

Discontinuous Galerkin Methods for Solving the Signorini Problem

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Abstract. We study several discontinuous Galerkin methods (DGMs) for solving the Signorini problem. A unified error analysis is provided for the methods. The error estimates are of optimal order for linear elements. A numerical example is reported to illustrate numerical convergence orders.

Keywords. Discontinuous Galerkin method, Signorini problem, error analysis

AMS Classification. 65N30, 49J40

1 Introduction

In this paper, we extend the ideas in [20], in which the discontinuous Galerkin (DG) methods for variational inequalities were analyzed, to solve the well-known Signorini problem. The initial DG method was introduced by Reed and Hill ([18]) for numerically solving the neutron transport equation. In the past two decades, DG methods have been widely used for a variety of partial differential equations, such as hyperbolic equations, convection-diffusion equations, Navier-Stokes equations, Hamilton-Jacobi equations and so on. We refer to [12] for a historical survey about DG methods.

DG methods provide discontinuous approximations by using the Galerkin method element by element, and transfer information between two neighboring elements through the use of numerical traces (numerical fluxes). Discontinuous property make DG methods easy to handle elements of arbitrary shapes and irregular meshes with hanging nodes, and flexible to construct local shape

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function spaces (*hp*-adaptivity). The increase of locality in discretization, which enhances the degree of parallelizability, is one of the main advantages. In addition, DG methods permit an easy treatment of nonhomogeneous boundary conditions, which greatly increases the robustness and accuracy of any boundary condition implementation.

For elliptic problems, there are two basic ways to construct DG methods. The first way is to replace the bilinear form of a weak formulation by a new bilinear form with a penalty term penalizing the inter-element discontinuity, see e.g. [4, 10, 15, 19]. The second one is to choose suitable numerical fluxes to make the DG schemes consistent, conservative and stable, see e.g. [5, 11, 13]. In [1] and [2], Arnold, Brezzi, Cockburn and Marini provided a unified error analysis of DG methods for elliptic problems and succeeded in building a bridge between these two families, establishing a framework to understand their properties, differences and the connections between them. In [20], a priori error estimates were established for the DG methods for solving an obstacle problem and a simplified frictional problem, which reach optimal order for linear elements. We will extend the ideas therein to solve the Signorini problem with DG methods.

The paper is organized as follows: In Section 2 we introduce the Signorini problem and the DG formulations for solving it. Then we show the consistency of the DG schemes, boundedness and stability of the bilinear forms in Section 3. In Section 4, we establish a priori error estimates for these DG methods. In the last section, we present results from a numerical example, paying particular attention to numerical convergence orders.

2 Signorini problem and DG formulations

2.1 Signorini problem and its weak formulation

The Signorini problem is an elastostatics problem describing the contact of a deformable body with a rigid frictionless foundation. It is an example of an elliptic variational inequality of the first kind. Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be an open bounded connected domain with a Lipschitz boundary Γ that is divided into three parts $\overline{\Gamma}_D$, $\overline{\Gamma}_F$ and $\overline{\Gamma}_C$ with Γ_D , Γ_F and Γ_C relatively open and mutually disjoint such that $\text{meas}(\Gamma_D) > 0$. The displacement $\mathbf{u} : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a vector valued function. The linearized strain tensor is

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^t).$$

Consider a homogeneous, isotropic, linearized elastic material. Then the stress tensor is

$$\boldsymbol{\sigma} = \lambda (\text{tr } \boldsymbol{\varepsilon}) \mathbf{I} + 2 \mu \boldsymbol{\varepsilon}, \tag{2.1}$$

where $\lambda > 0$ and $\mu > 0$ are Lamé coefficients. The linearized strain and stress tensors are second order symmetric tensors, which take values in \mathbb{S}^d , the space of second order symmetric tensors on \mathbb{R}^d with the inner product $\boldsymbol{\sigma} : \boldsymbol{\tau} = \sigma_{ij} \tau_{ij}$. Let $\boldsymbol{\nu}$ be the unit outward normal to Γ . For a vector \mathbf{v} , denote its normal component and tangential component by $v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}$ and $\mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}$ on

the boundary. Similarly for a tensor-valued function $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$, we define its normal component $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and tangential component $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu\boldsymbol{\nu}$. We have the decomposition formula

$$(\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{v} = (\sigma_\nu\boldsymbol{\nu} + \boldsymbol{\sigma}_\tau) \cdot (v_\nu\boldsymbol{\nu} + \boldsymbol{v}_\tau) = \sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau.$$

Given $\boldsymbol{f} \in [L^2(\Omega)]^d$, $\boldsymbol{g} \in [L^2(\Gamma_F)]^d$, the Signorini problem is to find a displacement field $\boldsymbol{u} : \Omega \rightarrow \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \rightarrow \mathbb{S}^d$ such that ([17])

$$\frac{1}{2\mu}\boldsymbol{\sigma} - \frac{\lambda}{2\mu(d\lambda + 2\mu)}\text{tr}(\boldsymbol{\sigma})\boldsymbol{I} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in } \Omega, \quad (2.2)$$

$$-\text{div } \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega, \quad (2.3)$$

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_D, \quad (2.4)$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_F, \quad (2.5)$$

$$u_\nu \leq 0, \quad \sigma_\nu \leq 0, \quad \sigma_\nu u_\nu = 0, \quad \boldsymbol{\sigma}_\tau = \mathbf{0} \quad \text{on } \Gamma_C. \quad (2.6)$$

Here (2.2) follows from the constitutive relation of the elastic material, (2.3) is the equilibrium equation, in which volume forces of density \boldsymbol{f} acts in Ω . Boundary condition (2.4) means that the body is clamped on Γ_D and so the displacement field vanishes there. Surface tractions of density \boldsymbol{g} act on Γ_F in (2.5). The body is in frictionless contact with a rigid foundation on Γ_C .

