A MOVING PSEUDO-BOUNDARY MFS FOR VOID DETECTION IN TWO-DIMENSIONAL THERMOELASTICITY

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Abstract. We investigate the numerical reconstruction of smooth star-shaped voids contained in a two-dimensional isotropic linear thermoelastic material from the knowledge a single non-destructive measurement of the temperature field and both the displacement and traction vectors (over-specified boundary data) on the outer boundary. The primary fields, namely the temperature and the displacement vector, which satisfy the thermo-mechanical equilibrium equations of linear thermoelasticity, are approximated using the method of fundamental solutions (MFS) in conjunction with a two-dimensional particular solution proposed in [37]. The fictitious source points are located both outside the (known) exterior boundary of the body and inside the (unknown) void. The inverse geometric problem reduces to finding the minimum of a nonlinear least-squares functional that measures the gap between the given and computed data, penalized with respect to both the MFS constants and the derivative of the radial polar coordinates describing the position of the star-shaped void. The interior source points are anchored to and move with the void during the iterative reconstruction procedure. The stability of the numerical method is investigated by inverting measurements contaminated with noise.

1. Introduction

Numerous problems in solid mechanics are characterised by the fact that at least one of the following conditions is entirely or partially unknown: (i) the governing partial differential equations; (ii) the geometry of the domain of interest; (iii) the material properties; and (iv) the boundary and initial conditions. Such problems are called inverse problems. Most of them are ill-posed and, consequently, are more difficult to solve than direct problems. It is also-well known that inverse problems are in general ill-posed in the sense of Hadamard [15], meaning that small errors in the boundary and or initial data may significantly amplify the errors in the solution.

Inverse geometric problems represent an important subclass of inverse problems and are characterised by a partial knowledge of the geometry of the solution domain, i.e. condition (ii) is incompletely known, whilst the remaining conditions (i), (iii) and (iv) are all known. In these problems, it is generally assumed that only a part of the boundary of the solution domain is known and accessible, while the remaining boundary is unknown and unaccessible and has to be determined from additional boundary data available on the accessible and known portion of the boundary or some internal measurements. In this study we refer to applications in the non-destructive evaluation of two-dimensional voids (e.g. rigid inclusions and cavities) in steady-state linear isotropic thermoelastic solids, for which the boundary temperature, displacements and tractions are measured on the accessible exposed surface, the boundary condition on the void surface is known, while the geometry of the void is to be determined.

The large majority of the theoretical and numerical studies devoted to the non-destructive detection of cavities and/or inclusions embedded in an elastic material available in the literature have considered either thermal [3, 5, 11, 12, 16, 18, 19, 21, 22, 25, 27, 28, 33, 39] or mechanical data [1, 2, 6, 7, 9, 17, 22, 24, 35, 38, 43, 44] measured on the accessible boundary and the corresponding boundary conditions on the unknown cavity/inclusion, i.e. either the heat conduction equation in isotropic solids or the equilibrium equations for isotropic linear elastic bodies have been considered. At the same time, to the best of our knowledge, there are only a few studies available in the

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literature that deal with the theoretical and/or numerical reconstruction of voids in an isotropic linear thermoelastic material, i.e. by measuring the thermo-mechanical fields at boundary and/or internal points [8, 10, 29, 30].

The method of fundamental solutions (MFS) [13, 14] is a meshless boundary collocation method [32] which may be used for the numerical solution of certain boundary value problems. This method, originally proposed in the mid 60s by Kupradze and Aleksidze [34] and introduced as a numerical method in the late 70s by Mathon and Johnston [41], has become increasingly popular primarily because of the ease with which it can be implemented. Due to this latter feature and the ideal suitability of the method for problems in which the shape of the boundary needs to be determined as part of the solution, the MFS has been used extensively for the numerical solution of inverse and related problems in the last decade. An extensive survey of applications of the MFS to inverse problems is provided in [23].

It should be mentioned that the MFS was used for the first time for the solution of inverse geometric problems in linear elasticity in [6], where the method of Kirsch and Kress [31] (originally proposed in the context of an exterior acoustic scattering problem) was adapted to the detection of rigid inclusions or cavities in an elastic body. The method of Alves and Martins [6] decomposes the inverse problem into a linear and ill-posed part in which a Cauchy problem is solved using the MFS, and a nonlinear part in which the unknown boundary of the void is sought as a zero level set for a rigid inclusion (or computed iteratively, in an optimization scheme for a class of approximating shapes, for a cavity). In contrast, in this paper we adopt a fully nonlinear MFS in which the nonlinear and ill-posed parts are dealt with simultaneously using a nonlinear regularized minimization. Moreover, in the method adopted herein the interior source points are free to move during the reconstruction procedure which is a further novel feature compared with earlier approaches.

Encouraged by the recent results obtained using the MFS for the detection of rigid inclusions and cavities (i.e. inner boundaries with prescribed homogeneous Dirichlet and Neumann boundary conditions, respectively) from Cauchy data measured on the outer boundary in the two- and three-dimensional Laplace equation [11, 21, 25] and the Lamé system in planar (two-dimensional) elasticity [24], we extend the MFS to the numerical reconstruction of voids in two-dimensional isotropic linear thermoelasticity. The paper is organized as follows. In Section 2, the inverse geometric problem considered herein is formulated mathematically. Section 3 is devoted to the MFS approximation for the aforementioned inverse problem via the MFS particular solution for the planar thermoelasticity system proposed in [37]. Implementational details on the penalized least-squares functional and the regularization method employed are given in Section 4. Four numerical examples associated with the numerical detection of a single void in planar thermoelasticity are thoroughly investigated in Section 5 and, furthermore, the proposed method is extended and applied to the numerical reconstruction of multiple voids in Section 6. Finally, some comments, conclusions and future related work are given in Section 7.

2. Mathematical Formulation

Consider a two-dimensional annular domain $\Omega = \Omega_2 \setminus \Omega_1$, where $\overline{\Omega}_1 \subset \Omega_2$, and $\Omega_1$, $\Omega_2$ are open bounded and simply-connected domains bounded by the interior and exterior smooth boundaries $\partial \Omega_1$ and $\partial \Omega_2$, respectively, see Figure 1. We further assume that $\Omega$ is occupied by an isotropic solid characterised by the thermal conductivity, $k$, the coefficient of linear thermal expansion, $\alpha_T$, Poisson’s ratio, $\nu$, and the shear modulus, $G$.

