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Periods of hyperelliptic integrals expressed in terms of θ-constants by means of Thomae formulae

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Expressions for the periods of first- and second-kind integrals on hyperelliptic curves are given in terms of θ-constants. They are derived with the help of Thomae's classical formulae and Picard–Fuchs equations for complete integrals as functions of the parameters of the curves. The example of genus 2 is considered in detail.

Keywords: Riemann θ-functions; θ-constants; Picard–Fuchs equations

1. Introduction

A large variety of mathematical and applied problems involving algebraic curves require the periods of Abelian integrals for an effective description of the underlying theory. For instance, frequencies and actions of completely integrable systems are often expressible in terms of complete hyperelliptic integrals, so that it is desirable to have a rapidly convergent series for their determination. Another application is the integration of KdV- and KP-type equations, where it is useful to express winding vectors in the Its–Matveev formula (1975) in terms of θ-constants (e.g. Dubrovin 1981; Mumford 1983). This approach was pioneered in the 1980s in the Russian school, following Novikov's conjecture on the link between KP-type equations and the classical Schottky problem, which led to its new solution using the concept of complete integrability. The periods of hyperelliptic integrals also appear in a wider context of work on integrable systems: the Riemann–Hilbert problem and the associated Schlesinger equation (e.g. Deift *et al.* 1999); Picard–Fuchs (Dullin *et al.* 2001); and Knizhnik–Zamolodchikov equations (Knizhnik 1989). The interrelation of these problems has recently attracted much attention in theoretical physics and applied mathematics.

The effective computation of complete elliptic integrals has a long history, which goes back to Gauss, Lagrange and Legendre computation of these integrals in terms of the arithmetic–geometric mean (e.g. Borwein & Borwein 1998). The generalization of this procedure to the case of genus 2 was recently developed by

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One contribution of 15 to a Theme Issue '30 years of finite-gap integration'.

Bost & Mestre (1989) and, surprisingly, was shown by Donagi & Livné (1999) to be impossible for genera higher than 3. On the surface, the question seems to have only practical relevance, but underlying there is obviously a deep mathematical background.

The present paper is based on two classical ingredients from this context. Thomae's formulae (1870) are one of them. In the work of Mestre (1991), they were combined with the arithmetic–geometric mean to obtain criteria for the reducibility of ultra-elliptic Jacobians. In the papers of Bershadsky & Radul (1987) and Nakayashiki (1997), they were generalized to the case of Z_N curves and applied by Bershadsky & Radul (1987) and Knizhnik (1989) for the calculation of correlation functions in the conformal field theory. In the works of Kitaev & Korotkin (1998), Korotkin (2000), Enolskii & Grava (2004) and Kokotov & Korotkin (2004) the Thomae formulae are used to construct the τ-function of the Schlesinger equation associated with hyperelliptic and more general curves. Recently, Thomae-type formulae for trigonal curve were implemented by Braden & Enolski (2006, 2007) to describe charge 3 monopole solution to Bogomolny equation. This list is far from being complete.

The other classical source of our investigations is Picard–Fuchs-type equations for the periods of Abelian integrals. Their use was pioneered by Fuchs (1871), Schlesinger (1895) and Bolza (1899). Our derivation is based on the variation procedure that goes back to Riemann, Thomae and Fuchs. To realize this procedure, we use Kleinian bi-differential (Klein 1886), which is explicitly given in algebraic form for hyperelliptic curves to represent derivatives of Abelian differentials with respect to branch points as residues. The application of the Kleinian bi-differential permits one to obtain Picard–Fuchs equations in closed algebraic form.

We shall be concerned with the expression of periods of Abelian integrals in terms of θ-constants of the associated curve. In this perspective, the modular variable τ is considered to be the principal parameter; all properties of the curve such as branch points and periods of first- or second-kind integrals are expressed as modular functions.

We have endeavoured to select appropriate language to display our results, and we hope that our paper could serve as an introduction to the classical parts of the subject. For further insight into the area, the reader should consult the modern classical literature (e.g. Mumford 1983; Arbarello *et al.* 1985), which gives much background on θ-functions, characteristics and provides a modern perspective within the theory of line bundles.

To outline our approach, we recall a number of well-known facts from the theory of elliptic functions. The periods 2ω, 2η and $2\omega'$, $2\eta'$ of Abelian integrals,

$$\begin{aligned} \omega &= -\int_{e_3}^{e_2} \frac{\mathrm{d}x}{y}, \quad \omega' = -\int_{e_2}^{e_1} \frac{\mathrm{d}x}{y} \quad \text{(first kind)}, \\ \eta &= \int_{e_3}^{e_2} \frac{x\,\mathrm{d}x}{y}, \quad \eta' = \int_{e_2}^{e_1} \frac{x\,\mathrm{d}x}{y} \quad \text{(second kind)}, \end{aligned} \tag{1.1}$$

on the curve (Weierstrass cubic)

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3), \quad e_1+e_2+e_3=0 \tag{1.2}$$

are expressible in terms of θ-constants (e.g. Bateman & Erdelyi 1955). In particular, when e_i are real and $e_1 > e_2 > e_3$, the following formulae hold:

$$2\omega = \frac{\pi\vartheta_2^2}{\sqrt{e_2 - e_3}} = \frac{\pi\vartheta_3^2}{\sqrt{e_1 - e_3}} = \frac{\pi\vartheta_4^2}{\sqrt{e_1 - e_2}}, \tag{1.3}$$

$$2\eta = -2\omega e_1 - \frac{1}{2\omega}\frac{\vartheta_2''}{\vartheta_2} = -2\omega e_2 - \frac{1}{2\omega}\frac{\vartheta_3''}{\vartheta_3} = -2\omega e_3 - \frac{1}{2\omega}\frac{\vartheta_4''}{\vartheta_4}. \tag{1.4}$$

The θ-constants ϑ_2, ϑ_3 and ϑ_4 are the values of the corresponding Jacobi θ-functions $\vartheta_i(v|\tau)$ at zero argument $v=0$; their derivatives $\partial\vartheta_i(v|\tau)/\partial v$ and $\partial^2\vartheta_i(v|\tau)/\partial v^2$ at $v=0$ are represented as ϑ_i' and ϑ_i'', respectively. The set of equations (1.3) can be interpreted as a special case of the first Thomae formula (see (3.1)). It can also be used to express the branch points in terms of θ-constants: the relations

$$\frac{e_2 - e_3}{e_1 - e_3} = \frac{\vartheta_2^4}{\vartheta_3^4}, \qquad \frac{e_1 - e_2}{e_1 - e_3} = \frac{\vartheta_4^4}{\vartheta_3^4}, \tag{1.5}$$

together with a normalization condition such as $e_1 + e_2 + e_3 = 0$, suffice to determine the e_i from the ϑ_i.

The set of equations (1.4) is derived from equations (1.3) using, in the general case, the Picard–Fuchs equations (4.3), Legendre's relation (2.4) and the 'heat equation' (2.18).

In the case of genus 1, the steps are as follows. First, take the derivative of the ith equation (1.3) with respect to e_i; for example,

$$\frac{\partial\omega}{\partial e_i} = \frac{2\omega}{\vartheta_2}\frac{\partial\vartheta_2}{\partial e_i} = \frac{2\omega}{\vartheta_2}\frac{\partial\vartheta_2}{\partial\tau}\frac{\partial\tau}{\partial e_i}. \tag{1.6}$$

Next, use the first of the Picard–Fuchs equations ($i \neq j \neq k = 1, 2, 3$)[1]

$$\frac{\partial\omega}{\partial e_i} = -\frac{1}{2}\frac{\eta + e_i\omega}{(e_i - e_j)(e_i - e_k)}, \tag{1.7}$$

$$\frac{\partial\eta}{\partial e_i} = \frac{1}{2}\frac{e_i\eta - (e_i^2 + e_j e_k)\omega}{(e_i - e_j)(e_i - e_k)}, \tag{1.8}$$

to eliminate $\partial\omega/\partial e_i$. These equations are obtained by taking the derivatives $\partial/\partial e_i$ of the two differentials $\mathrm{d}x/y$ and $x\,\mathrm{d}x/y$, respectively, using the fact that exact differentials do not contribute to the periods. For example, from

$$4(e_i - e_j)(e_i - e_k)\frac{\partial}{\partial e_i}\frac{1}{y} = 2\frac{x - e_i}{y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{y}{x - e_i}, \tag{1.9}$$

we find equation (1.7) because the last term is a total derivative.

