On Z-Isograded Isovector Space and Z-Isograded Isotensor Product

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Abstract:- In this paper $\hat{V}$ is a isovector space over isofield $\hat{K}$. The aim of the present paper is to study Z-isograded isovector space. Moreover we defined Z-graded isotensor product and study some properties of them.

Keywords: Isotopies, Isofield, Isovector space, Z-isograded isovector space, Isotensor product.

1 Introduction

Let $K(\alpha, +, \cdot)$ be a field and $\hat{I}$ be an object of a multiplication set (sometimes it is not in K). Then the set $\hat{K}(\hat{\alpha}, \hat{+}, \hat{\cdot}) = \{\alpha\hat{I}; \alpha \in K\}$ with the multiplication:

$$\begin{array}{rcl} \hat{\cdot} : \hat{K} \times \hat{K} & \to & \hat{K} \\ (\alpha\hat{I}, \beta\hat{I}) & \to & (\alpha \cdot \beta)\hat{I} \end{array}$$

and addition:

$$\begin{array}{rcl} \hat{+} : \hat{K} \times \hat{K} & \to & \hat{K} \\ (\alpha\hat{I}, \beta\hat{I}) & \to & (\alpha + \beta)\hat{I} \end{array}$$

is a field. This field is called an isofield and $\hat{\cdot}$ and $\hat{+}$ are called isomultiplication and addition respectively. By the above method we lift the unit I of K to the isounit $\hat{I}$.

In section one our study about Z-isograded isovector space, Z-isograded algebra and isograded isomodule over a isoalgebra and in section two, we defined Z-isograded isotensor product by two away and proved this two define together equality.

2 Spaces Z-isograded (Z_n-isograded)

$\hat{V}$ is an isovector space on the isofield $\hat{K}$ where $\hat{K}$ is $\hat{\mathbf{R}}$, $\hat{C}$ or any isofield with characteristic 2.

Definition 2.1. $(\hat{V}, W)$ is called an Z-isograded (Z_n-isograded) isovector space if $\hat{V} = \bigoplus_{i=1}^{\infty} \hat{V}_i$ where $\hat{V}_i$ is an isovector space on the isofield $\hat{K}_i$, which is isomorphic with a subspace of a vector space W.

The next example show that $\hat{W}$ is not isovector space isomorphic to $\hat{V}$.

Example 2.1. Let V be the vector space of polynomials with coeficient in the field $\mathbf{R}$. Let $S_i \in \mathbf{R}^+$, and let $\hat{\mathbf{R}}_i$ be the isofield with isounit $\hat{I}_i = \dfrac{(i-1)!}{S^i}$ where $i = 1, 2, \ldots$. Then $\hat{\mathbf{R}}_i$ is the isovector space on $\hat{\mathbf{R}}_i$. Now let $\hat{V} = \bigoplus_{i=1}^{\infty} \hat{R}_i$ where directsum is symbolic sum. The $(\hat{V}, \mathbf{R})$ is an Z-isograded isovector space.

Theorem 2.1. Let V be a Z-graded (Z_n-graded) vector space on the field K (K is $\mathbf{R}$, $\mathbf{C}$ or any field of characteristic 2) and let $\hat{K}$ be a isofield with isotopy $f : K \to \hat{K}$. Then there exists a Z-isograded (Z_n-isograded) isovector space $(\hat{V}, V)$ on the isofield $\hat{K}$ and an one-to-one and onto isotopy $\psi_V : V \to \hat{V}$ such that ψ_V preserves direct sum and isoscaler product.

Proof. Put $\hat{V} = \{\hat{v} = \hat{1} \cdot v | v \in V\}$ with scaler product

$$\begin{array}{rcl} \hat{K} \times \hat{V} & \to & \hat{V} \\ (\hat{k}, \hat{v}) & \to & \hat{k}\hat{\cdot}\hat{v} = k \hat{\cdot} v \end{array}$$

and addition

$$\begin{array}{rcl}\hat{V}\times\hat{V} & \rightarrow & \hat{V} \\ (\hat{u}_1,\hat{v}_2) & \rightarrow & \hat{v}_1+\hat{v}_2=\hat{1}\cdot(v_1+v_2)\end{array}.$$