In the framework of isotropic linear thermoelasticity, under the assumption of small displacements the stress tensor $\sigma = [\sigma_{ij}]_{1 \leq i,j \leq 2}$ is related to the displacement vector $u = (u_1, u_2)$ by means of the constitutive law of thermoelasticity

$$\sigma(x) = G \left[ (\nabla u(x) + \nabla u(x)^T) - \frac{2\mu}{1-2\nu} (\nabla \cdot u(x)) I \right] - \gamma T(x) I, \quad x = (x_1, x_2) \in \Omega. \quad (2.1)$$

In (2.1), $\nu = \nu$ for a plane strain state and $\nu = \nu/(1+\nu)$ for a plane stress state, while $\sigma_T = \alpha_T$ and $\sigma_T = \alpha_T (1+\nu)/(1+2\nu)$ for the plane strain and plane stress states, respectively, $\gamma = 2G \sigma_T (1+\nu)/(1-2\nu)$ and $I$ is the $2 \times 2$ identity matrix.
In the absence of body forces, one obtains the equilibrium equations of two-dimensional isotropic linear static thermoelasticity in terms of the displacement vector and the temperature (also known as the Navier-Lamé system of two-dimensional isotropic linear thermoelasticity), namely

\[ 0 = -\nabla \cdot \sigma(x) = \mathcal{L}u(x) + \gamma \nabla T(x), \quad x \in \Omega, \tag{2.2} \]

where \( \mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2)^\top \) is the partial differential operator associated with the Navier-Lamé system of isotropic linear elasticity, i.e.

\[ \mathcal{L}u(x) \equiv -G \left[ \nabla \cdot (\nabla u(x) + \nabla u(x)^\top) + \frac{2\nu}{1-2\nu} \nabla (\nabla \cdot u(x)) \right] , \quad x \in \Omega. \tag{2.3} \]

Further, we let \( n(x) = (n_1(x), n_2(x))^\top \) be the outward unit normal vector at \( \partial \Omega \) and \( t(x) \) be the traction vector at \( x \in \partial \Omega \) given by

\[ t(x) = \sigma(x) n(x), \quad x \in \partial \Omega. \tag{2.4} \]

On the outer boundary \( \partial \Omega_2 \) we prescribe both the displacement and traction vectors, i.e.

\[ u(x) = f_2(x), \quad x \in \partial \Omega_2. \tag{2.5a} \]

\[ t(x) = g_2(x), \quad x \in \partial \Omega_2. \tag{2.5b} \]

On the inner boundary, we have boundary conditions in the form

\[ \Lambda u(x) + (I - \Lambda) t(x) = \Lambda f_1(x) + (I - \Lambda) g_1(x), \quad x \in \partial \Omega_1, \tag{2.6} \]

where \( f_1 \) and \( g_1 \), are known displacement and traction vectors, respectively, which are usually taken to be zero. In (2.6), the matrix \( \Lambda = [\Lambda_{ij}]_{1 \leq i,j \leq 2} \) defines the type of boundary conditions available on \( \partial \Omega_1 \), where \( \Lambda_{ij} = \Lambda, \delta_{ij} \) and \( \Lambda_i \in \{0,1\} \) for \( i,j = 1,2 \), and \( \delta_{ij} \) is the Kronecker delta symbol. Note that \( \Lambda = I, \Lambda = 0, \Lambda = [\Lambda_{ij}]_{1 \leq i,j \leq 2} \) with \( \Lambda_1 = 1 \) and \( \Lambda_2 = 0 \), and \( \Lambda = [\Lambda_{ij}]_{1 \leq i,j \leq 2} \) with \( \Lambda_1 = 0 \) and \( \Lambda_2 = 2 \) cover all of the admissible boundary conditions on \( \partial \Omega_1 \) and correspond to Dirichlet, Neumann and mixed boundary conditions of the first (\( u_1|_{\partial \Omega_1} \) prescribed) and second (\( u_2|_{\partial \Omega_1} \) prescribed) kinds on \( \partial \Omega_1 \), respectively.

Next, we let \( q(x) \) be the normal heat flux at a point \( x \in \partial \Omega \) defined by

\[ q(x) = -\left( k \nabla T(x) \right) \cdot n(x), \quad x \in \partial \Omega. \tag{2.7} \]

In the absence of heat sources, the governing equation for steady heat conduction is given by

\[ -\nabla \cdot (k \nabla T(x)) = 0, \quad x \in \Omega. \tag{2.8} \]

On the outer boundary \( \partial \Omega_2 \) we prescribe the temperature, i.e.

\[ T(x) = v_2(x), \quad x \in \partial \Omega_2. \tag{2.9} \]

On the inner boundary, we have a boundary condition in the form

\[ \beta T(x) + (1 - \beta) q(x) = \beta v_1(x) + (1 - \beta) w_1(x), \quad x \in \partial \Omega_1, \quad \text{where} \quad \beta \in \{0,1\}. \tag{2.10} \]

In (2.10), \( v_1 \) and \( w_1 \) are prescribed boundary temperature and heat flux functions, respectively, which are usually taken to be zero.

In the above setting, the inverse problem under investigation requires reconstructing/identifying the unknown defect \( \Omega_1 \), the displacement \( u \) and the temperature \( T \) satisfying the equations (2.2), (2.5), (2.6), (2.8)-(2.10).

We note that if one replaces (2.5b) by the heat flux measurement

\[ q(x) = w_2(x), \quad x \in \partial \Omega_2 \tag{2.11} \]

then the resulting inverse thermoelastic problem can be decoupled into an inverse problem for the steady-heat conduction given by equations (2.8)-(2.10) and (2.11) which has been solved by the authors in [25], and a direct problem for linear thermoelasticity given by equations (2.2), (2.5a), (2.6), (2.8), and (5.3), [37].
In the case of a rigid inclusion, i.e. $f_1 = 0, \Lambda = I, v_1 = 0, \beta = 1$, or a cavity, i.e. $g_1 = 0, \Lambda = 0, w_1 = 0, \beta = 0$, equations (2.6) and (2.10) simplify to

$$ u(x) = 0, \quad T(x) = 0, \quad x \in \partial \Omega_1, \quad (2.12) $$

or

$$ t(x) = 0, \quad q(x) = 0, \quad x \in \partial \Omega_1, \quad (2.13) $$

respectively.

Assuming that $v_2$ and $g_2$ are not identically equal to zero, we point out that the question of uniqueness of solution of the inverse problem (2.2), (2.5), (2.6), (2.8)-(2.10) is still open. In any case, the inverse problem is still ill-posed since small errors in the data (2.5a), (2.5b) cause large errors in the output solution $(u, T, \partial \Omega_1)$.