[1] Picard–Fuchs equations are second-order equations, which result in the elimination of η or ω from (1.7) and (1.8). In the context of this paper, we shall call in this way the equivalent first-order system of differential equations.

In the third step, we use the Picard–Fuchs equations (which hold, respectively, for ω' and η' as well) and the Legendre relation

$$\eta\omega' - \omega\eta' = \frac{\pi \mathrm{i}}{2}, \tag{1.10}$$

to compute

$$\frac{\partial \tau}{\partial e_1} = \frac{\partial}{\partial e_1}\frac{\omega'}{\omega} = \frac{1}{4\omega^2}\frac{\pi \mathrm{i}}{(e_1 - e_2)(e_1 - e_3)}. \tag{1.11}$$

Finally, inserting this and the heat equation

$$4\pi \mathrm{i}\frac{\partial \vartheta_i(v|\tau)}{\partial \tau} = \frac{\partial^2 \vartheta_i(v|\tau)}{\partial v^2}, \tag{1.12}$$

into (1.6), we obtain the first of equations (1.4). Adding all three equations and using $e_1 + e_2 + e_3 = 0$, we obtain

$$\eta = -\frac{1}{12\omega}\left(\frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_4''}{\vartheta_4}\right). \tag{1.13}$$

The second Thomae formula (3.4) allows for a generalization of Jacobi's derivative formula

$$\vartheta_1' = \pi\vartheta_2\vartheta_3\vartheta_4, \tag{1.14}$$

to higher genera, where it is called Riemann–Jacobi formula. Taking its derivative with respect to τ and using the heat equation again, one finds

$$\frac{\vartheta_1'''}{\vartheta_1'} = \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3} + \frac{\vartheta_4''}{\vartheta_4}, \tag{1.15}$$

which turns (1.13) into the remarkably simple relation

$$\eta = -\frac{1}{12\omega}\frac{\vartheta_1'''}{\vartheta_1'}. \tag{1.16}$$

In what follows we shall generalize the above derivations to higher genera.

The paper is organized as follows. In §2 we introduce canonical differentials and recall known facts about θ-functions and the theory of their characteristics. In §3 we discuss the classical Thomae formulae (without proofs) and the Riemann–Jacobi relation. In §4 we derive the appropriate Picard–Fuchs equations, and on their basis give expressions for the periods of meromorphic differentials in terms of θ-constants. In §5, we consider the cases of genera 2 and 3 as examples.

2. Preliminaries

Let $V(x, y)$ be the hyperelliptic curve given by the equation

$$y^2 = \sum_{i=0}^{2g+1}\lambda_i x^i = 4\prod_{k=1}^{2g+1}(x - e_k) = R(x), \tag{2.1}$$

realized as a two-sheeted covering over the Riemann sphere branched in the points $(e_k, 0)$, $k \in \mathcal{G} = \{1, \ldots, 2g+1\}$, with $e_j \neq e_k$ for $j \neq k$, and at infinity, $e_{2g+2} = \infty$. Note we do not require the e_k to be real. However, when they are real, we find

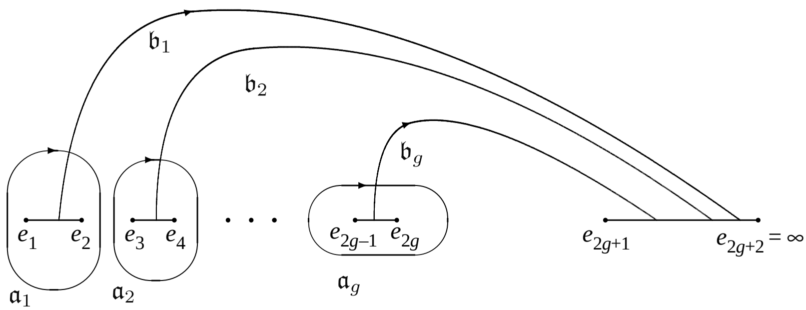

Figure 1. A homology basis on a Riemann surface of the hyperelliptic curve of genus g with real branch points $e_1, \ldots, e_{2g+2} = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i} for $i = 1, \ldots, g+1$. The $\mathfrak{b}$-cycles are completed on the lower sheet (the picture on lower sheet is just flipped horizontally).

it convenient to order them according to $e_1 < e_2 < e_3 < \cdots < e_{2g+1}$, i.e. in the opposite way as compared with the Weierstrass ordering (figure 1).

(*a*) *Canonical differentials*

Given a canonical homology basis $\mathfrak{a}_1, \ldots, \mathfrak{a}_g; \mathfrak{b}_1, \ldots, \mathfrak{b}_g$ as shown in figure 1, choose canonical holomorphic differentials (first kind) $\mathrm{d}\boldsymbol{u}^t = (\mathrm{d}u_1, \ldots, \mathrm{d}u_g)$ and associated meromorphic differentials (second kind) $\mathrm{d}\boldsymbol{r}^t = (\mathrm{d}r_1, \ldots, \mathrm{d}r_g)$ in such a way that their periods

$$2\omega = \left(\oint_{\mathfrak{a}_k} \mathrm{d}u_i\right)_{i,k=1,\ldots,g} \qquad 2\omega' = \left(\oint_{\mathfrak{b}_k} \mathrm{d}u_i\right)_{i,k=1,\ldots,g}, \tag{2.2}$$

$$2\eta = \left(-\oint_{\mathfrak{a}_k} \mathrm{d}r_i\right)_{i,k=1,\ldots,g} \qquad 2\eta' = \left(-\oint_{\mathfrak{b}_k} \mathrm{d}r_i\right)_{i,k=1,\ldots,g}, \tag{2.3}$$

satisfy the generalized Legendre relation

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}^t = -\frac{1}{2}\pi \mathrm{i} \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}. \tag{2.4}$$

Such a basis of differentials can be realized as follows (Baker 1897, p. 195):

$$\mathrm{d}\boldsymbol{u}(x, y) = \frac{\mathcal{U}(x)\mathrm{d}x}{y}, \; \mathcal{U}_i(x) = x^{i-1}, \; i = 1, \ldots, g, \tag{2.5}$$

$$\mathrm{d}\boldsymbol{r}(x, y) = \frac{\mathcal{R}(x)\mathrm{d}x}{4y}, \quad \mathcal{R}_i(x) = \sum_{k=i}^{2g+1-i} (k+1-i)\lambda_{k+1+i}x^k, \quad i = 1, \ldots, g. \tag{2.6}$$

It is convenient to use normalized holomorphic differentials,

$$\mathrm{d}\boldsymbol{v} = (2\omega)^{-1}\mathrm{d}\boldsymbol{u}. \tag{2.7}$$

Definition 2.1. The Kleinian bi-differential $\mathrm{d}\omega(x, y; z, w)$ is defined on $V \times V$ by the following properties:

— it is symmetric,

$$\mathrm{d}\omega(x, y; z, w) = \mathrm{d}\omega(z, w; x, y); \tag{2.8}$$

— it has only one pole of second order along the diagonal $(x,y)=(z,w)$, in the vicinity of which

$$\mathrm{d}\omega(x, y; z, w) = \left(\frac{1}{(\xi-\eta)^2} + O(1)\right)\mathrm{d}\xi\mathrm{d}\eta, \tag{2.9}$$

where ξ and η are the local coordinates of the points (x, y) and (z, w), respectively, in the neighbourhood of a point (X, Y).

