

PARTIAL SUMS OF FUNCTIONS OF BOUNDED TURNING

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We determine conditions under which the partial sums of the Libera integral operator of functions of bounded turning are also of bounded turning.

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1. Introduction. Let \mathcal{A} denote the family of functions f which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and are normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathcal{U}. \quad (1.1)$$

For $0 \leq \alpha < 1$, let $\mathcal{B}(\alpha)$ denote the class of functions f of the form (1.1) so that $\Re(f') > \alpha$ in \mathcal{U} . The functions in $\mathcal{B}(\alpha)$ are called functions of bounded turning (cf. [4]). By the Nashiro-Warschowski theorem (see, e.g., [3]), the functions in $\mathcal{B}(\alpha)$ are univalent and also close-to-convex in \mathcal{U} .

For f of the form (1.1), the Libera integral operator F is given by

$$F(z) = \frac{2}{z} \int_0^z f(\zeta) d\zeta = z + \sum_{k=2}^{\infty} \frac{2}{k+1} a_k z^k. \quad (1.2)$$

The n th partial sums $F_n(z)$ of the Libera integral operator $F(z)$ are given by

$$F_n(z) = z + \sum_{k=2}^n \frac{2}{k+1} a_k z^k. \quad (1.3)$$

In [6] it was shown that if $f \in \mathcal{A}$ is starlike of order α , $\alpha = 0.294, \dots$, so is the Libera integral operator F . We also know that (see, e.g., [1]) there are functions which are univalent or spiral-like in \mathcal{U} so that their Libera integral operators are not univalent or spiral-like in \mathcal{U} . Li and Owa [5] proved that if $f \in \mathcal{A}$ is univalent in \mathcal{U} , then $F_n(z)$ is starlike in $|z| < 3/8$. The number $3/8$ is sharp. In this note we make use of a result of Gasper [2] to provide a simple proof for the following theorem.

MAIN THEOREM. *If $1/4 \leq \alpha < 1$ and $f \in \mathcal{B}(\alpha)$, then $F_n \in \mathcal{B}((4\alpha - 1)/3)$.*

2. Preliminary lemmas. To prove our Main theorem, we will need the following three lemmas. The first lemma is due to Gasper (see [2, Theorem 1]) and the third lemma

is a well-known and celebrated result (cf. [3]) that can be derived from the Herglotz' representation for positive real part functions.

LEMMA 2.1. *Let θ be a real number and let m and k be natural numbers. Then*

$$\frac{1}{3} + \sum_{k=1}^m \frac{\cos(k\theta)}{k+2} \geq 0. \quad (2.1)$$

LEMMA 2.2. *For $z \in \mathcal{U}$,*

$$\Re \left(\sum_{k=1}^m \frac{z^k}{k+2} \right) > -\frac{1}{3}. \quad (2.2)$$

PROOF. For $0 \leq r < 1$ and for $0 \leq |\theta| \leq \pi$, write $z = re^{i\theta} = r(\cos(\theta) + i \sin(\theta))$. By DeMoivre's law and the minimum principle for harmonic functions, we have

$$\Re \left(\sum_{k=1}^m \frac{z^k}{k+2} \right) = \sum_{k=1}^m \frac{r^k \cos(k\theta)}{k+2} > \sum_{k=1}^m \frac{\cos(k\theta)}{k+2}. \quad (2.3)$$

Now by Abel's lemma (cf. Titchmarsh [7]) and condition (2.1) of Lemma 2.1 we conclude that the right-hand side of (2.3) is greater than or equal to $-1/3$. \square

LEMMA 2.3. *Let $P(z)$ be analytic in \mathcal{U} , $P(0) = 1$ and let $\Re(P(z)) > 1/2$ in \mathcal{U} . For functions Q analytic in \mathcal{U} , the convolution function $P * Q$ takes values in the convex hull of the image on \mathcal{U} under Q .*

The operator “ $*$ ” stands for the Hadamard product or convolution of two power series $f(z) = \sum_{k=1}^{\infty} a_k z^k$ and $g(z) = \sum_{k=1}^{\infty} b_k z^k$ denoted by $(f * g)(z) = \sum_{k=1}^{\infty} a_k b_k z^k$.

3. Proof of Main theorem. Let f be of the form (1.1) and belong to $\mathcal{B}(\alpha)$ for $1/4 \leq \alpha < 1$. Since $\Re(f'(z)) > \alpha$, we have

$$\Re \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1} \right) > \frac{1}{2}. \quad (3.1)$$

Applying the convolution properties of power series to $F'_n(z)$, we may write

$$\begin{aligned} F'_n(z) &= 1 + \sum_{k=2}^n \frac{2k}{k+1} a_k z^{k-1} \\ &= \left(1 + \frac{1}{2(1-\alpha)} \sum_{k=2}^{\infty} k a_k z^{k-1} \right) * \left(1 + (1-\alpha) \sum_{k=2}^n \frac{4}{k+1} z^{k-1} \right) \\ &= P(z) * Q(z). \end{aligned} \quad (3.2)$$

From Lemma 2.2 for $m = n-1$, we obtain

$$\Re \left(\sum_{k=2}^n \frac{z^{k-1}}{k+1} \right) > -\frac{1}{3}. \quad (3.3)$$

Applying a simple algebra to inequality (3.3) and $Q(z)$ in (3.2) yields

$$\Re(Q(z)) = \Re\left(1 + (1 - \alpha) \sum_{k=2}^n \frac{4}{k+1} z^{k-1}\right) > \frac{4\alpha - 1}{3}. \quad (3.4)$$

On the other hand, the power series $P(z)$ in (3.2) in conjunction with the condition (3.1) yield $\Re(P(z)) > 1/2$. Therefore, by Lemma 2.3, $\Re(F'_n(z)) > (4\alpha - 1)/3$. This concludes the Main theorem.

REMARK 3.1. The Main theorem also holds for $\alpha < 1/4$. We also note that $\mathcal{B}(\alpha)$ for $\alpha < 0$ is no longer a bounded turning family.

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