In the remaining of the paper, the purpose is to obtain an accurate and stable numerical solution. This is achieved using a nonlinear constrained MFS regularization procedure, as described in the sequel.

3. THE METHOD OF FUNDAMENTAL SOLUTIONS (MFS)

Assuming that $\kappa$ is constant, we first approximate the temperature $T$ by a linear combination of fundamental solutions of the two-dimensional Laplace operator, namely,

$$ T_N(c, \xi; x) = -\frac{1}{2\pi\kappa} \sum_{k=1}^{N_1+N_2} c_k \ln |x - \xi_k|, \quad x \in \overline{\Omega}. \quad (3.1) $$

The sources $(\xi_k)_{k=1}^{N_1}$ are located in $\Omega_1$ and the sources $(\xi_k)_{k=N_1+1}^{N_1+N_2}$ are in $\mathbb{R}^2 \setminus \overline{\Omega}_2$. More specifically, $(\xi_k)_{k=1}^{N_1+N_2}$ are placed on a (moving) pseudo-boundary $\partial \Omega'_1$ similar (contraction) to $\partial \Omega_1$, while $(\xi_k)_{k=N_1+1}^{N_1+N_2} \in \mathbb{R}^2 \setminus \overline{\Omega}_2$ are also placed on a (moving) pseudo-boundary $\partial \Omega'_2$ similar (dilation) to $\partial \Omega_2$. The arrangement of the sources may be seen in Figure 1. In (3.1), the sources $(\xi_k)_{k=1}^{N_1+N_2}$, where $N = N_1 + N_2$, are not preassigned and are determined as part of the solution of the problem, as will be described in the iterative process presented in the sequel. Such a moving pseudo-boundary approach in the MFS for inverse problems was used for the first time in [25].

The MFS approximation for the particular solution $u^{(p)}$ of the non-homogeneous equilibrium equations (2.2) in $\mathbb{R}^2$ is given by

$$ u_N^{(p)}(c, \xi; x) = -\frac{\gamma}{8\pi\kappa G} \left( \frac{1}{1-\nu} \right)^{N_1+N_2} \sum_{k=1}^{N_1+N_2} c_k \ln |x - \xi_k|, \quad x \in \mathbb{R}^2 \setminus \bigcup_{k=1}^{N_1+N_2} \{ \xi_k \}. \quad (3.2) $$

From approximation (3.2), the corresponding particular traction vector $t^{(p)}$ on the boundary $\partial \Omega$ is approximated as

$$ t_N^{(p)}(c, \xi; x) = -\frac{\gamma}{4\pi\kappa} \left( \frac{1}{1-\nu} \right)^{N_1+N_2} \sum_{k=1}^{N_1+N_2} c_k \left[ \frac{1}{1-2\nu} \ln |x - \xi_k| + \frac{\nu}{1-2\nu} \right] n(x) + \frac{(x - \xi_k) \cdot n(x)}{|x - \xi_k|^2} \left( x - \xi_k \right), \quad x \in \partial \Omega, \quad (3.3) $$

while the term $(t^{(p)} - \gamma T n)$ is approximated on $\partial \Omega$ by

$$ t_N^{(p)}(c, \xi; x) - \gamma T_N(x) n(x) $$

$$ = -\frac{\gamma}{4\pi\kappa} \left( \frac{1}{1-\nu} \right)^{N_1+N_2} \sum_{k=1}^{N_1+N_2} c_k \left[ \ln |x - \xi_k| + \frac{\nu}{1-2\nu} \right] n(x) - \frac{(x - \xi_k) \cdot n(x)}{|x - \xi_k|^2} \left( x - \xi_k \right), \quad x \in \partial \Omega, \quad (3.4) $$
In the case of the Cauchy-Navier system associated with the two-dimensional isotropic linear elasticity, the fundamental solution matrix $U = [U_{ij}]_{1 \leq i, j \leq 2}$ for the displacement vector is given by [4]

$$U_{ij}(x, \xi) = \frac{1}{8\pi G(1-\nu)} \left[-(3-4\nu) \ln |x - \xi| \delta_{ij} + \frac{x_i - \xi_i}{|x - \xi|} \frac{x_j - \xi_j}{|x - \xi|}\right], \quad x \in \overline{\Omega}, \quad i, j = 1, 2, \quad (3.5)$$

where $x = (x_1, x_2) \in \overline{\Omega}$ is a collocation point and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \overline{\Omega}$ is a source point. On differentiating equation (3.5) with respect to $x_k$, $k = 1, 2$, one obtains the derivatives of the fundamental solution for the displacement vector, denoted by $\partial x^k U_{ij}(x, \xi)$, where $\partial x^k \equiv \partial / \partial x_k$. By combining Hooke’s law for isotropic linear elasticity and equation (3.5), the fundamental solution matrix $T = [T_{ij}]_{1 \leq i, j \leq 2}$ for the traction vector in the case of two-dimensional isotropic linear elasticity is obtained as [4]

$$T_{1k}(x, \xi) = \frac{2G}{1-2\nu} \left[ (1-\nu) \partial x^i U_{1k}(x, \xi) + \nu \partial x^j U_{2k}(x, \xi) \right] n_1(x) + G \left[ \partial x^i U_{1k}(x, \xi) + \partial x^j U_{2k}(x, \xi) \right] n_2(x),$$

$$T_{2k}(x, \xi) = G \left[ \partial x^i U_{1k}(x, \xi) + \partial x^j U_{2k}(x, \xi) \right] n_1(x) + \frac{2G}{1-2\nu} \left[ \nu \partial x^i U_{1k}(x, \xi) + (1-\nu) \partial x^j U_{2k}(x, \xi) \right] n_2(x), \quad (3.6)$$

for $x \in \partial \Omega$, $k = 1, 2$. Analogously to the MFS for the thermal problem, we approximate the displacement vector, $u^{(H)}$, associated with the homogeneous equilibrium equation (2.2) by a linear combination of the displacement fundamental solutions (3.5), namely,