It is known by Klein (1886) that this bi-differential can be realized as

$$\mathrm{d}\omega(x, y; z, w) = \frac{2yw + F(x, z)}{4(x-z)^2}\frac{\mathrm{d}x}{y}\frac{\mathrm{d}z}{w}, \tag{2.10}$$

where

$$F(x, z) = 2w^2 + 2(x-z)w\frac{\mathrm{d}w}{\mathrm{d}z} + (x-z)^2\boldsymbol{\mathcal{U}}^t(x)\boldsymbol{\mathcal{R}}(z). \tag{2.11}$$

Below we shall use the equivalence

$$\frac{2yw + F(x, z)}{y(x-z)^2} = 2\frac{\partial}{\partial x}\frac{y+w}{2(z-x)} + \frac{\boldsymbol{\mathcal{U}}^t(z)\boldsymbol{\mathcal{R}}(x)}{y}. \tag{2.12}$$

We denote by $\mathrm{Jac}(V)$ the *Jacobian* of the curve V, i.e. the factor $\mathbb{C}^g/\Gamma$, where $\Gamma = 2\omega \oplus 2\omega'$ is the lattice generated by the periods of canonical holomorphic differentials.

Let $\mathcal{D}$ be a divisor of degree 0, $\mathcal{D} = \mathcal{X} - \mathcal{Z}$, with $\mathcal{X}$ and $\mathcal{Z}$, the effective divisors, of degree $\deg \mathcal{X} = \deg \mathcal{Z} = n$, given by $\mathcal{X} = \{(x_1, y_1), \ldots, (x_n, y_n)\} \in (V)^n$ and $\mathcal{Z} = \{(z_1, w_1), \ldots, (z_n, w_n)\} \in (V)^n$, where $(V)^n$ is the nth symmetric power of V. The Abel map

$$\mathfrak{A} : (V)^n \to \mathrm{Jac}(V) \tag{2.13}$$

puts into correspondence the divisor $\mathcal{D}$, with fixed $\mathcal{Z}$, and the point $\boldsymbol{u} \in \mathrm{Jac}(V)$, according to

$$\boldsymbol{u} = \int_{\mathcal{Z}}^{\mathcal{X}} \mathrm{d}\boldsymbol{u}, \quad \text{or} \quad u_i = \sum_{k=1}^{n}\int_{z_k}^{x_k} \mathrm{d}u_i, \quad i = 1, \ldots, g. \tag{2.14}$$

(For notational convenience, we write x_k instead of (x_k, y_k), etc. whenever there is no danger of confusion.)

(*b*) *θ-functions*

Let $\mathcal{H}_g = \{\tau^t = \tau, \quad \mathrm{Im}\, \tau \geq 0\}$ be the Siegel half-space of degree g, where $\tau = \omega^{-1}\omega'$ is the period matrix. The hyperelliptic θ-function, $\theta : \mathrm{Jac}(V) \times \mathcal{H}_g \to \mathbb{C}$, with characteristics

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \boldsymbol{\varepsilon}'^t \\ \boldsymbol{\varepsilon}^t \end{bmatrix} = \begin{bmatrix} \varepsilon'_1 & \ldots & \varepsilon'_g \\ \varepsilon_1 & \ldots & \varepsilon_g \end{bmatrix} \in \mathbb{R}^{2g}, \tag{2.15}$$

is defined as the Fourier series

$$\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^g} \exp \pi\mathrm{i}\{(\boldsymbol{m}+\boldsymbol{\varepsilon}')^t\tau(\boldsymbol{m}+\boldsymbol{\varepsilon}') + 2(\boldsymbol{v}+\boldsymbol{\varepsilon})^t(\boldsymbol{m}+\boldsymbol{\varepsilon}')\}, \tag{2.16}$$

and has the periodicity properties

$$\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}+\boldsymbol{n}+\tau\boldsymbol{n}'|\tau) = \exp\left\{-2\mathrm{i}\pi\boldsymbol{n}'^t\left(\boldsymbol{v}+\frac{1}{2}\tau\boldsymbol{n}'\right)\right\}\exp\{2\mathrm{i}\pi(\boldsymbol{n}^t\boldsymbol{\varepsilon}'-\boldsymbol{n}'^t\boldsymbol{\varepsilon})\}\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}|\tau). \tag{2.17}$$

The θ-functions satisfy the heat equation

$$\frac{\partial^2}{\partial v_k \partial v_l}\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}|\tau) = 2(1+\delta_{kl})\pi\mathrm{i}\frac{\partial}{\partial \tau_{kl}}\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}|\tau). \tag{2.18}$$

In all the following, the values ε_k, ε'_k will either be 0 or 1/2. It is then typographically convenient to introduce the notation

$$\theta[\boldsymbol{\varepsilon}] \equiv \theta\left[\frac{1}{2}2\boldsymbol{\varepsilon}\right] =: \theta[2\boldsymbol{\varepsilon}]_2. \tag{2.19}$$

The equality (2.17) implies that

$$\theta[\boldsymbol{\varepsilon}](-\boldsymbol{v}|\tau) = \mathrm{e}^{-4\pi\mathrm{i}\boldsymbol{\varepsilon}^t\boldsymbol{\varepsilon}'}\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}|\tau), \tag{2.20}$$

and therefore the function $\theta[\boldsymbol{\varepsilon}](\boldsymbol{v}|\tau)$, with characteristics $[\boldsymbol{\varepsilon}]$ of only half-integers, is even when $4\boldsymbol{\varepsilon}^t\boldsymbol{\varepsilon}'$ is an even integer and odd otherwise. Correspondingly, $[\boldsymbol{\varepsilon}]$ is called even or odd, and among the 4^g half-integer characteristics, there are $1/2(4^g+2^g)$ even and $1/2(4^g-2^g)$ odd characteristics.

Definition 2.2. The non-vanishing values of θ-functions with even characteristics, at $\boldsymbol{v}=\boldsymbol{0}$, are called θ-constants of the first kind,

$$\theta[\boldsymbol{\varepsilon}](\boldsymbol{0}|\tau) =: \theta[\boldsymbol{\varepsilon}] \quad \text{for even } [\boldsymbol{\varepsilon}]; \tag{2.21}$$

the non-vanishing values of the first derivatives of θ-functions with odd characteristics, at zero argument, are called non-singular θ-constants of the second kind,

$$\frac{\partial}{\partial v_k}\theta[\delta](\boldsymbol{v}|\tau)|_{v=0} =: \theta_k[\delta], \qquad k=1,\ldots,g, \quad \text{for odd } [\delta]. \tag{2.22}$$

We stress that the derivatives are taken with respect to *normalized* variables v_i. This will turn out to be important for the validity of formula (3.4).

(c) *Characteristics*

Identify each branch point e_j of the curve V with a vector

$$\mathfrak{A}_j = \int_\infty^{e_j} \mathrm{d}\boldsymbol{v} =: \varepsilon_j + \tau\varepsilon'_j \in \mathrm{Jac}(V), \tag{2.23}$$

where $\mathrm{d}\boldsymbol{v}$ is the vector of normalized holomorphic differentials. Modulo 1, all components of the vectors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ are either 0 or 1/2. The $2\times g$ matrices

$$[\mathfrak{A}_j] = \begin{bmatrix} \varepsilon_j'^t \\ \varepsilon_j^t \end{bmatrix} = \begin{bmatrix} \varepsilon_{j1}' & \dots & \varepsilon_{jg}' \\ \varepsilon_{j_1} & \dots & \varepsilon_{j_g} \end{bmatrix} \tag{2.24}$$

will serve as a basis for the characteristics of the θ-functions to be discussed.