If

$$\begin{array}{rcl}\psi_V:V & \rightarrow & \hat{V} \\ v & \rightarrow & \hat{1}\cdot v=\hat{v}\end{array}$$

then ψ_V preserves isoscaler product and directsum.□

Example 2.2. Let $V=\{\phi:\mathbf{R}\rightarrow\mathbf{R}|\phi$ is a solution of differential equation $y^4+y=0\}$. Moreover let $0\neq S\in\mathbf{R}$ be given. Assume that $\hat{V}_S=\langle S\rangle\oplus\langle 1\rangle$. Then $\hat{V}_S$ is a Z_2-isograded isovector space on the isofield $\hat{\mathbf{R}}$ with isounit $\hat{I}=\dfrac{1}{S^2+1}$ and isoscaler product $\hat{\cdot}=\cdot$. The corresponding isotopy $\hat{I}_S:V\rightarrow\hat{V}$ is defined by $\hat{I}_S(C_1\phi_1+C_2\phi_2)=\hat{C}_1S+\hat{C}_21$, where $\phi_1(x)=\sin x$ and $\phi_2(x)=\cos x$. This isotopy is deduced by laplac transform.

Definition 2.2. An element of $\hat{V}_i$ is called a isohomogenious element of degree i.

Definition 2.3. A Z-isograded (Z_n-isograded) isovector space $(\hat{A},B)$ on the isofield $\hat{K}$ called a Z-isograded (Z_n-isograded) isoalgebra if:

i) B is a Z-graded (Z_n-graded) algebra

ii) B_i is isomorphic with $\hat{A}_i$ where

$$\hat{A}=\bigoplus_{i=1}^{\infty}\hat{A}_i \quad , \quad B=\bigoplus_{i=1}^{\infty}B_i$$

iii) $\hat{A}_i\hat{\cdot}\hat{A}_j\subseteq\hat{A}_{i+j}$ ($\hat{A}_i\hat{\cdot}\hat{A}_j\subseteq\hat{A}_{i+j}(modn)$).

Example 2.3. Let $V=\bigoplus_{i=1}^{\infty}C_i[x]$ which $C_i[x]=\{rx^i|r\in\mathbf{R}\}$, V is Z-graded vector space and let be $\hat{V}=\hat{C}_i[x]$ which $\hat{C}_i[x]=\{\hat{r}x^i|\hat{r}\in\hat{R}\}$ then $(\hat{V},V)$ is Z-isograded isoalgebra.

Example 2.4. Let $\hat{K}$ be an isofield and $\hat{K}[x]$ be the space of isopolynomials. Then $\hat{K}[x]=(\hat{K}[x])_0\oplus(\hat{K}[x])_1$, where $(\hat{K}[x])_0$ is the isosubspace of polynomials of even degree and $(\hat{K}[x])_1$ is the isosubspace of polynomial of odd degree. Then $(\hat{K}[x],K[x])$ is a Z_2-isograded isoalgebra.

Theorem 2.2. Let B a Z-graded (Z_n-graded) algebra on the field K and let $\hat{K}$ be an isofield of K with isotopy $f:K\rightarrow\hat{K}$. Then there exists an Z-isograded (Z_n-isograded) isoalgebra $(\hat{B},B)$ and an isotopy B to $\hat{B}$.

Proof. We state the proof for the case Z-graded. The case Z_n-graded is similar. Let $B=\bigoplus_{i=1}^{\infty}B_i$. Moreover let $\hat{B}_i=\{bf(1)|b\in B_i\}$ where $bf(1)$ is only a symbol correspond to $b\in B$. $\hat{B}_i$ with the scaler product

$$\begin{array}{rcl}\hat{K}\times\hat{B}_i & \rightarrow & \hat{B}_i \\ (\hat{k},bf(1)) & \rightarrow & (k\cdot b)f(1)\end{array}$$

and sum:

$$\begin{array}{rcl}\hat{B}_i\times\hat{B}_i & \rightarrow & \hat{B}_i \\ (bf(1),cf(1)) & \rightarrow & (b+c)f(1)\end{array}$$

and multiplication

$$\begin{array}{rcl}\hat{B}_i\times\hat{B}_i & \rightarrow & \hat{B}_i \\ (bf(1),cf(1)) & \rightarrow & (b\cdot c)f(1)\end{array}$$

is an isoalgebra.