$$u^{(H)}_{N}(d, \xi; x) = \sum_{k=1}^{N_1 + N_2} \begin{bmatrix} U_{11}(x, \xi) & U_{12}(x, \xi) \\ U_{21}(x, \xi) & U_{22}(x, \xi) \end{bmatrix} \begin{bmatrix} d_{1k}^i \\ d_{2k}^i \end{bmatrix}, \quad x \in \overline{\Omega}, \quad (3.7)$$
As a result, the outer boundary collocation and source points are chosen as
\[ \eta \]
The coefficients \( \eta \) are chosen in a linear combination of the traction fundamental solutions (3.6), namely,
\[
 t_N^{(H)}(d, \xi; x) = \sum_{k=1}^{N_1+N_2} \begin{bmatrix} T_{11}(x, \xi) & T_{12}(x, \xi) \\ T_{21}(x, \xi) & T_{22}(x, \xi) \end{bmatrix} \begin{bmatrix} d_k^1 \\ d_k^2 \end{bmatrix}, \quad x \in \partial \Omega. \tag{3.8}
\]
Without loss of generality, we shall assume that the (known) fixed exterior boundary \( \partial \Omega_2 \) is a circle of radius \( R \).
As a result, the outer boundary collocation and source points are chosen as
\[
x_{N_1+k} = R(\cos \tilde{\vartheta}_k, \sin \tilde{\vartheta}_k), \quad \xi_{N_1+k} = \eta_{\text{ext}} R(\cos \tilde{\vartheta}_k, \sin \tilde{\vartheta}_k), \quad \text{where} \quad \tilde{\vartheta}_k = \frac{2\pi(k-1)}{N_2}, \quad k = \overline{1,N_2},
\tag{3.9}
\]
and the (unknown) parameter \( \eta_{\text{ext}} \in (1,S) \) with \( S > 1 \) prescribed.
We further assume that the unknown boundary \( \partial \Omega_1 \) is a smooth, star-like curve with respect to a centre which has unknown coordinates \( (X, Y) \). This means that its equation in polar coordinates can be written as
\[
x = X + r(\vartheta) \cos \vartheta, \quad y = Y + r(\vartheta) \sin \vartheta, \quad \vartheta \in [0,2\pi),
\tag{3.10}
\]
where \( r \) is a smooth \( 2\pi \)-periodic function.
The discretized form of (3.10) for \( \partial \Omega_1 \) becomes
\[
r_k = r(\vartheta_k), \quad k = \overline{1,N_1} \tag{3.11}
\]
and we choose the inner boundary collocation and source points as
\[
x_k = (X, Y) + r_k (\cos \vartheta_k, \sin \vartheta_k), \quad \xi_k = (X, Y) + \eta_{\text{in}} r_k (\cos \vartheta_k, \sin \vartheta_k), \quad \text{where} \quad \vartheta_k = \frac{2\pi(k-1)}{N_1}, \quad k = \overline{1,N_1},
\tag{3.12}
\]
where the (unknown) parameter \( \eta_{\text{in}} \in (0,1) \).

4. Implementational details
The coefficients \( (c_k)_{k=1,N_1+N_2} \) and \( (d_k^j)_{j=1,2} \), the radii \( (r_k)_{k=1,N_1} \) in (3.11), the contraction and dilation coefficients \( \eta_{\text{in}} \) and \( \eta_{\text{ext}} \) in (3.9) and (3.12), and the coordinates of the centre \( (X,Y) \) can be determined by imposing the boundary conditions (2.5a), (2.5b), (2.6), (5.3) and (2.10) in a least-squares sense. This leads to the minimization of the functional
\[
S(c,d,r,\eta,C) := \sum_{j=1}^{N_1} \left[ 2T_N(c,\xi;x_j) + (1-\beta)q(c,\xi;x_j) - \beta v_1(x_j) - (1-\beta)w_1(x_j) \right]^2
\]
\[
+ \sum_{j=N_1+1}^{N_1+N_2} \left[ T_N(c,\xi;x_j) - v_2(x_j) \right]^2
\]
\[
+ \sum_{j=1}^{N_1} \left[ \Lambda_1 \left( t_{11}^{(H)}(d,\xi;x_j) + t_{12}^{(H)}(d,\xi;x_j) - \gamma T_N(x_j)n_1 \right) - \Lambda_1 f_1(x_j) - (1-\Lambda_1) g_1(x_j) \right]^2
\]
\[
+ \sum_{j=1}^{N_1} \left[ \Lambda_2 \left( t_{21}^{(H)}(d,\xi;x_j) + t_{22}^{(H)}(d,\xi;x_j) - \gamma T_N(x_j)n_2 \right) - \Lambda_2 f_2(x_j) - (1-\Lambda_2) g_2(x_j) \right]^2
\]
\[
+ \sum_{j=N_1+1}^{N_1+N_2} \left[ \left( t_{11}^{(H)}(d,\xi;x_j) + t_{12}^{(H)}(d,\xi;x_j) - \gamma T_N(x_j)n_1 \right) - g_1(x_j) \right]^2
\]
\[
+ \sum_{j=N_1+1}^{N_1+N_2} \left[ \left( u_{2N}^{(H)}(d, \xi; x_j) + u_{2N}^{(P)}(c, \xi; x_j) \right) - f_2(x_j) \right]^2 + \sum_{j=N_1+1}^{N_1+N_2} \left[ \left( t_{2N}^{(H)}(d, \xi; x_j) + t_{2N}^{(P)}(c, \xi; x_j) - \tau T_N(x_j) n_2 \right) - g_2(x_j) \right]^2 \\
+ \lambda_1 |c|^2 + \lambda_2 (|d|^2 + |\tilde{d}|^2) + \lambda_3 \sum_{\ell=2}^{N_1} (r_{\ell} - r_{\ell-1})^2, \quad (4.1)
\]

where \( \lambda_1, \lambda_2, \lambda_3 \geq 0 \) are regularization parameters to be prescribed, \( c = (c_1, c_2, \ldots, c_N) \), \( d^1 = (d_1^1, d_2^1, \ldots, d_N^1) \), \( d^2 = (d_1^2, d_2^2, \ldots, d_N^2) \), \( r = (r_1, r_2, \ldots, r_N) \), \( \eta = (\eta_{\text{int}}, \eta_{\text{ext}}) \) and \( C = (X, Y) \).

The minimization of functional (4.1) is carried out using the MATLAB\textsuperscript{®} \cite{MATLAB} optimization toolbox routine lsqnonlin which solves nonlinear least squares problems. This routine does not require the user to provide the gradient and, in addition, it offers the option of imposing lower and upper bounds on the elements of the vector of unknowns \( (c, d, r, \eta, C) \) through the vectors \( \mathbf{lb} \) and \( \mathbf{ub} \). We can thus easily impose the constraints \( 0 < r_i < 1, \quad i = 1, N_1, 0 < \eta_{\text{int}} < 1, 1 < \eta_{\text{ext}} < S \) for an appropriately chosen \( S \) and \(-R < X < R, \quad -R < Y < R\).