Let us identify the half-periods $\mathfrak{A}_i$, $i=1, \dots, 2g+1$ (e.g. Farkas & Kra 1980, p. 303). Evidently, $[\mathfrak{A}_{2g+2}]=[0]$. Using the notation $\boldsymbol{f}_k=(1/2)(\delta_{1k},\dots,\delta_{gk})^t$ (where δ_{ik} is the Kronecker symbol) and $\boldsymbol{\tau_k}$ for the kth column vector of the matrix $\boldsymbol{\tau}$, we find

$$\mathfrak{A}_{2g+1} = \mathfrak{A}_{2g+2} - \sum_{k=1}^{g} \int_{e_{2k-1}}^{e_{2k}} \mathrm{d}\boldsymbol{v} = \sum_{k=1}^{g} \boldsymbol{f}_k, \quad \rightarrow [\mathfrak{A}_{2g+1}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix},$$

$$\mathfrak{A}_{2g} = \mathfrak{A}_{2g+1} - \int_{e_{2g+1}}^{e_{2g}} \mathrm{d}\boldsymbol{v} = \sum_{k=1}^{g} \boldsymbol{f}_k + \boldsymbol{\tau}_g, \quad \rightarrow [\mathfrak{A}_{2g}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 \end{bmatrix},$$

$$\mathfrak{A}_{2g-1} = \mathfrak{A}_{2g} - \int_{e_{2g}}^{e_{2g-1}} \mathrm{d}\boldsymbol{v} = \sum_{k=1}^{g-1} \boldsymbol{f}_k + \boldsymbol{\tau}_g, \quad \rightarrow [\mathfrak{A}_{2g-1}] = \frac{1}{2}\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{bmatrix}.$$

Continuing in the same manner, we get for arbitrary $k>1$

$$[\mathfrak{A}_{2k+2}] = \frac{1}{2}\begin{bmatrix} \overbrace{0 \quad 0 \quad \dots \quad 0}^{k} & 1 & 0 & \dots & 0 \\ 1 \quad 1 \quad \dots \quad 1 & 1 & 0 & \dots & 0 \end{bmatrix},$$

$$[\mathfrak{A}_{2k+1}] = \frac{1}{2}\begin{bmatrix} \overbrace{0 \quad 0 \quad \dots \quad 0}^{k} & 1 & 0 & \dots & 0 \\ 1 \quad 1 \quad \dots \quad 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and finally,

$$[\mathfrak{A}_2] = \frac{1}{2}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix}, \quad [\mathfrak{A}_1] = \frac{1}{2}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \end{bmatrix}.$$

The characteristics with even indices, corresponding to the branch points e_{2n}, $n=1, \dots, g$, are odd (except for $[\mathfrak{A}_{2g+2}]$ which is zero); the others are even. The vector of Riemann constants has the form

$$\boldsymbol{K}_\infty = -\sum_{k=1}^{g} \int_\infty^{e_{2k}} \mathrm{d}\boldsymbol{v} = -\sum_{k=1}^{g} \mathfrak{A}_{2k}. \tag{2.25}$$

(See Farkas & Kra 1980, p. 305, for a proof.)

The $2g+2$ characteristics $[\mathfrak{A}_i]$ serve as a basis for the construction of all 4^g possible half-integer characteristics $[\boldsymbol{\varepsilon}]$. There is a one-to-one correspondence between these $[\boldsymbol{\varepsilon}]$ and partitions of the set $\bar{\mathcal{G}}=\{1,\dots,2g+2\}$ of indices of the branch points (Baker 1897, p. 271; Fay 1973, p. 13). Note the distinction between $\mathcal{G}$ and $\bar{\mathcal{G}}=\mathcal{G}\cup\{\infty\}$. The partitions of interest are

$$\bar{\mathcal{G}}=\mathcal{I}_m\cup\mathcal{J}_m,\quad \mathcal{I}_m=\{i_1,\ldots,i_{g+1-2m}\},\quad \mathcal{J}_m=\{j_1,\ldots,j_{g+1+2m}\},\tag{2.26}$$

where m is any integer between 0 and $[(g+1)/2]$. The corresponding characteristic $[\boldsymbol{\varepsilon}_m]$ is defined by the vector

$$\boldsymbol{E}_m=\sum_{k=1}^{g+1-2m}\mathfrak{A}_{i_k}+\boldsymbol{K}_\infty=:\boldsymbol{\varepsilon}_m+\boldsymbol{\tau}\boldsymbol{\varepsilon}'_m.\tag{2.27}$$

It can be seen that characteristics with even m are even and with odd m odd. There are $1/2\binom{2g+2}{g+1}$ different partitions with $m=0$, $\binom{2g+2}{g-1}$ different with $m=1$, and so on down to $\binom{2g+2}{1}=2g+2$ for even g and $m=g/2$, or $\binom{2g+2}{0}=1$ for odd g and $m=(g+1)/2$. (In the last case, the set $\mathcal{I}_{(g+1)/2}$ is the empty set, and the only possible characteristic derives from $\boldsymbol{E}_{(g+1)/2}=\boldsymbol{K}_\infty$.) One may check that the total number of even (odd) characteristics is indeed $2^{2g-1}\pm2^{g-1}$.

According to the Riemann theorem on the zeros of θ-functions, $\theta(\boldsymbol{E}_m+\boldsymbol{v})$ vanishes to order m at $\boldsymbol{v}=0$. We shall call the characteristics with $m=0$ *non-singular even characteristics* and those with $m=1$ *non-singular odd characteristics*. The remaining characteristics, i.e. those with $m>1$, are called *singular characteristics*. In the following we deal only with non-singular characteristics.

The correspondence between characteristics and partitions suggests the notation $\theta\{\mathcal{I}_m\}(\boldsymbol{v}|\boldsymbol{\tau})\equiv\theta[\varepsilon_m](\boldsymbol{v}|\boldsymbol{\tau})$; we shall use whatever is more convenient.

Let us now take care of the fact that $e_{2g+2}=\infty$ is taken as the base point of the Abel map. It belongs to $\mathcal{I}_m$ for even m and to $\mathcal{J}_m$ for odd m. As we are only interested in $m=0$ and $m=1$, we decide to omit the point ∞ from consideration and define

$$\mathcal{I}_0=\{i_1,\ldots,i_g\}\in\mathcal{G},\quad \mathcal{I}_1=\{i_1,\ldots,i_{g-1}\}\in\mathcal{G}.\tag{2.28}$$

That permits us to denote non-singular θ-constants of the first kind as $\theta(i_1,\ldots,i_g)$ or $\theta\{\mathcal{I}_0\}$, and non-singular θ-constants of the second kind as $\theta_k(i_1,\ldots,i_{g-1})$ or $\theta_k\{\mathcal{I}_1\}$.

Let us put in correspondence with each partition $\mathcal{I}_m\cup\mathcal{J}_m$, $m=0$, 1, the product of Vandermonde determinants

$$\Delta(\mathcal{I}_m)=\prod_{i_k<i_l\in\mathcal{I}_m}(e_{i_k}-e_{i_l})\prod_{j_k<j_l\in\mathcal{J}_m}(e_{j_k}-e_{j_l}),\tag{2.29}$$

and denote as $s_k(\mathcal{I}_1)$ the elementary symmetric functions of degree k of elements e_i, with $i\in\mathcal{I}_1$, i.e.

$$s_0(\mathcal{I}_1)=1,\quad s_1(\mathcal{I}_1)=\sum_{i\in\mathcal{I}_1}e_i,\ldots s_{g-1}(\mathcal{I}_1)=\prod_{i\in\mathcal{I}_1}e_i.\tag{2.30}$$

3. Thomae formulae

In this section we exhibit classical results of Thomae (1870). The Thomae formula which links branch points with non-singular even θ-constants, i.e. θ-constants of the first kind, is well known and widely used. We shall also implement another Thomae formula, written for θ-constants of the second kind. We refer to these formulae as first and second Thomae theorems. The important Riemann–Jacobi derivative relation which generalizes to higher genera Jacobi's relation (1.14) follows from the second Thomae theorem.

The θ-constants of the first kind are expressed in terms of branch points and periods of holomorphic integrals as follows.

Theorem 3.1 (first Thomae theorem). *Let $\mathcal{I}_0 \cup \mathcal{J}_0$ be a partition of the set $\mathcal{G}=\{1,\ldots,2g+1\}$ of indices of the finite branch points of the hyperelliptic curve V. Then the following formula is valid*:

$$\theta^4\{\mathcal{I}_0\} = \pm\frac{(\det 2\omega)^2}{\pi^{2g}}\varDelta(\mathcal{I}_0). \tag{3.1}$$

The proof can be found in many places (e.g. Thomae 1870; Bolza 1899; Fay 1973; Mumford 1983). There are $\binom{2g+1}{g}$ different possibilities to choose the set $\mathcal{I}_0$. For $g=1$, this gives the three equations (1.3); the case $g=2$ is explicitly discussed in §5.

Among various corollaries of the Thomae formula, we shall single out the following two.

