VERTEX-DISJOINT STARS IN GRAPHS

KATSUHIRO OTA

Department of Mathematics
Keio University
Yokohama, 223-8522 Japan

e-mail: ohta@math.keio.ac.jp

Abstract

In this paper, we give a sufficient condition for a graph to contain vertex-disjoint stars of a given size. It is proved that if the minimum degree of the graph is at least $k + t - 1$ and the order is at least $(t + 1)k + O(t^2)$, then the graph contains $k$ vertex-disjoint copies of a star $K_{1,t}$. The condition on the minimum degree is sharp, and there is an example showing that the term $O(t^2)$ for the number of uncovered vertices is necessary in a sense.

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1. Introduction

We consider only undirected graphs without loops or multiple edges. For a graph $G$, we denote by $V(G)$, $E(G)$ and $\delta(G)$ the vertex set, the edge set and the minimum degree of $G$, respectively.

For a graph $F$ and a positive integer $k$, $kF$ denotes the vertex-disjoint union of $k$ copies of $F$. A spanning subgraph isomorphic to $kF$ for some integer $k$ is called an $F$-factor. There are several results concerning minimum degree conditions for a graph to have an $F$-factor for several specific graphs $F$. The result of Corrádi and Hajnal [3] implies that $\delta(G) \geq \frac{2}{3}|V(G)|$ suffices for the existence of a $K_3$-factor. (When we consider an $F$-factor of a graph $G$, we always assume that $|V(G)|$ is a multiple of $|V(F)|$). Dirac [4] generalized this result by showing that if $\delta(G) \geq \frac{1}{2}(|V(G)| + k)$,
then $G$ contains $k$ vertex-disjoint triangles for any integer $k$ with $3k \leq |V(G)|$. Enomoto, Kaneko and Tuza [7] proved for $F = P_3$ (the path of order three) that $\delta(G) \geq \frac{1}{2} |V(G)|$ is sufficient for the existence of an $F$-factor if we assume that $G$ is connected. Hajnal and Szemerédi [8] proved that for $F = K_t$, $\delta(G) \geq \frac{1}{t+1} |V(G)|$ suffices. More generally, Alon and Yuster [2] proved an asymptotic result, which states that $\delta(G) \geq \left( \frac{\chi(F)^{-1}}{\chi(F)^2} + o(1) \right) |V(G)|$ assures the existence of an $F$-factor, where $\chi(F)$ denotes the chromatic number of $F$.

On the other hand, if we want to find $k$ vertex-disjoint copies of $F$ in a graph $G$ of order slightly larger than $k|F|$, and if $F$ admits a $\chi(F)$-coloring in which some color classes are tiny, then a much weaker condition may guarantee the existence. Komlós [9] (and Alon and Fischer [1] for bipartite case) have proved that the required minimum degree of $G$ is

$$\frac{1}{\chi(F) - 1} \left( \chi(F) - 2 + \frac{\alpha}{|V(F)|} \right) |V(G)|,$$

where $\alpha$ is the smallest possible color class size in any $\chi(F)$-coloring of $F$.

In the case where $F$ is a star $K_{1,t}$, Alon and Yuster’s result implies that $\delta(G) \geq (\frac{1}{t} + o(1)) |V(G)|$ is sufficient for the existence of a $K_{1,t}$-factor, and Komlós, Alon and Fischer’s result implies that $\delta(G) \geq \frac{1}{t+1} |V(G)|$ is sufficient for the existence of $k$ copies of $K_{1,t}$ if $|V(G)|$ is large. In this paper, we prove the following theorem, in which the required minimum degree of $G$ does not depend on $|V(G)|$. The proof is given in the next section.

**Theorem 1.** Let $t$ be an integer with $t \geq 3$. If $G$ is a graph of order $n \geq (t + 1)k + 2t^2 - 3t - 1$ with minimum degree at least $k + t - 1$, then $G$ contains $k$ vertex-disjoint copies of $K_{1,t}$.

The minimum degree condition in the theorem cannot be replaced by any weaker condition even if the order of the graph is assumed much larger. To see this, let $H$ be a $t - 1$ regular graph of order sufficiently large, and let $G$ be obtained from $H$ by adding $k - 1$ new vertices which are joined to all other vertices. Then $\delta(G) = k + t - 2$. Since any $K_{1,t}$ subgraph of $G$ must contain one of the new vertices, $G$ does not contain $k$ vertex-disjoint copies of $K_{1,t}$.