**Remark.** In computation, noisy data are generated from equations (2.5a)-(2.5b) as
\[
u^s(x_j) = (1 + \rho_f p_f) f_2(x_j), \quad \nu^t(x_j) = (1 + \rho_g p_g) g_2(x_j), \quad j = N_1 + 1, N_1 + N_2, \quad (4.2)
\]

where \( \rho_f \) and \( \rho_g \) represent the percentage of noise added to the Dirichlet and Neumann boundary data on \( \partial \Omega_2 \), respectively, and \( \rho_f \) is a pseudo-random noisy variable drawn from a uniform distribution in \([-1, 1]\) using the MATLAB\textsuperscript{®} command \texttt{-1+2*rand(1,N)}). In our numerical experiments it was observed that the effect of noise added to the Dirichlet boundary data was similar to that of perturbing the Neumann data. As a result, in the next results section we only present results for noisy Neumann data, i.e. \( \rho_g = p \neq 0 \) and \( \rho_f = 0 \).

5. **Numerical Examples**

In all numerical examples we take \( \kappa = 1 \).

5.1. **Example 1.** We first consider an example for which the exact solution is known. Here we consider the case where \( X = Y = 0, R = R_{\text{out}}, \) and \( \beta = 1, \quad \Lambda = I. \) The exact solution is defined by the domains
\[
\Omega_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R_{\text{int}}^2 < R_{\text{out}}^2 \}, \quad \Omega_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < R_{\text{out}}^2 \}
\]
and
\[
T^{(\text{an})}(x) = \frac{T_{\text{out}}}{R_{\text{out}} - R_{\text{int}}} \log \left( \frac{\|x\|}{R_{\text{int}}} \right) + \frac{T_{\text{int}}}{R_{\text{out}} - R_{\text{int}}} \log \left( \frac{R_{\text{out}}}{\|x\|} \right), \quad x \in \overline{\Omega}, \quad (5.2a)
\]
\[
q^{(\text{an})}(x) = -\kappa \frac{T_{\text{out}} - T_{\text{int}}}{R_{\text{out}} - R_{\text{int}}} \frac{x \cdot n(x)}{\|x\|^2}, \quad x \in \partial \Omega, \quad (5.2b)
\]
\[
u^{(\text{an})}(x) = \frac{\gamma}{2} \left( \frac{1 - 2\nu}{1 - \nu} \right) \frac{T_{\text{out}} - T_{\text{int}}}{R_{\text{out}} - R_{\text{int}}} \log \|x\| + \frac{1 - \nu}{1 + \nu} - \frac{W}{2} \frac{1}{\|x\|^2} \frac{x}{2G}, \quad x \in \overline{\Omega}, \quad (5.2c)
\]
\[

t^{(\text{an})}(x) = \begin{cases} 
-\sigma_{\text{out}} n(x), & x \in \Gamma_{\text{out}} = \left\{ x \in \partial \Omega : \|x\| = R_{\text{out}} \right\}, \\
-\sigma_{\text{int}} n(x), & x \in \Gamma_{\text{int}} = \left\{ x \in \partial \Omega : \|x\| = R_{\text{int}} \right\},
\end{cases} \quad (5.2d)
\]

where
\[
V = -\frac{\sigma_{\text{out}}}{R_{\text{out}}^2 - R_{\text{int}}^2} R_{\text{out}}^2 - \frac{\sigma_{\text{int}}}{R_{\text{int}}^2} R_{\text{int}}^2, \quad W = \frac{\sigma_{\text{out}} - \sigma_{\text{int}}}{R_{\text{out}}^2 - R_{\text{int}}^2} \left( \frac{1}{1 - \nu} \log \frac{R_{\text{out}}}{R_{\text{int}}} + 1 \right),
\]
\[
\sigma^{(\text{H})} = \sigma_{\text{out}} - \gamma T_{\text{out}} + \frac{\gamma}{2} \frac{T_{\text{out}} - T_{\text{int}}}{\log \left( \frac{R_{\text{out}}}{R_{\text{int}}} \right)} \left( \frac{1}{1 - \nu} \log \frac{R_{\text{out}}}{R_{\text{int}}} + 1 \right),
\]
\[
\sigma^{(H)}_{\text{int}} = \sigma_{\text{int}} - \gamma T_{\text{int}} + \frac{\gamma}{2} \frac{T_{\text{out}} - T_{\text{int}}}{\log \left( \frac{R_{\text{out}}}{R_{\text{int}}} \right)} \left( \frac{1}{1 - \nu} \log R_{\text{int}} + 1 \right).
\]

We consider the case where \( R_{\text{int}} = 0.5 \) and \( R = R_{\text{out}} = 1 \), \( T_{\text{int}} = 1^\circ \text{C} \), \( T_{\text{out}} = 2^\circ \text{C} \), \( \sigma_{\text{int}} = 1.0 \times 10^{10} \text{N/m}^2 \), \( \sigma_{\text{out}} = 2.0 \times 10^{10} \text{N/m}^2 \), \( \nu = 0.34 \), \( \nu = \nu/(1 + \nu) \) and \( G = 4.8 \times 10^{10} \text{N/m}^2 \).

In Figure 2 we present the reconstructed curves for \( \partial \Omega \) for various numbers of degrees of freedom \( N_1 = N_2 \) obtained in 40 iterations, no noise and no regularization. From this figure it can be seen that very accurate and convergent numerical results are obtained. We choose the initial vector of unknowns \( x_0 = (c_0, d_0, r_0, \eta_{\text{int}}, \eta_{\text{ext}}, X_0, Y_0) = (0, 0.05, 1.1, 0, 0) \). In Figure 3 we present the reconstructed curves with a noise level of \( p = 10\% \) after 40 iterations and various regularization parameters \( \lambda_1 = \lambda_2 \) for \( \lambda_3 = 0 \), with \( N_1 = N_2 = 48 \). From this figure it can be seen that stable and reasonably accurate results for the circular rigid inclusion are obtained for \( \lambda_3 = 0 \) and \( \lambda_1 = \lambda_2 \in [10^{-5}, 10^{-4}] \). Figure 4 shows the corresponding results with various regularization parameters \( \lambda_3 \) for \( \lambda_1 = \lambda_2 = 0 \). From this figure it can be seen that stable and accurate results are obtained for \( \lambda_1 = \lambda_2 = 0 \) and \( \lambda_3 \in [10^{-2}, 10^0] \). Moreover, by comparing Figures 3 and 4 we observe that regularization with \( \lambda_3 \) produces more stable and accurate solutions than regularization with \( \lambda_1 = \lambda_2 \). Nevertheless, the choice of the regularization parameters \( \lambda_1, \lambda_2, \lambda_3 \) other than trial and error is very tedious and difficult and is an issue on which more research needs to be undertaken in the future.