Corollary 3.1. *Let $\mathcal{S}=\{i_1,\ldots,i_{g-1}\}$ and $\mathcal{T}=\{j_1,\ldots,j_{g-1}\}$ be two disjoint sets of non-coinciding integers taken from the set $\mathcal{G}$ of indices of the finite branch points. Then for any two $k\neq l$ from the set $\mathcal{G}\backslash(\mathcal{S}\cup\mathcal{T})$, the following formula is valid*:

$$\frac{e_l-e_m}{e_k-e_m} = \epsilon\frac{\theta^2\{k,\mathcal{S}\}\theta^2\{k,\mathcal{T}\}}{\theta^2\{l,\mathcal{S}\}\theta^2\{l,\mathcal{T}\}}, \tag{3.2}$$

where m is the remaining number when $\mathcal{S}$, $\mathcal{T}$, k and l are taken away from $\mathcal{G}$, and $\epsilon^4=1$.

Corollary 3.2. *Let $\mathcal{I}_0=\{i_1,\ldots,i_g\}$ and $\mathcal{J}_0=\{j_1,\ldots,j_{g+1}\}$ be the partitions. Choose $k,n\in\mathcal{I}_0$ and $i,j\in\mathcal{J}_0$. Define the sets $\mathcal{S}_k=\mathcal{I}_0\backslash\{k\}$, $\mathcal{S}_{k,n}=\mathcal{I}_0\backslash\{k,n\}$ and $\mathcal{T}_{i,j}=\mathcal{J}_0\backslash\{i,j\}$. Then*

$$\frac{\prod_{j_l\in\mathcal{J}_0}(e_k-e_{j_l})}{\prod_{i_l\in\mathcal{I}_0, i_l\neq k}(e_k-e_{i_l})(e_k-e_n)^2} = \frac{\pm\theta^4\{i,\mathcal{S}_k\}\theta^4\{j,\mathcal{S}_k\}\theta^4\{n,\mathcal{T}_{i,j}\}}{\theta^4\{i,j,\mathcal{S}_{k,n}\}\theta^4\{i,\mathcal{T}_{i,j}\}\theta^4\{j,\mathcal{T}_{i,j}\}}. \tag{3.3}$$

Recent discussions concerning the proof of (3.2) can be found in Takase (1996) and Koizumi (1997), where the formula is called 'generalization of Rosenhain's normal form'. There is considerable freedom in the choice of combination of characters on the right-hand side of formula (3.2). Farkas (1971) has used this to derive in an elegant and simple way Schottky's conditions for the case of hyperelliptic curves of genus 4.

The signs '$\pm$' and the values of ϵ should be determined in each particular case by some limiting procedure (e.g. Fay 1973).

The Thomae paper (1870; see also Krazer & Wirtinger 1915) contains another set of formulae expressing the non-singular θ-constants of the second kind in terms of branch points and periods of Abelian differentials.

Theorem 3.2 (second Thomae theorem). *Let $\mathcal{I}_1 \cup \mathcal{J}_1$ be a partition of the set $\mathcal{G}$ of indices of the finite branch points, and $v_1, \ldots, v_g$ the normalized holomorphic integrals. Then the θ-constants of the second kind are given by the formula*

$$\frac{\partial}{\partial v_j}\theta\{\mathcal{I}_1\}(\boldsymbol{v}|\tau)|_{\boldsymbol{v}=0} =: \theta_j\{\mathcal{I}_1\} = 2\epsilon\sqrt{\frac{\det 2\omega}{\pi^g}}\Delta(\mathcal{I}_1)^{1/4}\sum_{i=1}^{g}\omega_{ij}s_{g-i}(\mathcal{I}_1), \quad j=1,\ldots,g, \tag{3.4}$$

where $s_l(\mathcal{I}_1)$ is the elementary symmetric function of degree l associated with the set $\mathcal{I}_1$ of indices of the branch points.

It is convenient to rewrite this Thomae theorem in matrix form. To do that, we introduce for any set of non-singular odd characteristics, $[\delta_1], \ldots, [\delta_g]$ the Jacobi matrix

$$D[\delta_1,\ldots,\delta_g] = \begin{pmatrix} \theta_1[\delta_1] & \theta_1[\delta_2] & \ldots & \theta_1[\delta_g] \\ \vdots & \vdots & \ldots & \vdots \\ \theta_g[\delta_1] & \theta_g[\delta_2] & \ldots & \theta_g[\delta_g] \end{pmatrix}. \tag{3.5}$$

Theorem 3.3. *Let $\mathcal{I}_0=\{i_1,\ldots,i_g\}$ and $\mathcal{J}_0=\{j_1,\ldots,j_{g+1}\}$ be the sets of a partition $\mathcal{I}_0\cup\mathcal{J}_0=\mathcal{G}$. Define the g sets $\mathcal{S}_k=\mathcal{I}_0\backslash\{i_k\}$ and use the correspondence $[\delta_k]\Leftrightarrow\{\mathcal{S}_k\}$, $k=1,\ldots,g$, for non-singular odd characteristics. Then*

$$D[\delta_1,\ldots,\delta_g] = \epsilon\sqrt{\frac{\det 2\omega}{\pi^g}}2\omega^t SM, \tag{3.6}$$

where $\epsilon^8=1$; the matrices S and M are given as

$$S = (s_{g-i}(\mathcal{S}_k))_{k,i=1,\ldots,g}, \quad M = \mathrm{diag}\left(\sqrt[4]{\Delta(\mathcal{S}_1)},\ldots,\sqrt[4]{\Delta(\mathcal{S}_{\mathrm{g}})}\right), \tag{3.7}$$

where the $s_{g-i}(\mathcal{S}_k)$ are the symmetric functions.

Moreover, by choosing any $n\in\mathcal{S}_k$ and $i,j\in\mathcal{J}_0$, the formula (3.6) is transformed to

$$D[\delta_1,\ldots,\delta_g] = \epsilon 2\omega^t SN, \tag{3.8}$$

with

$$N = \theta\{\mathcal{J}_0\}\times\mathrm{diag}\left(\ldots,\sqrt{e_k-e_n}\frac{\theta\{i,\mathcal{S}_k\}\theta\{j,\mathcal{S}_k\}\theta\{n,\mathcal{T}_{i,j}\}}{\theta\{i,j,\mathcal{S}_{k,n}\}\theta\{i,\mathcal{T}_{i,j}\}\theta\{j,\mathcal{T}_{i,j}\}},\ldots\right)_{k=1,\ldots,g},$$

where we defined the sets $\mathcal{S}_{k,n}:=\mathcal{I}_0\backslash\{k,n\}$, $\mathcal{T}_{i,j}:=\mathcal{J}_0\backslash\{i,j\}$.

Proof. The formula (3.6) is the matrix version of formula (3.4). To write (3.6) in the form (3.8), we use (3.1) to obtain for every $k=1, \ldots, g$

$$\sqrt{\frac{\det 2\omega}{\pi^g}}\sqrt[4]{\Delta(\mathcal{S}_k)} = \epsilon\theta\{\mathcal{I}_0\}\sqrt[4]{\frac{\Delta(\mathcal{S}_k)}{\Delta(\mathcal{I}_0)}}.$$

The quotient under the sign of the fourth root is exactly the left-hand side of the equality (3.3). ■

As an immediate corollary of the second Thomae theorem, we obtain the following theorem.

Theorem 3.4 (Riemann–Jacobi formula). *Fix g different positive integers $\{i_1,\ldots,i_g\}=:\mathcal{I}_0$ of the set $\mathcal{G}$, and let $\mathcal{J}_0=\{j_1,\ldots,j_{g+1}\}$ be the complementary set. Define the g sets $\mathcal{S}_k=\mathcal{I}_0\backslash\{i_k\}$ and use the correspondence $[\delta_k]\Leftrightarrow\{\mathcal{S}_k\}$,*

$k=1, \ldots, g$, for non-singular odd characteristics. Similarly, define $g+2$ sets $\mathcal{T}_l$, where $\mathcal{T}_0=\mathcal{I}_0$ and $\mathcal{T}_l=\mathcal{J}_0\backslash\{j_l\}$, $l=1, \ldots, g+1$, and use the correspondence $[\varepsilon_l] \Leftrightarrow \{\mathcal{T}_l\}$ with the $g+2$ non-singular even characteristics. Then the following formula is valid:

$$\det D[\delta_1, \ldots, \delta_g] = \pm\pi^g\theta[\varepsilon_0]\theta[\varepsilon_1]\cdots\theta[\varepsilon_{g+1}]. \tag{3.9}$$

Proof. Compute the determinant of both sides of the matrix equality (3.6)

$$\det D[\delta_1, \ldots, \delta_g] = \epsilon(\det 2\omega)^{(g+2)/2}\pi^{-g^2/2}\det M \det S.$$

One can see that the product

$$\det M^4 \det S^4 = \prod_{l=0}^{g+1} \varDelta(\mathcal{T}_l), \tag{3.10}$$

where the partitions $\mathcal{T}_l$ are given in the formulation of the theorem. The final formula follows immediately after expressing each $\varDelta(\mathcal{T}_l)$ in terms of θ-constants $\theta\{\mathcal{T}_l\}$ by formula (3.1). Our analysis enables us to give the exact value of ϵ from which it follows that the only remaining ambiguity in (3.9) is the $\pm$ sign, corresponding to the antisymmetry of the determinant. ■

Formula (3.9) was called generalized Riemann–Jacobi formula by Fay (1979). Its general theory, including non-hyperelliptic curves, was developed in the series of works by Igusa (1979, 1980, 1982). In the elliptic case $g=1$, it reduces to (1.14).