Now put $\hat{B}=\bigoplus_{i=0}^{\infty}\hat{B}_i$ then $(\hat{B},B)$ is an Z-isograded isoalgebra. The corresponding is $\psi:B\longrightarrow\hat{B}$.□

Remark 2.1. Since in the proof of Theorem 2.2, Z-isograde (Z_n-isograded) $\hat{B}$ has related to f, So by changing f we can construct different Z-isograded (Z_n-isograded) isoalgebra.

Theorem 2.3. Let $(\hat{A},B)$ be a Z-isograded (Z_n-isograded) isoalgebra and let $\hat{C}\subset\hat{A}$. Moreover let $\hat{C}=\bigoplus_{i=0}^{\infty}\hat{C}_i$ ($\hat{C}=\bigoplus_{i=0}^{n-1}\hat{C}_i$), where $\hat{C}_i$ is a isoalgebra on isofield $\hat{K}$ and let there exists an one-to-one and onto isotopy from $\hat{C}_i$ to $B_i\cap D$ where D is a subalgebra of B, for $i=0,1,2,\ldots$ (for $i=0,1,2,\ldots,n-1$) then $(\hat{C},D)$ is a Z-isograded (Z_n-isograded) isoalgebra.

Proof. We state the proof only for the case Z-isograded. The other cases is similar.

Let $i\in\{0,1,2,\ldots\}$ be given. Then $B_i\cap D$ is an algebra on K. For $i,j\in\{0,1,2,\ldots\}$ and $x\in(B_i\cap D)$, $y\in B_j\cap D$ we have $xy\in B_iB_j\subset B_{i+j}$ and $xy\in D$ so $xy\in(B_{i+j}\cap D$. Hence $(B_i\cap D)(B_j\cap D)\subseteq B_{i+j}\cap D$. Thus $\hat{C}$ is a Z-isograded isoalgebra.

Definition 2.4. $(\hat{C},D)$ stated in Theorem 2.3, is called a Z-isograded subisoalgebra of $(\hat{A},B)$.

Remark 2.2. There exists an one-to-one correspondence between the set of subalgebras of B and the set of Z-isograded subisoalgebras of $(\hat{A},B)$ up to one-to-one and onto isotopic.

Example 2.5. Let $(\hat{V},V)$ be Z-isograded isoalgebra Example 2.3, let $D=C_{2i}[x]$ and $\hat{C}=\hat{C}_{2i}[x]$ then $(\hat{C},D)$ is a Z-isograded isosubalgebra of $(\hat{V},V)$.

Theorem 2.4. Let $(\hat{A},B)$ be a Z-isograded isoalgebra, where B is an algebra on K. Define on $\hat{A}$ multiplication

$$\begin{array}{rcl} \hat{*} : \hat{A}\times\hat{A} & \to & \hat{A} \\ (\sum_{i=0}^{n-1}\hat{a}_i, \sum_{j=0}^{n-1}\hat{b}_j) & \longmapsto & \sum_{i=0}^{2n-1}\hat{C}_i \end{array}$$

where $\hat{C}_i = \sum_{k=0}^{i}\hat{a}_k\hat{\cdot}\hat{b}_{i-k}$ and define addition

$$\begin{array}{rcl} \hat{+} : \hat{A}\times\hat{A} & \to & \hat{A} \\ (\sum_{i=0}^{n-1}\hat{a}_i, \sum_{j=0}^{n-1}\hat{b}_j) & \to & \sum_{i=0}^{n-1}(\widehat{a_i + b_i}) \end{array}$$

and scaler product

$$\begin{array}{rcl} \hat{\cdot} : K\times\hat{A} & \to & \hat{A} \\ (k, \sum_{i=0}^{n-1}\hat{a}_i) & \longmapsto & \sum_{i=0}^{n-1}K\hat{\cdot}a_i \end{array}$$

Then $\hat{A}$ is an algebra on K.

Proof. The proof is straightforward.

Remark 2.3. If in Theorem 2.4, the sum on indices is in (mod n), then Theorem 2.4, is correct for Z_n-isograded isoalgebras.

Notation. The K-algebra $\hat{A}$ introduced in Theorem 2.4 is denoted by $\mathcal{A}$.

Definition 2.5. A Z-isograded (Z_n-isograded) isoalgebra $(\hat{A}, B)$ is called a commutative Z-isograded (Z_n-isograded) isoalgebra if $\hat{a}\hat{*}\hat{b} = (-1)^{i+j}\hat{b}\hat{*}\hat{a}$ where $\hat{a}\in\hat{A}_i$ and $\hat{b}\in\hat{A}_j$.