![Figure 2](image-url)

**Figure 2.** Example 1: Results for various numbers of degrees of freedom, no noise and no regularization.

### 5.2. Example 2

We now consider Example 1 again with the exact solution given by (5.1) and (5.2), but in this case with \( \beta = 0 \), \( \Lambda = 0 \). All the other details are the same as in Example 1. Figures 5 and 6 correspond to Figures 3 and 4 of Example 1. A comparison of Figures 3 and 4 of Example 1 to Figures 5 and 6 of Example 2 indicates that larger values of the regularization parameters are required for obtaining stable and accurate solutions.
5.3. Example 3. In this example, we consider a more complicated peanut-shaped rigid inclusion whose boundary $\partial \Omega_1$ is described by $X = Y = 0$, $R = 1$ and the radial parametrization
\[
r(\vartheta) = \frac{3}{4} \sqrt{\cos^2(\vartheta) + 0.25 \sin^2(\vartheta)}, \quad \vartheta \in [0, 2\pi),
\] (5.4)
which was considered in [20, 25]. Since in this case no analytical solution is available, the input traction data (2.5b) is numerically simulated by solving the direct problem with boundary conditions given by equation (2.12) for the input outer boundary temperature, namely,

\[ T(R, \vartheta) = v_2(\vartheta) = e^{-\cos^2 \vartheta}, \quad \vartheta \in [0, 2\pi). \] (5.5)
and outer boundary displacements (2.5a), namely,
\[ u_1(x_1, x_2) = f_2(x_1, x_2) = x_1/100, \quad u_2(x_1, x_2) = f_2(x_1, x_2) = x_2/100, \quad (x_1, x_2) \in \partial \Omega, \] (5.6)
when \( \partial \Omega \) is given by (5.4), using the MFS with \( N_1 = N_2 = 250 \). In order to avoid committing an inverse crime, the inverse solver is applied using \( N_1 = N_2 = 48 \). In Figure 7 we present the results obtained for different numbers of iterations, no regularization, and no noise. From this figure it can be seen that a convergent and accurate numerical solution for the rigid peanut-shape inclusion (5.4) is obtained. In Figures 8 and 9 we present the reconstructed curves with a noise level of \( p = 5\% \) after 100 iterations and various regularization parameters \( \lambda_2 \) for \( \lambda_1 = \lambda_3 = 0 \), and with \( \lambda_3 \) for \( \lambda_1 = \lambda_2 = 0 \), respectively. From these figures it can be seen that regularization with \( \lambda_2 \in [5 \times 10^{-4}, 10^{-3}] \) for \( \lambda_1 = \lambda_3 = 0 \) produces good reconstructions of the desired peanut shape (5.4), while regularization with \( \lambda_3 \) for \( \lambda_1 = \lambda_2 = 0 \) remains close to the initial guess (probably stuck at a local minimum) for all values of \( \lambda_3 \).

**Figure 7.** Example 3: Results for various numbers of iterations for no noise and no regularization.

**5.4. Example 4.** We next consider a rigid inclusion \( \Omega_1 \) described by \( X = 0.5, Y = -1, R = 3.5 \) and the radial parametrization
\[ r(\vartheta) = 1.52 - 0.24 \sin(3\vartheta), \quad \vartheta \in [0, 2\pi). \] (5.7)
This example, which was considered in [40] for the Stokes equations in slow viscous flow, is more difficult than the previous examples the coordinates of the centre of \( \Omega_1 \) are also unknown. The traction data (2.5b) is simulated by solving the direct problem with the data as in Example 3. In Figure 10 we present the results obtained for different numbers of iterations, no regularization, and no noise. As in Example 3, from Figure 10 we observe that a convergent and reasonably accurate reconstruction of the shape (5.7) is obtained. Next, for \( p = 5\% \) noisy data, the shapes obtained with \( \lambda_1 = \lambda_2 \) for \( \lambda_3 = 0 \), \( \lambda_2 \) for \( \lambda_1 = \lambda_3 = 0 \), and \( \lambda_3 \) for \( \lambda_1 = \lambda_2 = 0 \), after 100 iterations, are presented in Figures 11-13, respectively. From these, it can be observed that regularization with \( \lambda_3 \) produces more accurate results than with \( \lambda_1 = \lambda_2 \) and with \( \lambda_3 \).
6. Extension to multiple voids

The MFS analysis performed so far showed the successful implementation of this method for the identification of a single void. In this section, we extend the analysis to multiple voids which may contain both cavities and rigid inclusions. For the sake of clarity, we describe the formulation for the case of two voids. Therefore, we consider...
the inverse problem consisting of the governing equations (2.2) and (2.7) in a domain $\Omega$ subject to the boundary conditions (2.5a), (2.5b) and (5.3) on the outer boundary $\partial\Omega_2$ and the boundary conditions on the inner boundaries $\partial\Omega^\ell_1$, $\ell = a, b$

\[
\Lambda \, u(x) + (I - \Lambda) \, t(x) = \Lambda \, f^\ell_1(x) + (I - \Lambda) \, g^\ell_1(x), \quad x \in \partial\Omega^\ell_1,
\]  

(6.1)
and

$$\beta T(x) + (1 - \beta)q(x) = \beta v_1^1(x) + (1 - \beta)v_1^2(x) \quad x \in \partial \Omega_1^1, \text{ where } \beta \in \{0, 1\}. \quad (6.2)$$

Here $\Omega_1^a$ and $\Omega_1^b$ are two disjoint voids, such that $\Omega_1^a \cup \Omega_1^b = \Omega_1$ and $\overline{\Omega_1^a} \cap \overline{\Omega_1^b} = \emptyset$. 

**Figure 12.** Example 4: Results for noise $p = 5\%$ and regularization with $\lambda_2$ for $\lambda_1 = \lambda_3 = 0$. 

**Figure 13.** Example 4: Results for noise $p = 5\%$ and regularization with $\lambda_3$ for $\lambda_1 = \lambda_2 = 0$. 