By inverting equation (3.8), we obtain the periods of the first kind and their inverse matrix ρ, i.e. the normalizing constants for the holomorphic differentials (see (2.7)) in terms of θ-constants,

$$2\omega = \epsilon(S^t)^{-1}N^{-1}D^t[\delta_1, \ldots, \delta_g], \quad \rho := (2\omega)^{-1} = \epsilon D[\delta_1, \ldots, \delta_g]^{-1}NS^t. \tag{3.11}$$

4. Derivation of the periods of the second kind

Periods of the second kind can be obtained from the Picard–Fuchs equations for the derivatives with respect to the branch points e_i of the two sets of periods ω and η. Bolza (1899) described how to derive them by a variation procedure which goes back to Riemann, Thomae and Fuchs; it was generalized in terms of Rauch's formula (e.g. Rauch 1959; Fay 1992), which in the case of the hyperelliptic curve (2.1) reads

$$\frac{\partial}{\partial e_k}\mathrm{d}\boldsymbol{v}(x, y) = -2\,\mathrm{Res}|_{z=e_k}\mathrm{d}\omega_{\mathrm{norm}}(z, w; x, y)\mathrm{d}\boldsymbol{v}(z, w), \tag{4.1}$$

where $\mathrm{d}\boldsymbol{v}(x, y)$ is the vector of normalized holomorphic differentials and $\mathrm{d}\omega_{\mathrm{norm}}(z, w; x, y)$ is the normalized Kleinian bi-differential

$$\begin{aligned} &\mathrm{d}\omega_{\mathrm{norm}}(z, w; x, y) = \mathrm{d}\omega(z, w; x, y) + \mathrm{d}\boldsymbol{u}^t(z, w)\varkappa\,\mathrm{d}\boldsymbol{u}(x, y), \\ &\varkappa^t = \varkappa = \eta(2\omega)^{-1}, \quad \oint_{\mathfrak{a}_l}\mathrm{d}\omega_{\mathrm{norm}}(z, w; x, y) = 0, \quad l = 1, \ldots, g. \end{aligned} \tag{4.2}$$

In the proof below we use the same set of ideas using the Klein bi-differential (2.10) because our aim is to derive differential equations, with respect to the branch points, in the space of periods ω, ω', η and η' of the non-normalized differentials (2.5) and (2.6).

Theorem 4.1. *For an arbitrary branch point e_l, the following equations are valid*:

$$\frac{\partial}{\partial e_l}\begin{pmatrix}\omega & \omega' \\ \eta & \eta'\end{pmatrix} = \begin{pmatrix}\alpha_l & \beta_l \\ \gamma_l & -\alpha_l^t\end{pmatrix}\begin{pmatrix}\omega & \omega' \\ \eta & \eta'\end{pmatrix}, \tag{4.3}$$

where

$$\alpha_l = -\frac{1}{2}\left\{\frac{1}{R'(e_l)}\mathcal{U}(e_l)\mathcal{R}^t(e_l) - M_l\right\}, \tag{4.4}$$

$$\beta_l = -2\left\{\frac{1}{R'(e_l)}\mathcal{U}(e_l)\mathcal{U}^t(e_l)\right\}, \tag{4.5}$$

$$\gamma_l = \frac{1}{8}\left\{\frac{1}{R'(e_l)}\mathcal{R}(e_l)\mathcal{R}^t(e_l) - N_l\right\}, \tag{4.6}$$

with

$$M_l = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ e_l & 1 & 0 & \dots & 0 & 0 \\ e_l^2 & e_l & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ e_l^{g-2} & e_l^{g-3} & \dots & e_l & 1 & 0 \end{pmatrix} \tag{4.7}$$

and

$$N_l = e_l\left(M_l Q_l + Q_l M_l^t\right) + Q_l, \quad Q_l = \operatorname{diag}\left(\dots, \frac{\mathcal{R}_k(e_l)}{\mathcal{U}_{k+1}(e_l)}, \dots\right). \tag{4.8}$$

Proof. First, we consider the equation for $\partial\omega/\partial e_l$. It is obtained by integrating the following equivalence over cycles $\mathfrak{a}_l$:

$$R'(e_i)\frac{\partial}{\partial e_i}\frac{\mathcal{U}_m(x)}{y} = \frac{\mathcal{U}_m(e_i)}{2y}\{\mathcal{U}^t(e_i)\mathcal{R}(x) - \mathcal{R}^t(e_i)\mathcal{U}(x)\}$$
$$-\mathcal{U}_m(e_i)\frac{\partial}{\partial x}\frac{y}{x-e_i} + \frac{R'(e_i)}{2y}\sum_{j=1}^{m-1}\mathcal{U}_{m-j}(x)\mathcal{U}_j(e_i). \tag{4.9}$$

To prove this, substitute $(z,w)=(e_i,0)$ in equation (2.12), which leads to the relation

$$-\frac{\partial}{\partial x}\frac{y}{(x-e_i)} + \frac{\mathcal{U}^t(e_i)\mathcal{R}(x)}{2y} = \frac{F(x,e_i)}{2(x-e_i)^2 y}; \tag{4.10}$$

insert

$$F(x,e_i) = (x-e_i)R'(e_i) + (x-e_i)^2\mathcal{U}^t(x)\mathcal{R}(e_i), \tag{4.11}$$

from equation (2.11) to obtain the equality

$$-\frac{\partial}{\partial x}\frac{y}{(x-e_i)} + \frac{\mathcal{U}^t(e_i)\mathcal{R}(x)}{2y} - \frac{\mathcal{R}^t(e_i)\mathcal{U}(x)}{2y} = \frac{R'(e_i)}{2(x-e_i)y} \equiv R'(e_i)\frac{\partial}{\partial e_i}\frac{1}{y}, \tag{4.12}$$

which is (4.9) for $m=1$. The validity of (4.9) for $m=2,\dots$ follows from the equivalence

$$\frac{\partial}{\partial e_i}\frac{x^{m-1}}{y} = e_i^{m-1}\frac{\partial}{\partial e_i}\frac{1}{y} + \frac{1}{2}\sum_{j=1}^{m-1} e_i^{j-1}\frac{x^{m-j-1}}{y}, \tag{4.13}$$

which can be proved inductively.

Compute now the periods $2\omega_{m,l} = \oint_{\mathfrak{a}_l}(\mathcal{U}_m(x)/y)\mathrm{d}x$ from (4.9),

$$\frac{\partial \omega_{m,l}}{\partial e_i} = -\frac{2\mathcal{U}_m(e_i)}{R'(e_i)}\sum_{k=1}^{g}\left\{\mathcal{U}_k(e_i)\eta_{k,l} + \frac{1}{4}\mathcal{R}_k(e_i)\omega_{k,l}\right\} + \frac{1}{2}\sum_{j=1}^{m-1}\omega_{m-j,l}\mathcal{U}_j(e_i). \tag{4.14}$$

The upper left block of (4.3) is nothing but this formula written in matrix form.