On the other hand, the following example shows that the term $O(t^2)$ for the number of uncovered vertices is necessary. Let $k_1 + k_2 + \cdots + k_t = k - 1$ so that $|k_i - k_j| \leq 1$ for any $i$ and $j$. We define the graph $G$ to be the vertex-disjoint union of the complete graphs $K_{(t+1)k_1+t}$, $K_{(t+1)k_2+t}$, $\ldots$, $K_{(t+1)k_t+t}$. Theorem 1 still holds.
Then, \(|V(G)| = (t + 1)(k - 1) + t^2 = (t + 1)k + t^2 - t - 1\) and \(\delta(G) = (t + 1)\left\lfloor \frac{k - 1}{t} \right\rfloor + t - 1 \geq k + t - 1\) (if \(k \gg t\)). However, it is obvious that \(G\) contains at most \(k - 1\) copies of \(K_{1,t}\).

This example suggests that the same conclusion as in Theorem 1 follows if \(|V(G)| \geq (t + 1)k + t^2 - t\). In fact, it is known to be true for \(t \leq 3\). The case \(t = 1\) is an easy exercise. The case \(t = 3\) is proved in [6].

**Theorem 2** [6]. *If \(G\) is a graph with \(|V(G)| \geq 4k + 6\) and \(\delta(G) \geq k + 2\), then \(G\) contains \(k\) vertex-disjoint copies of \(K_{1,3}\).*

The case \(t = 2\) can be proved in the following way. We use the following theorem due to Enomoto [5].

**Theorem 3** [5]. *Let \(G\) be a connected graph of order \(n\) and \(n = n_1 + \cdots + n_k\) with \(n_i \geq 2\) (\(1 \leq i \leq k\)). If \(\delta(G) \geq k\), then \(V(G)\) can be partitioned into \(V_1, \ldots, V_k\) so that for each \(i\), \(|V_i| = n_i\) and \(V_i\) induces a subgraph without isolated vertices.*

**Corollary 4.** *Let \(G\) be a connected graph of order \(n\), and \(k\) be an integer with \(3k \leq n\). If

\[
\delta(G) \geq \begin{cases} 
  k, & \text{if } n - 3k \text{ is even,} \\
  k + 1, & \text{if } n - 3k \text{ is odd,}
\end{cases}
\]

then \(G\) contains \(k\) vertex-disjoint copies of \(P_3\).*

**Proof.** If \(n - 3k\) is even, then put \(n_1 = \cdots = n_{k-1} = 3\) and \(n_k = n - 3k + 3\), and apply Theorem 3. If \(n - 3k\) is odd, then by deleting one vertex from \(G\) so that the resulting graph is connected, we can apply the previous case. \(\blacksquare\)

Now we can prove the following theorem.

**Theorem 5.** *Let \(G\) be a graph of order \(n\) with \(n \geq 3k + 2\). If \(\delta(G) \geq k + 1\), then \(G\) contains \(k\) vertex-disjoint copies of \(P_3\).*

**Proof.** If \(G\) is connected, or if \(G\) has a component of order at least \(3k\), then the result follows immediately from Corollary 4. Suppose that \(G\) is disconnected and each component is order less than \(3k\). Note that by Corollary 4, each component \(C\) of \(G\) contains \(\lceil |V(C)|/3 \rceil\) vertex-disjoint copies of \(P_3\). Also, since \(\delta(G) \geq k + 1\), each component has at least \(k + 2\) vertices.
If $G$ has at least three components, then $G$ contains at least $3\lceil \frac{k+2}{3} \rceil \geq k$ copies of $P_3$, and we are done. If $G$ consists of two components of orders $n_1$ and $n_2$, then the number of vertex-disjoint copies of $P_3$ in $G$ is at least $\lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2}{3} \rceil \geq \lceil \frac{3k+2-4}{3} \rceil = k$.

However, for the general case of the stronger statement, we need more crucial argument than the one used in this paper.

2. Proof of Theorem 1

Let $t$ be an integer with $t \geq 3$, and let $G$ be a graph of order at least $(t + 1)k + 2t^2 - 3t - 1$ and minimum degree at least $k + t - 1$.

We use the following notation and terminology. For $S \subseteq V(G)$, we write $\langle S \rangle$ for the subgraph of $G$ induced by $S$. For disjoint vertex sets $S$ and $T$, we denote the set of edges joining $S$ and $T$ by $E(S, T)$.