$\lambda_2 = 0, \lambda_2 = 10^{-4}, \lambda_2 = 10^{-3}$

$\lambda_3 = 0, \lambda_3 = 10^{-5}, \lambda_3 = 10^{-4}$

$\lambda = 8 \times 10^{-4}, \lambda = 10^{-3}, \lambda = 10^{-2}$

$\lambda_3 = 3 \times 10^{-4}, \lambda_3 = 10^{-3}, \lambda_3 = 10^{-2}$
The temperature $T$ is now approximated by

$$T_N(c, \xi; x) = -\frac{1}{2\pi k} \sum_{k=1}^{N_1+N_2+N_3} c_k \ln |x - \xi_k|, \quad x \in \bar{\Omega}.$$ (6.3)

The sources $(\xi_k)_{k=1,N_1+N_2+N_3}$ are located outside the solution domain $\Omega$, i.e. in $\Omega_1 \cup (\mathbb{R}^2 \setminus \bar{\Omega}_2)$. In particular, $(\xi_k)_{k=1,N_1} \in \Omega_1^a$ are placed on a (moving) pseudo-boundary $\partial \Omega_1^a$ similar (contraction) to $\partial \Omega_1$, $(\xi_k)_{k=N_1+1,N_1+N_2} \in \Omega_1^b$ are placed on a (moving) pseudo-boundary $\partial \Omega_1^b$ similar (contraction) to $\partial \Omega_1^b$, while $(\xi_k)_{k=N_1+N_2+1,N_1+N_2+N_3} \in \mathbb{R}^2 \setminus \bar{\Omega}_2$ are also placed on a (moving) pseudo-boundary $\partial \Omega_2$ similar (dilation) to $\partial \Omega_2$. The situation is depicted in Figure 14. The contraction parameters for $\partial \Omega_1^a$ and $\partial \Omega_1^b$ are taken to be $\eta_{\text{int}}^a \in (0, 1)$ and $\eta_{\text{int}}^b \in (0, 1)$, respectively. Note that from now on we shall denote $N = N_1 + N_2 + N_3$.

![Figure 14. Geometry of the problem with two inclusions. The crosses (+) denote the source points.](image)

The MFS approximation for the particular solution $u^{(p)}$ is now given by

$$u^{(p)}_N(c, \xi; x) = -\frac{\pi}{8\pi kG} \left( \frac{1 - 2\eta}{1 - \eta} \right)^{N_1+N_2+N_3} \sum_{k=1}^{N_1+N_2+N_3} c_k (x - \xi_k) \ln |x - \xi_k|, \quad x \in \mathbb{R}^2 \setminus \bigcup_{k=1}^{N_1+N_2+N_3} \{\xi_k\}. \quad (6.4)$$

with the expressions (3.3), (3.4), (3.7) and (3.8), defined accordingly with $N_1 + N_2$ replaced by $N_1 + N_2 + N_3$.

The outer boundary collocation and source points are chosen as

$$x_{N_1+N_2+k} = R(\cos \tilde{\vartheta}_k, \sin \tilde{\vartheta}_k), \quad \xi_{N_1+N_2+k} = \eta_{\text{ext}} R(\cos \tilde{\vartheta}_k, \sin \tilde{\vartheta}_k), \quad \text{where} \quad \tilde{\vartheta}_k = \frac{2\pi(k-1)}{N_3}, \quad k = 1, N_3, \quad (6.5)$$

where $\eta_{\text{ext}}$ is defined as in (3.9).

We further assume that the unknown boundaries $\partial \Omega_1^a$ and $\partial \Omega_1^b$ are a smooth, star-like curves with respect to their centres which have unknown coordinates $(X^a, Y^a)$ and $(X^b, Y^b)$, respectively. This means that their equations in polar coordinates can be written as

$$x = X^a + r^a(\vartheta) \cos \vartheta, \quad y = Y^a + r^a(\vartheta) \sin \vartheta, \quad (6.6)$$

$$x = X^b + r^b(\vartheta) \cos \vartheta, \quad y = Y^b + r^b(\vartheta) \sin \vartheta, \quad \vartheta \in [0, 2\pi), \quad (6.7)$$
where \( r^a \) and \( r^b \) are smooth \( 2\pi \)-periodic functions.

The discretized forms of (6.6) and (6.7) for \( \partial \Omega_1^a \) and \( \partial \Omega_1^b \) become

\[
    r_k^a = r^a(\theta_k), \quad k = 1, N_1 \quad \text{and} \quad r_k^b = r^b(\theta_k) \quad k = 1, N_2. \tag{6.8}
\]

We choose the inner boundary collocation and source points as

\[
    x_k = (X^a, Y^a) + r_k^a (\cos \theta_k, \sin \theta_k), \quad k = 1, N_1 \tag{6.9}
\]

\[
    x_k = (X^b, Y^b) + r_k^b (\cos \theta_k, \sin \theta_k), \quad k = N_1 + 1, N_1 + N_2. \tag{6.10}
\]

\[
    \xi_k = (X^a, Y^a) + \eta_{\text{int} k} r_k^a (\cos \theta_k, \sin \theta_k), \quad k = 1, N_1, \tag{6.11}
\]

\[
    \xi_k = (X^b, Y^b) + \eta_{\text{int} k} r_k^b (\cos \theta_k, \sin \theta_k), \quad k = N_1 + 1, N_1 + N_2. \tag{6.12}
\]

The coefficients \((c_k)_{k=1, N_1+N_2+N_3}, (d_k^l)_{k=1, N_1+N_2+N_3, j=1, 2}\), the radii \((r_k^a)_{k=1, N_1}, (r_k^b)_{k=1, N_2}\) in (6.8), the contraction and dilation coefficients \(\eta_{\text{int}}, (r_k^b)_{k=1, N_2}\) and \(\eta_{\text{ext}}\), and the coordinates of the centres \((X^a, Y^a), (X^b, Y^b)\) can be now determined by imposing the boundary conditions in a least-squares sense, i.e., by minimizing the functional

\[
    S(c, d, r, \eta, C) := \sum_{j=1}^{N_1} \left[ \beta_1 T_N(c, \xi, x_j) + (1 - \beta_1) q(c, \xi, x_j) - \beta_1 v_1^a(x_j) - (1 - \beta_1) w_1^a(x_j) \right]^2
\]

\[
    + \sum_{j=N_1}^{N_1+N_2} \left[ \beta_2 T_N(c, \xi, x_j) + (1 - \beta_2) q(c, \xi, x_j) - \beta_2 v_1^b(x_j) - (1 - \beta_2) w_1^b(x_j) \right]^2
\]

\[
    + \sum_{j=N_1+N_2}^{N_1+N_2+N_3} \left[ T_N(c, \xi, x_j) - v_2(x_j) \right]^2
\]