Next, we derive the equation for $\partial\eta/\partial e_l$ in an analogous way, using the equivalence which may be checked by direct computing,

$$\begin{aligned} R'(e_i)\frac{\partial}{\partial e_i}\frac{\mathcal{R}_m(x)}{y} &= \frac{\mathcal{R}_m(e_i)}{2y}\{\mathcal{U}^T(e_i)\mathcal{R}(x) - \mathcal{R}^T(e_i)\mathcal{U}(x)\} \\ &\quad -\mathcal{R}_m(e_i)\frac{\partial}{\partial x}\frac{y}{x-e_i} - \frac{R'(e_i)}{2y}\left(\frac{\mathcal{R}_m(x)-\mathcal{R}_m(e_i)}{x-e_i} + 2\frac{\partial}{\partial e_i}\mathcal{R}_m(x)\right). \end{aligned} \tag{4.15}$$

The expression in the bracket of the last term can be written as

$$\frac{-1}{\mathcal{U}_{m+1}(e_i)}\left(\sum_{k=1}^{m}\mathcal{U}_{m-k+1}(x)\mathcal{U}_k(e_i) + \sum_{k=m+1}^{g}(\mathcal{U}_k(x)\mathcal{R}_k(e_i) - \mathcal{R}_k(x)\mathcal{U}_k(e_i))\right).$$

The periods $2\eta_{m,l} = -\oint_{\mathfrak{a}_l}(\mathcal{R}_m(x)/y)\mathrm{d}x$ are obtained by integration,

$$\begin{aligned} \frac{\partial \eta_{m,l}}{\partial e_i} &= \frac{\mathcal{R}_m(e_i)}{2R'(e_i)}\sum_{k=1}^{g}\left(\mathcal{U}_k(e_i)\eta_{k,l} + \frac{1}{4}\mathcal{R}_k(e_i)\omega_{k,l}\right) - \frac{1}{2}\sum_{k=m+1}^{g} U_{k-m}(e_i)\eta_{k,l} \\ &\quad -\frac{1}{8}\left(\sum_{k=m+1}^{g}\omega_{k,l}\frac{\mathcal{R}_k(e_i)}{\mathcal{U}_m(e_i)} + \sum_{k=1}^{m}\omega_{m-k+1,l}\mathcal{U}_{k-m}(e_i)\right). \end{aligned} \tag{4.16}$$

Written in matrix form, this is the lower left block of equation (4.3). The equations for the derivatives of ω' and η' are obtained in the same way. ■

Corollary 4.1. *The following variation formula is valid*:

$$\frac{\partial \tau}{\partial e_l} = \frac{\mathrm{i}\pi}{2}\omega^{-1}\beta_l(\omega^t)^{-1}, \quad e = 1,\dots,2g+1, \tag{4.17}$$

where β_l is given in equation (4.5).

This is a consequence of equations (4.3) and (4.4) for $\tau = \omega^{-1}\omega'$; it was derived by Thomae (1870) and has been proved again in many places. For the case of genus 1, the explicit formula is given in (1.11).

We are now in a position to give expressions for the second-kind periods in terms of θ-constants.

Theorem 4.2. *Choose any g different positive integers* $\{i_1, \ldots, i_g\} =: \mathcal{I}_0$ *from the set* $\mathcal{G}$, *and let* $e_{i_1}, \ldots, e_{i_g}$ *be the corresponding branch points. Then the period matrix* η *is given as*

$$B(\mathcal{I}_0)\eta = \sum_{i_l \in \mathcal{I}_0} \frac{\partial \omega}{\partial e_{i_l}} - A(\mathcal{I}_0)\omega, \tag{4.18}$$

where

$$B(\mathcal{I}_0) = \sum_{i_l \in \mathcal{I}_0} \beta(e_{i_l}), \quad A(\mathcal{I}_0) = \sum_{i_l \in \mathcal{I}_0} \alpha(e_{i_l}),$$

and the matrix $B(\mathcal{I}_0)$ *is invertible.*

Proof. Equation (4.18) follows from (4.3), and it is straightforward to check that

$$\det B(\mathcal{I}_0) = (-2)^g \frac{\prod_{i_l < i_k \in \mathcal{I}_0} (e_{i_l} - e_{i_k})^2}{\prod_{i_n \in \mathcal{I}_0} R'(e_{i_n})} \neq 0.$$

■

The derivative $\partial\omega/\partial e_l$ can be calculated with the help of formula (4.17), (3.11) and the heat equation (2.18).

Formula (4.18) can be applied as follows. Let $\lambda_{2g} = \sum_k e_k = 0$. Define the matrices

$$C(\mathcal{I}_0) = \sum_{i_k \in \mathcal{I}_0} e_{i_k} A(\mathcal{I}_0)^{-1} B(\mathcal{I}_0), \quad D(\mathcal{I}_0) = \sum_{i_k \in \mathcal{I}_0} e_{i_k} A(\mathcal{I}_0)^{-1}. \tag{4.19}$$

Then by taking a suitable sum of the $\binom{2g+1}{g}$ equations (4.18), we obtain

$$\eta = \left(\sum_{\mathcal{I}_0} C(\mathcal{I}_0) \right)^{-1} \sum_{\mathcal{I}_0} D(\mathcal{I}_0) \sum_{i_k \in \mathcal{I}_0} \frac{\partial \omega}{\partial e_{i_k}}, \tag{4.20}$$

where the summation is over all subsets $\mathcal{I}_0$ of the set $\mathcal{G}$ of indices. For genus 1, this formula reduces to (1.16).

5. Examples

(a) *Example g=1*

The case of genus 1 was covered in §1.

(b) *Example g=2*

Consider the hyperelliptic curve V of genus 2,

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)(x - e_4)(x - e_5). \tag{5.1}$$

The homology basis of the curve is fixed by defining the set of half-periods corresponding to the branch points, following the recipe of §5*c*:

$$[\mathfrak{A}_1] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad [\mathfrak{A}_2] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad [\mathfrak{A}_3] = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$[\mathfrak{A}_4] = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_5] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad [\mathfrak{A}_6] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The characteristics of the vector of Riemann constants $\boldsymbol{K}_\infty$ is given as

$$[\boldsymbol{K}_\infty] = [\mathfrak{A}_2] + [\mathfrak{A}_4] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The correspondence between characteristics $\{^*\}$, $\{^{*,*}\}$ and $\begin{bmatrix} * & * \\ * & * \end{bmatrix}_2$ is given as

$$\{1\} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}_2, \quad \{2\} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}_2, \quad \{3\} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}_2,$$

$$\{4\} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}_2, \quad \{5\} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}_2, \quad \{6\} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_2,$$

and

$$\{1{,}2\} = \begin{bmatrix} 1\,1 \\ 1\,1 \end{bmatrix}_2, \{1{,}3\} = \begin{bmatrix} 0\,0 \\ 1\,1 \end{bmatrix}_2, \{1{,}4\} = \begin{bmatrix} 0\,0 \\ 1\,0 \end{bmatrix}_2, \{1{,}5\} = \begin{bmatrix} 0\,1 \\ 1\,0 \end{bmatrix}_2, \{2{,}3\} = \begin{bmatrix} 0\,0 \\ 0\,1 \end{bmatrix}_2$$

$$\{2{,}4\} = \begin{bmatrix} 0\,0 \\ 0\,0 \end{bmatrix}_2, \{2{,}5\} = \begin{bmatrix} 0\,1 \\ 0\,0 \end{bmatrix}_2, \{3{,}4\} = \begin{bmatrix} 1\,1 \\ 0\,0 \end{bmatrix}_2, \{3{,}5\} = \begin{bmatrix} 1\,0 \\ 0\,0 \end{bmatrix}_2, \{4{,}5\} = \begin{bmatrix} 1\,0 \\ 0\,1 \end{bmatrix}_2.$$

For any $k \neq l \neq p, q, r$ from the set $\{1, \ldots, 5\}$ (there are 10 different possibilities), the following representation is valid (see theorem 3.3):

$$2\omega = \frac{T_{pqr}}{\theta\{k,l\}(e_k - e_l)^{3/2}} \begin{pmatrix} 1 & -1 \\ -e_l & e_k \end{pmatrix} \begin{pmatrix} \dfrac{1}{T_l} & 0 \\ 0 & \dfrac{1}{T_k} \end{pmatrix} \begin{pmatrix} \theta_1\{l\} & \theta_2\{l\} \\ \theta_1\{k\} & \theta_2\{k\} \end{pmatrix}, \tag{5.2}$$

where

$$T_k = \theta\{p,k\}\theta\{q,k\}\theta\{r,k\}, \quad T_l = \theta\{p,l\}\theta\{q,l\}\theta\{r,l\},$$
$$T_{pqr} = \theta\{p,q\}\theta\{p,r\}\theta\{q,r\}. \tag{5.3}$$

Note that the modular form of the weight 5, χ_5, is given in this notation as

$$\chi_5 = \prod_{\text{even}[\boldsymbol{\varepsilon}]} \theta[\boldsymbol{\varepsilon}] = T_k T_l T_{pqr} \theta\{k,l\},$$

with $k \neq l \neq p \neq q \neq r \in \{1, 2, 3, 4, 5\}$.