Remark 2.4. Let $(\hat{A}, B)$ be a Z-isograded isovector space, then $\hat{A}$ with $\hat{+}, \hat{\cdot}$ are define Theorem 2.4 is a vector space on K. We denote it by $\mathcal{A}$.

Definition 2.6. Let $(\hat{A}, B)$ and $(\hat{C}, D)$ be two Z-isograded (Z_n-isograded) isovector space on the field K. Then the linear transformation $T : \mathcal{A} \longrightarrow \mathcal{C}$ is called homogeneous of degree s if $T(\hat{a}_i) = \hat{C}_{i+s}(T(\hat{a}_i) = \hat{C}_{i+s}(mod\ n))$, where $\hat{a}_i \in \hat{A}_i$.

Example 2.7. Let $(\bigoplus_{i=0}^{\infty}\hat{C}_{2i}[x], \bigoplus_{i=0}^{\infty}\hat{C}_{2i}[x])$ and $(\bigoplus_{i=0}^{\infty}\hat{C}_{2i+1}[x], \bigoplus_{i=0}^{\infty}\hat{C}_{2i+1}[x])$ be two Z-isograded isovector space on the field K. Then $T : \bigoplus_{i=0}^{\infty}\hat{C}_{2i}[x] \to \bigoplus_{i=0}^{\infty}\hat{C}_{2i+1}[x]$ with the relation $T(\{\hat{r}x^{2i}\}) = \hat{r}x^{2i+1}$ is a linear transformation homogeneous of degree 1.

Theorem 2.6. Let $A = \bigoplus_{i=0}^{\infty}A_i$ be a $\mathcal{A}$-algebra and $V = \bigoplus_{i=0}^{\infty}V_i$ is a graded vector space and moreover V_i is A_i-module and suppose $\hat{a}\hat{\cdot}\hat{v} = \hat{d}_0 + \ldots + \hat{d}_n$ where $d_i = \sum_{k=0}^{i}a_k v_{i-k}$ then $\hat{A}\hat{\cdot}\hat{V} \subseteq \hat{V}\&\hat{A}_i\hat{\cdot}\hat{V}_j \subseteq \hat{V}_{i+j}$, i.e. $\hat{V}$ is a isograded isomodule over $\hat{A}$.

Proof. Since V is Z-graded vector space on field K, then by Theorem 2.1, $(\hat{V}, V)$ is a Z-isograded isovector space on isofiled $\hat{K}$. Since A is z-graded algebra on field K, thus by Theorem 2.2, $(\hat{A}, A)$ is z-isograded isoalgebra on isofiled $\hat{K}$. Let product " $\cdot$ " be defined as follows:

$$\begin{array}{rcl} \hat{\cdot} : \hat{A}\times\hat{V} & \to & \hat{V} \\ (\hat{a}, \hat{v}) & \longmapsto & \hat{a}\hat{\cdot}\hat{v} = (a\hat{\cdot}v) \end{array}$$

By this product, $\hat{A}\hat{\cdot}\hat{V} \subseteq \hat{V}$. Therefore, the proof is hold.

Definition 2.7. $\hat{V}$ in Theorem 2.6, is called isograded isomodule over $\hat{A}$.

3 Isotensor product of two Z- isograded(Z_n-isograded) isovector spaces (Bilinear forms)

Let $(\hat{V}, W), (\hat{U}, G)$ be two Z-isograded (Z_n-isograded) isovector space and $\hat{V} = \bigoplus_{i=0}^{\infty}\hat{V}_i, \hat{U} = \bigoplus_{i=0}^{\infty}\hat{U}_i$ $(\hat{V} = \bigoplus_{i=0}^{n-1}\hat{V}_i, \hat{U} = \bigoplus_{i=0}^{n-1}\hat{U}_i)$ and let $\psi_i : \hat{V}_i \to W_i \subseteq W$, $\phi_i : \hat{U}_i \to G_i \subseteq G$ be isomorphism. Which W_i and G_i respect are subspaces W and G. Let $T : W\times G \to K$ be bilinear forms. Then we define $\hat{T} : \hat{V}\times\hat{U} \to K$ by following:

Let $\hat{\alpha}\in\hat{V}$, $\hat{\beta}\in\hat{U}$, $\hat{\alpha} = \sum_{i=0}^{d}a_{s_i}$, $\hat{\beta} = \sum_{i=0}^{d}b_{s_i}$

$$\begin{aligned} \hat{T}(\hat{\alpha}, \hat{\beta}) &= \hat{T}(\sum_{i=0}^{d}a_{s_i}, \sum_{i=0}^{d}b_{s_i}) \\ &= T(\sum_{i=0}^{d}\psi_{s_i}(a_{s_i}), \sum_{i=0}^{d}\phi_{s_i}(b_{s_i})) \\ &= \sum_{i=0}^{d}\sum_{i=0}^{d}T(\psi_{s_i}(a_{s_i}), \phi_{s_i}(b_{s_i})) \end{aligned}$$

Theorem 3.1. $\hat{T}$ is bilinear forms.

Proof. Let $\hat{\alpha}\in\hat{V}$, $\hat{\beta}\in\hat{U}$, $r\in k$, $\hat{\alpha} = \sum_{i=0}^{d}a_{s_i}$, $\hat{\beta} = \sum_{i=0}^{d}b_{s_i}$. First we show that $\hat{T}$ have a linear property

$$
\begin{aligned}
\hat{T}(r\cdot\hat{\alpha},\hat{\beta}) &= \hat{T}(\sum_{i=0}^{d} ra_{s_i},\sum_{i=0}^{d} b_{s_i}) \\
&= T(\sum_{i=0}^{d}\psi_{s_i}(ra_{s_i}),\sum_{i=0}^{d}\phi_{s_i}(b_{s_i})) \\
&= T(r\sum_{i=0}^{d}\psi_{s_i}(a_{s_i}),\sum_{i=0}^{d}\phi_{s_i}(b_{s_i})) \\
&= rT(\sum_{i=0}^{d}\psi_{s_i}(a_{s_i}),\sum_{i=0}^{d}\phi_{s_i}(b_{s_i})) \\
&= r\hat{T}(\hat{\alpha},\hat{\beta})
\end{aligned}
$$

Let $\hat{\gamma}=\sum_{i=0}^{d} C_{s_i}$. Now we show that $\hat{T}$ have a additive property.

$$
\begin{aligned}
\hat{T}(\hat{\alpha}+\hat{\gamma},\hat{\beta}) &= \hat{T}(\sum_{i=0}^{d} a_{s_i}+\sum_{i=0}^{d} C_{s_i},\sum_{i=0}^{d} b_{s_i}) \\
&= \hat{T}(\sum_{i=0}^{d}(a_{s_i}+C_{s_i}),\sum_{i=0}^{d} b_{s_i}) \\
&= T(\sum_{i=0}^{d}\psi_{s_i}(a_{s_i}+c_{s_i}),\sum_{i=0}^{d} b_{s_i}) \\
&= T(\sum_{i=0}^{d}\psi_{s_i}(a_{s_i}) \\
&\quad +\sum_{i=0}^{d}\psi_{s_i}(c_{s_i}),\sum_{i=0}^{d}\phi_{s_i}(d_{s_i})) \\
&= T(\sum_{i=0}^{d}\psi_{s_i}(a_{s_i}),\sum_{i=0}^{d}\phi_{s_i}(b_{s_i})) \\
&\quad +T(\sum_{i=0}^{d}\psi_{s_i}(c_{s_i}),\sum_{i=0}^{d}\phi_{s_i}(b_{s_i})) \\
&= \hat{T}(\hat{\alpha},\hat{\beta})+\hat{T}(\hat{\gamma},\hat{\beta})
\end{aligned}
$$

Therefore $\hat{T}$ is bilinear forms.

Definition 3.2. Let $\hat{F}$ be space of all biliner forms on $\hat{V}\times\hat{U}$. Then $(\hat{F})^*$, i.e. the dual space of $\hat{F}$, is called the isotensor product of $\hat{V}$ and $\hat{U}$. We denote this space by $\hat{V}\hat{\otimes}\hat{U}$. Moreover, for all $\hat{\alpha}\in\hat{V}$ and $\hat{\beta}\in\hat{U}$, the isotensor product $\hat{t}=\hat{\alpha}\hat{\otimes}\hat{\beta}$ is the element of $\hat{V}\hat{\otimes}\hat{U}$ which is defined by $\hat{t}(\hat{T})=\hat{T}(\hat{\alpha},\hat{\beta})$ for every bilinear forms $\hat{T}$ on $\hat{V}\times\hat{U}$.

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