We consider a partition $V(G) = X \cup Y \cup Z$ satisfying the following conditions:
(a) $|X| = (t + 1)p$ and $X$ contains $p$ vertex-disjoint copies of $K_{1,t}$, say $C_1, C_2, \ldots, C_p$.
(b) The vertices of $Y$ can be labelled $y_1, y_2, \ldots, y_q$ so that for each $r$ ($1 \leq r \leq q$), $|N_G(y_r) \cap Z| \geq rt + (2t - 1)$.

Note that $X = Y = \emptyset$ and $Z = V(G)$ satisfy the above conditions with $p = q = 0$. We choose such a partition so that $p + q$ is maximum, and subject to this condition, $q$ is maximum possible.

Claim 1. For any subset $A \subseteq Z$ with $|A| \leq 2t - 1$, $G - X - A$ contains $q$ vertex-disjoint copies of $K_{1,t}$. In particular, $G$ contains $p + q$ vertex-disjoint copies of $K_{1,t}$.

Proof. By the condition (b), it follows that $|N_G(y_r) \cap (Z - A)| \geq rt + 2t - 1 - |A| \geq rt$ for each $1 \leq r \leq q$. Therefore we can complete to take $q$ stars in $\langle Y \cup (Z - A) \rangle$ whose centers are $y_1, y_2, \ldots, y_q$, respectively.

Claim 2. For any subset $A \subseteq Z$ with $|A| \leq 2t - 1$, $\langle X \cup A \rangle$ does not contain $p + 1$ vertex-disjoint copies of $K_{1,t}$.

Proof. Suppose that $\langle X \cup A \rangle$ contains $(p + 1)K_{1,t}$. Then by Claim 1, $G$ contains $(p + q + 1)K_{1,t}$. Let $V(G) = X' \cup Y' \cup Z'$ be a partition such that $X'$
is the set of vertices contained in \((p + q + 1)K_{1,t}\) and \(Y' = \emptyset\). This partition satisfies the condition (a) and (b), and contradicts the maximality of \(p + q\).

\[\frac{\text{Claim 3.}}{\text{In particular, we have the following.}}\]

**Claim 3.** The maximum degree of \(\langle Z \rangle\) is less than \(t\).

Let \(a\) be the center of any star \(C_i\) in \(X\). If \(|N_G(a) \cap Z| \geq tq + 2t - 1\), then we put \(X' = X - V(C_i), Y' = Y \cup \{a\}\) with \(y_{q+1} = a, \text{ and } Z' = Z \cup (V(C_i) - \{a\})\). Then \(X'\) contains \(p - 1\) vertex-disjoint \(K_{1,t}\)'s, and \(|N_G(y_{q+1}) \cap Z'| = |N_G(a) \cap Z| + t \geq t(q + 1) + 2t - 1\). This contradicts the maximality of \(q\). Hence we have

\[
\begin{align*}
(1) \quad |N_G(a) \cap Z| & \leq tq + 2t - 2. \\
\end{align*}
\]

By a similar argument, for each leaf \(b\) of any star \(C_i\) in \(X\), we have

\[
(2) \quad |N_G(b) \cap Z| \leq tq + 3t - 3.
\]

**Claim 4.** For each \(1 \leq i \leq p, |E(C_i, Z)| \leq \max\{tq + t^2 + t - 2, 2t^2 - 2t\}\).

**Proof.** Let \(a\) be the center and \(b_1, b_2, \ldots, b_t\) be the leaves of \(C_i\).

1. **Case 1.** \(|E(a, Z)| \geq t + 1\).

   In this case, each \(b_j\) is adjacent to at most \(t - 1\) vertices in \(Z\). For otherwise, we can take \(A \subset N(b_j) \cap Z\) with \(|A| = t\) and \(z \in N(a) \cap Z - A\) so that \(\{b_j\} \cup A\) and \(\{V(C_i) - \{b_j\}\} \cup \{z\}\) contain a \(K_{1,t}\). This contradicts Claim 2. Hence \(|E(b_j, Z)| \leq t - 1\) for all \(j\) \((1 \leq j \leq t)\). Then, since \(|E(a, Z)| \leq tq + 2t - 2\) by \((1)\),

\[
|E(V(C_i), Z)| = |E(a, Z)| + \sum_{j=1}^{t} |E(b_j, Z)| \\
\leq tq + 2t - 2 + t(t - 1) = tq + t^2 + t - 2.
\]

2. **Case 2.** \(1 \leq |E(a, Z)| \leq t\).