Choosing $k=2$ and $l=4$, and normalizing the curve with $e_2=0$ and $e_4=1$, we obtain the explicit result

$$2\omega = \frac{T_{135}^2}{\chi_5} \begin{pmatrix} T_{135}\theta_1 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_2 & T_{135}\theta_2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}_2 \\ T_4\theta_1 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}_2 & T_4\theta_2 \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}_2 \end{pmatrix}, \tag{5.4}$$

where

$$T_2 = \theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}_2 \theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}_2 \theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}_2,$$
$$T_4 = \theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}_2 \theta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}_2 \theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}_2, \tag{5.5}$$
$$T_{135} = \theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}_2 \theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_2 \theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_2.$$

In the derivation of this result, we used the equality (i=1, 2)

$$\theta_i\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\theta\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \theta_i\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}\theta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\theta\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$
$$= \theta_i\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\theta\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\theta\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},$$

equivalently,

$$\theta_i\{2\}T_4 - \theta_i\{4\}T_2 = \theta_i\{6\}T_{135},$$

which can be derived from addition theorems as given (e.g. Baker 1897, p. 342); a complete set of such relations can be found in Forsyth (1882).

The entries to the inverse matrix $\rho = 2\omega^{-1}$ are the normalizing constants of the holomorphic differentials. Its columns represent the so-called winding vectors in the Its–Matveev formula (1975) for the genus-2 solution of the KdV equation. For the case of g=2, the general formula (3.11) for ρ reduces to

$$\rho = (2\omega)^{-1} = \frac{\sqrt{e_k - e_l}}{\pi^2 T_{pqr}^2}\begin{pmatrix}\theta_2\{k\} & -\theta_2\{l\}\\ -\theta_1\{k\} & \theta_1\{l\}\end{pmatrix}\begin{pmatrix}T_l & 0\\ 0 & T_k\end{pmatrix}\begin{pmatrix}e_k & 1\\ e_l & 1\end{pmatrix}. \tag{5.6}$$

With the normalization e_2=0 and e_4=1, this coincides with the formulae given in the Rosenhain memoir of Rosenhain (1851, p. 75),

$$\rho = \frac{1}{\pi^2 T_{135}^2}\begin{pmatrix} -T_4\theta_2\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}_2 & T_{135}\theta_2\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}_2 \\ T_4\theta_1\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}_2 & -T_{135}\theta_1\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}_2\end{pmatrix}. \tag{5.7}$$

The set of periods 2η can be computed with the help of (4.18).

The special case g=2 of the Riemann–Jacobi formula (3.9) was discovered by Rosenhain (1851).

Theorem 5.1 (Rosenhain). *Let* $[\delta_1], \ldots, [\delta_6]$ *be the six odd characteristics and* $[\delta_1]$ *and* $[\delta_2]$ *any two of them. With the remaining four* $[\delta_{i+2}]$, i=1, …, 4, *and* $[\boldsymbol{\varepsilon}_i] = [\delta_1] + [\delta_2] + [\delta_{i+2}](\mathrm{mod})1$, *there exist 15 relations*

$$D[\delta_1, \delta_2] = \theta_1[\delta_1]\theta_2[\delta_2] - \theta_2[\delta_1]\theta_1[\delta_2] = \pm\pi^2\theta[\varepsilon_1]\theta[\varepsilon_2]\theta[\varepsilon_3]\theta[\varepsilon_4]. \tag{5.8}$$

(c) Example g=3

Consider briefly the hyperelliptic curve V of genus 3,

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3)(x-e_4)(x-e_5)(x-e_6)(x-e_7). \tag{5.9}$$

The homology basis of the curve is again fixed by defining the set of half-periods corresponding to the branch points

$$[\mathfrak{A}_1] = \frac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \quad [\mathfrak{A}_2] = \frac{1}{2}\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}, \quad [\mathfrak{A}_3] = \frac{1}{2}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix},$$

$$[\mathfrak{A}_4] = \frac{1}{2}\begin{bmatrix}0 & 1 & 0\\ 1 & 1 & 0\end{bmatrix}, \quad [\mathfrak{A}_5] = \frac{1}{2}\begin{bmatrix}0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}, \quad [\mathfrak{A}_6] = \frac{1}{2}\begin{bmatrix}0 & 0 & 1\\ 1 & 1 & 1\end{bmatrix},$$

$$[\mathfrak{A}_7] = \frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 1 & 1 & 1\end{bmatrix}, \quad [\mathfrak{A}_8] = \frac{1}{2}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

For any choice of three different numbers k, l and n from the set $\{1, \ldots, 7\}$ (there are 35 different possibilities), the following representation is valid (see theorem 3.3):

$$S = \begin{pmatrix} e_l e_n & e_k e_n & e_k e_l \\ e_l + e_n & e_k + e_n & e_k + e_l \\ 1 & 1 & 1 \end{pmatrix}, \tag{5.10}$$

$$\begin{aligned} N = \theta\{k,l,n\} \times \mathrm{diag}\Bigg(& \sqrt{e_k - e_n}\,\frac{\theta\{p,l,n\}\theta\{q,l,n\}\theta\{l,r,s\}}{\theta\{p,q,n\}\theta\{p,r,s\}\theta\{q,r,s\}}, \\ & \sqrt{e_l - e_n}\,\frac{\theta\{p,k,n\}\theta\{q,k,n\}\theta\{n,r,s\}}{\theta\{p,q,k\}\theta\{p,r,s\}\theta\{q,r,s\}}, \\ & \sqrt{e_n - e_k}\,\frac{\theta\{p,k,l\}\theta\{q,k,l\}\theta\{k,r,s\}}{\theta\{p,q,l\}\theta\{p,r,s\}\theta\{q,r,s\}}\Bigg). \end{aligned} \tag{5.11}$$

It is always possible to normalize the curve with $e_k=0$ and $e_l=1$; then according to the corollary 3.1,

$$e_n = \frac{\theta^2\{l,p,q\}\theta^2\{l,r,s\}}{\theta^2\{n,p,q\}\theta^2\{n,r,s\}},$$

and therefore (5.11) is expressible entirely in terms of θ-constants. Using this in the formulae (3.11) and (4.18), we arrive at the final expression for periods in terms of θ-constants.

6. Conclusion

For the class of hyperelliptic curves, we have recalled the lines of thought that allow one to express the periods of integrals in terms of θ-constants. The main ingredients were Thomae's two formulae and Picard–Fuchs equations. It should be clear how to derive explicit results for curves of any genus.

These results can be generalized in various directions. First, we believe that the Rosenhain formula (5.7) for periods of the first kind can be given in the same form for higher genera where the entries to the matrix are proportional to minors of the Jacobian $D[\delta_1, \ldots, \delta_g]$ (see (3.5)). Second, it should be possible to simplify the expression (4.18) to make it more similar to the Weierstrass formula (1.16).

Third, the formulae (5.6) obtained for the winding vectors could be useful to prove the equivalence of the classical Schottky conditions, in the spirit of the derivation given by Farkas (1971), with Dubrovin's relations involving winding vectors and θ-derivatives which he derived in the framework of KdV theory (Dubrovin 1981).

Moreover, since the first Thomae formula has shown to hold also for certain non-hyperelliptic curves (Bershadsky & Radul 1987; Nakayashiki 1997; Enolskii & Grava 2006), there is hope that some of the material developed here can be generalized to those cases.

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