sigma = 0.05\) for datasets containing between 4000 and 7000 sample points. In Table 6 the mean shape parameter error is reported for each combination of tested parameters. While the recovery tended to be quite accurate at low and moderate levels of noise (e.g., \(\sigma < 0.2\)), both RMS and shape parameter errors increased as noise increased. For example, at \(\sigma = 0.20\), mean \(E_{rms}\) varied between 0.3 (for the large paraboloids) and 1.5 (for the small paraboloids). A plot of the RMS error as a function of \(\sigma\) is shown in Fig. 5 for paraboloids in standard orientation, with vertex at (10, 10, 10) and shape parameters \(a = 10.0, b = 8.0\). The observed errors were on par with that for comparably-sized hyperboloids fit with the Hall et al. [4] mechanism. Performance degradation was graceful as noise increased, although the rate of degradation did increase somewhat at the highest noise level.

5.2.1.1. Sampling density A series of experiments was also conducted to test the impact of sampling density on the fitting. A summary of the fitting error for Monte-Carlo testing on a synthetic paraboloid with parameters \(a = 10.0, b = 8.0\) in the standard pose, with point locations perturbed by additive noise with \(\sigma = 0.05\), is shown in Table 7. In these cases, points were collected on \(z = c_1\) planes, for \(c_1\) values that ranged from 0 to 30 units from the vertex, with a separation between sampling planes of 1 to 15 units, as shown in the table. The angular separation between points in each plane varied between 20 and 90° and is also shown in the table.
The error $E_{rms}$ in fit remained low even when the sampling resolution was low, although the shape parameter error $E_p$ grew as the number of points used to perform the fitting decreased.

We also tested the impact of sampling from only part of the surface. We found that reasonable recovery generally required samples to be collected from over half of the surface. If most of the surface was sampled, high accuracy was still achievable. For example, Monte-Carlo testing on a little more than a hemi-paraboloid with parameters $a = 10.0$, $b = 8.0$ in the standard pose, with point locations perturbed by additive noise with $\sigma = 0.05$, with sampling performed with parameters of $18^\circ$ angular separation and a sampling plane separation of 2 units in $z$ (i.e., 208 data points), produced a mean $E_{rms}$ of 0.066 and a mean $E_p$ of 0.0011. (In this case, the back $135^\circ$ of the paraboloid was not sampled.)

### 5.2.1.2. Pose perturbation

We have also found the fitting to be robust with respect to position of the paraboloid’s vertex. For example, a series of Monte-Carlo tests involving synthetic datasets with additive noise at the $\sigma = 0.05$ level of a paraboloid with shape parameters $a = 10.0$ and $b = 2.0$, oriented in the standard orientation but translated by factors of $(0, 10, 10)$, $(0, 0, 10)$, $(10, 5, 5)$, and $(100, 5, 5)$ from the standard position was conducted. For these tests, all translational factors produced comparable fitting results. Specifically, the mean $E_{rms}$ errors were small and nearly identical (ranging from 0.184 to 0.187) and the shape parameter errors were uniformly small (ranging from 0.001 to 0.006). The recovered positions were also accurate. The maximal absolute error in any translational coordinate component across all these experiments was 0.10 units. The maximum standard deviation in any coordinate component position was 0.03 units.

The fitting is less robust to orientational perturbation, however. For example, Monte-Carlo testing involving synthetic datasets of a paraboloid with shape parameters $a = 10.0$ and $b = 2.0$, positioned at $(10, 10, 10)$ but rotated about the $y$-axis, with additive noise at the $\sigma = 0.05$ level, resulted in fits with $E_{rms}$ errors matching those for the standard orientation trials. However, parameter shape error increased substantially (from an average of 0.002 in the standard orientation, to 0.021 when rotated $11.25^\circ$, to 0.083 when rotated $22.5^\circ$, and to 0.203 when rotated $33.75^\circ$). Since the linear least-squares formulation does not attempt to minimize the orthogonal distance to the quadric surface, it cannot always produce an optimal fitting. However, as mentioned above, calculation of the (orthogonal) distance of a point to a quadric is, in general, not a solvable problem.

### 5.2.2. Paraboloids with defects

Our fitting technique has also been benchmarked for paraboloid surfaces with defects. First, Monte-Carlo testing on synthetically-generated paraboloid surfaces in standard pose with $a = 15.0$ and $b = 10.0$ for data within the range of $z \in [0, 30]$ was performed. A defect of height 0.5 and spatial extent 1.0 was imposed in a random location on the surface for each of 500 Monte-Carlo trials. The recovered shape parameters from applying our strategy are summarized in Table 8. On average, the reference surface recovery was completed in three iterations, using a convergence criterion of terminating when the change in maximum deviation from the reference surface in successive iterations was less than 0.03. The process was completed in 0.3 s on our workstation.

Real data was also collected using our 3D digitizer from an egg-like object that contained a large paraboloid surface with an excess material defect of approximate height 6 mm and extent 12 mm. The surface’s measurable parameters were approximately $a = b = 8.5$ mm. A range image synthesized from the digitized data of the object is shown in Fig. 6. The defect is the whitish blob near the center of the figure. The recovered parameters for this scenario are shown in Table 9. The iterative process was successful in improving reference surface recovery. The CPU time to perform the 25 iterations required for convergence was 3.2 s on our workstation.
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### References


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