\[
    + \sum_{j=1}^{N_1} \left[ \Lambda_1 \left( u_1^{(h)}(d, \xi, x_j) + u_1^{(p)}(c, \xi, x_j) \right) + (1 - \Lambda_1) \left( t_1^{(h)}(d, \xi, x_j) + t_1^{(p)}(c, \xi, x_j) - \gamma T_N(x_j) n_1 \right) - \Lambda_1 f_1^a(x_j) - (1 - \Lambda_1) g_1^a(x_j) \right]^2
\]

\[
    + \sum_{j=N_1+1}^{N_1+N_2} \left[ \Lambda_2 \left( u_2^{(h)}(d, \xi, x_j) + u_2^{(p)}(c, \xi, x_j) \right) + (1 - \Lambda_2) \left( t_2^{(h)}(d, \xi, x_j) + t_2^{(p)}(c, \xi, x_j) - \gamma T_N(x_j) n_2 \right) - \Lambda_2 f_2^a(x_j) - (1 - \Lambda_2) g_2^a(x_j) \right]^2
\]

\[
    + \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} \left[ \Lambda_1 \left( u_1^{(h)}(d, \xi, x_j) + u_1^{(p)}(c, \xi, x_j) \right) + (1 - \Lambda_1) \left( t_1^{(h)}(d, \xi, x_j) + t_1^{(p)}(c, \xi, x_j) - \gamma T_N(x_j) n_1 \right) - \Lambda_1 f_1^b(x_j) - (1 - \Lambda_1) g_1^b(x_j) \right]^2
\]

\[
    + \sum_{j=N_1+1}^{N_1+N_2+1} \left[ \Lambda_2 \left( u_2^{(h)}(d, \xi, x_j) + u_2^{(p)}(c, \xi, x_j) \right) + (1 - \Lambda_2) \left( t_2^{(h)}(d, \xi, x_j) + t_2^{(p)}(c, \xi, x_j) - \gamma T_N(x_j) n_2 \right) - \Lambda_2 f_2^b(x_j) - (1 - \Lambda_2) g_2^b(x_j) \right]^2
\]

\[
    + \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} \left[ \left( u_1^{(h)}(d, \xi, x_j) + u_1^{(p)}(c, \xi, x_j) \right) - f_2(x_j) \right]^2 + \sum_{j=N_1+N_2+1}^{N_1+N_2+N_3} \left[ \left( t_1^{(h)}(d, \xi, x_j) + t_1^{(p)}(c, \xi, x_j) - \gamma T_N(x_j) n_1 \right) - g_2(x_j) \right]^2
\]

\[
    + \lambda_1 |c|^2 + \lambda_2 \left( |d|^2 + |d|^2 \right) + \lambda_3 \sum_{j=1}^{N_1} \left( r_k^a - r_{k-1}^a \right)^2 + \lambda_4 \sum_{j=1}^{N_2} \left( r_k^b - r_{k-1}^b \right)^2, \tag{6.13}
\]

where \(\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0\) are regularization parameters to be prescribed, \(c = (c_1, c_2, \ldots, c_N)\), \(d^1 = (d_1^1, d_2^1, \ldots, d_N^1)\), \(d^2 = (d_1^2, d_2^2, \ldots, d_N^2)\), \(r^a = (r_1^a, r_2^a, \ldots, r_N^a)\), \(r^b = (r_1^b, r_2^b, \ldots, r_N^b)\), \(\eta = (\eta_{\text{int}}, \eta_{\text{ext}})\), and \(C = (X^a, Y^a, X^b, Y^b)\).

The number of unknowns is \(4N_1 + 4N_2 + 3N_3 + 7\) and the number of boundary collocation equations \(3N_1 + 3N_2 + 5N_3\), and thus we need to take \(2N_3 \geq N_1 + N_2 + 7\).
6.1. Example 5. We consider the case when two rigid inclusions $\Omega^a_1$ and $\Omega^b_1$ are present. $\Omega^a_1$ is a disk of radius 1, with centre $X^a = 1, Y^a = -1$, while $\Omega^b_1$ is described by the radial parametrization

$$r(\vartheta) = \frac{1 + 0.8 \cos(\vartheta) + 0.2 \sin(2\vartheta)}{1 + 0.7 \cos(\vartheta)}, \quad \vartheta \in [0, 2\pi),$$

(6.14)

and has centre $X^b = -1, Y^b = 1$. In this example $R = 3.5$. The Neumann data (2.5b) is simulated numerically by solving the direct problem subject to the boundary conditions (6.1) with $A = I$ and $f^a_1 = f^b_1 = 0$, and (6.2) with $\beta = 1$ and $u^a_0 = u^b_0 = 0$ for the input outer boundary temperature (5.5) and the outer boundary displacements (5.6), using the MFS with $N_1 = N_2 = N_3 = 400$. In order to avoid committing an inverse crime, the inverse solver is applied using $N_1 = N_2 = 32, N_3 = 64$.

Figures 15-17 correspond to Figures 11-13 of Example 4 but for $p = 3\%$ noisy data. From Figures 15 and 16 it can be seen that when regularizing with $\lambda_1 = \lambda_2$ for $\lambda_3 = 0$ and with $\lambda_2$ for $\lambda_1 = \lambda_3 = 0$ the results are very similar. Figure 17 reveals that regularization with $\lambda_3$ for $\lambda_1 = \lambda_2 = 0$ yields slightly better results in retrieving the two-component rigid inclusion.

![Figure 15. Example 5: Results for noise $p = 3\%$ and regularization with $\lambda_1 = \lambda_2$ for $\lambda_3 = 0$.](image)

7. Conclusions

Two-dimensional steady-state reconstructions of voids embedded in a thermoelastic host material have been accomplished using a dynamic MFS with regularization. This nonlinear and ill-posed inverse geometric problem has been recast as a constrained minimization problem which has been solved using the MATLAB optimization toolbox routine lsqnonlin. The results show that the numerical solution is accurate for exact data, and stable for noisy data. The selection of regularization parameters was based on trial and error, but nevertheless more research is needed in order to select more rigorously the multiple regularization parameters in the non linear Tikhonov method. Three-dimensional thermoelastic reconstructions can be attempted in the future by combining the efforts of [36] and [26].
Figure 16. Example 5: Results for noise $p = 3\%$ and regularization with $\lambda_2$ for $\lambda_1 = \lambda_3 = 0$.

Figure 17. Example 5: Results for noise $p = 3\%$ and regularization with $\lambda_3$ for $\lambda_1 = \lambda_2 = 0$.

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