   If \(|E(b_j, Z)| \geq t + 1\) for some \(j\) \((1 \leq j \leq t)\), then we can take \(z \in N(a) \cap Z\) and \(A \subset N(b_j) \cap Z - \{z\}\) with \(|A| = t\) so that \(\{V(C_i) \cup A \cup \{z\}\}\) contains \(2K_{1,t}\), a contradiction. Hence \(|E(b_j, Z)| \leq t\) for all \(j\). Thus, \(|E(V(C_i), Z)| \leq (t + 1)t \leq 2t^2 - 2t\), since \(t \geq 3\).
Case 3. \(|E(a, Z)| = 0\).
If each leaf of \(C_i\) is adjacent to at most \(2t - 2\) vertices in \(Z\), then we have 
\(|E(V(C_i), Z)| \leq 2t^2 - 2t\). Hence we may assume that there exists a vertex \(b_h\) \((1 \leq h \leq t)\) with \(|E(b_h, Z)| \geq 2t - 1\). If \(|E(b_j, Z)| \geq t - 1\) for some \(j\) with \(j \neq h\), then we can take \(A \subset N(b_j) \cap Z\) with \(|A| = t - 1\) and \(A' \subset N(b_h) \cap Z - A\) with \(|A'| = t\) so that \(\langle\{b_h, b_j, a\} \cup A \cup A'\rangle\) contains \(2K_1, t\)'s, a contradiction. Hence \(|E(b_j, Z)| \leq t - 2\) for all \(j \neq h\). Since \(|E(b_h, Z)| \leq t^q + 3t - 3\) by (2),
we have \(|E(V(C_i), Z)| \leq t^q + 3t - 3 + (t - 1)(t - 2) < t^q + t^2 + t - 2\).
This completes the proof of Claim 4.

Now, we shall estimate the number of edges joining \(X\) and \(Z\) in two ways,
by assuming that \(G\) does not contain \(k\) vertex-disjoint \(K_1, t\)'s. By Claim 3,
each vertex in \(Z\) is adjacent to at least \((k + t - 1) - (t - 1) - q = k - q\)
vertices of \(X\). Hence,
\[|E(X, Z)| \geq (k - q)|Z| = (k - q)(n - (t + 1)p - q).\]
On the other hand, it follows from Claim 4 that
\[|E(X, Z)| = \sum_{i=1}^{p} |E(V(C_i), Z)| \leq p \cdot \max\{t^q + t^2 + t - 2, 2t^2 - 2t\}.\]

If \(t^q + t^2 + t - 2 \geq 2t^2 - 2t\), or equivalently if \(q \geq t - 2\), then
\[(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(t^q + t^2 + t - 2),\]
and hence
\[(k - q)(n - q) \leq p((t + 1)k + t^2 + t - 2 - q).\]
By Claim 1, we may assume that \(p + q \leq k - 1\). Thus the above inequality implies that
\[(k - q)(n - q) \leq (k - 1 - q)((t + 1)k + t^2 + t - 2 - q),\]
and hence
\[n \leq (t + 1)k + t^2 + t - 2 - \frac{(t + 1)k + t^2 + t - 2 - q}{k - q}\]
\[= (t + 1)k + 2t^2 - 3t - 1 - \left(t^2 - 4t + 3 + \frac{(t - 1)k + t^2 + t - 2 + q}{k - q}\right)\]
\[< (t + 1)k + 2t^2 - 3t - 1.\]
This contradicts the assumption that $n \geq (t + 1)k + 2t^2 - 3t - 1$.

If $q \leq t - 3$, then since $tq + t^2 + t - 2 < 2t^2 - 2t$,

$$(k - q)(n - (t + 1)p - q) \leq |E(X, Z)| \leq p(2t^2 - 2t),$$

and hence

$$(k - q)(n - q) \leq p((t + 1)(k - q) + 2t^2 - 2t).$$

Since $p + q \leq k - 1$,

$$(k - q)(n - q) \leq (k - 1 - q)((t + 1)(k - q) + 2t^2 - 2t),
\quad n \leq (t + 1)k + 2t^2 - 3t - 1 - tq - \frac{2t^2 - 2t}{k - q}
\quad < (t + 1)k + 2t^2 - 3t - 1.$$

This is a contradiction.

This completes the proof of Theorem 1. 

\[\blacksquare\]

References


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