For a tensor valued function $\boldsymbol{\sigma}$, define its divergence by

$$\text{div } \boldsymbol{\sigma} = (\partial_j \sigma_{ij})_{1 \leq i \leq d}.$$

Then, for any symmetric tensor $\boldsymbol{\sigma}$ and any vector field \boldsymbol{v} , both being continuously differentiable over $\overline{\Omega}$, we have the following integration by parts formula:

$$\int_{\Omega} \text{div } \boldsymbol{\sigma} \cdot \boldsymbol{v} \, dx = \int_{\Gamma} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \boldsymbol{v} \, ds - \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \, dx. \quad (2.7)$$

To give the weak formulation of the Signorini problem, we define

$$Q = \{ \boldsymbol{\tau} = (\tau_{ij}) \in L^2(\Omega)^{d \times d} \mid \tau_{ij} = \tau_{ji}, 1 \leq i, j \leq d \}, \quad (2.8)$$

$$V = \{ \boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v} = \mathbf{0} \text{ on } \Gamma_D \}, \quad (2.9)$$

$$K = \{ \boldsymbol{v} \in V \mid v_\nu \leq 0 \text{ a.e. on } \Gamma_C \}. \quad (2.10)$$

The admissible set K is non-empty, closed and convex. Following a standard argument ([16]), we proceed to derive a weak formulation of the problem (2.2)–(2.6). For an arbitrary smooth vector-valued function $\boldsymbol{v} \in K$, multiplying the equation (2.3) by $(\boldsymbol{v} - \boldsymbol{u})$ and integrating over Ω , we obtain by (2.7)

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}) \, dx = \int_{\Omega} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, dx + \int_{\Gamma} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, ds.$$

We use the boundary conditions (2.4) and (2.5) to deduce the equality

$$\int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{u}) \, dx = \int_{\Omega} \boldsymbol{f} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, dx + \int_{\Gamma_F} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, ds + \int_{\Gamma_C} \boldsymbol{\sigma}\boldsymbol{\nu} \cdot (\boldsymbol{v} - \boldsymbol{u}) \, ds. \quad (2.11)$$

Over Γ_C , we have

$$\boldsymbol{\sigma}\boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}) = \sigma_\nu(v_\nu - u_\nu) + \boldsymbol{\sigma}_\tau \cdot (\mathbf{v}_\tau - \mathbf{u}_\tau) = \sigma_\nu v_\nu,$$

where the boundary conditions (2.6) are used. Note that over Γ_C , we also have $\sigma_\nu \leq 0$ and $v_\nu \leq 0$. Then

$$\boldsymbol{\sigma}\boldsymbol{\nu} \cdot (\mathbf{v} - \mathbf{u}) \geq 0 \quad \text{on } \Gamma_C.$$

Therefore, we derive from (2.11) that

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u}) \, dx \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) \, dx + \int_{\Gamma_F} \mathbf{g} \cdot (\mathbf{v} - \mathbf{u}) \, ds.$$

From the boundary conditions (2.4) and (2.6), we know $\mathbf{u} \in K$. By the above argument, the variational formulation of the Signorini problem (2.2)–(2.6) is: Find a displacement field $\mathbf{u} \in K$ such that

$$a(\boldsymbol{\sigma}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v} - \mathbf{u})) \geq \ell(\mathbf{v} - \mathbf{u}) \quad \forall \mathbf{v} \in K, \quad (2.12)$$

where $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{u})$ is given by (2.1), the bilinear form $a(\cdot, \cdot)$ and the linear form $\ell \in V'$ are

$$a(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} \, dx \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in Q, \quad (2.13)$$

$$\ell(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Gamma_F} \mathbf{g} \cdot \mathbf{v} \, ds \quad \forall \mathbf{v} \in V. \quad (2.14)$$

This problem has a unique solution ([17]).

2.2 Notations and DG formulations

For definiteness, in the following, we only consider the case $d = 2$, although the discussion can be adapted to the three dimensional case. Given a bounded domain $D \subset \mathbb{R}^2$ and a positive integer m , $H^m(D)$ is the usual Sobolev space with the corresponding norm $\|\cdot\|_{m,D}$ and semi-norm $|\cdot|_{m,D}$. Let $\mathbf{u} = (u_1, u_2)^t \in [H^m(D)]^2$ and define the corresponding norm and semi-norm by $\|\mathbf{u}\|_{m,D}^2 = \sum_{i=1}^2 \|u_i\|_{m,D}^2$ and $|\mathbf{u}|_{m,D}^2 = \sum_{i=1}^2 |u_i|_{m,D}^2$. Similarly, $\boldsymbol{\tau} \in [L^2(\Omega)]_s^{2 \times 2}$ is a matrix-valued function with each component $\tau_{ij} \in L^2(\Omega)$ and $\tau_{12} = \tau_{21}$. We assume Ω is a polygonal domain and consider a regular family of triangulations of $\bar{\Omega}$ by $\{\mathcal{T}_h\}_h$ that are compatible with the boundary splitting $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_F \cup \bar{\Gamma}_C$, i.e., if an element edge has a non-empty intersection with one of the sets Γ_D , Γ_F and Γ_C , then the edge lies entirely in the corresponding closed set $\bar{\Gamma}_D$, $\bar{\Gamma}_F$, or $\bar{\Gamma}_C$. Let $h_K = \text{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. Denote by \mathcal{E}_h the union of the boundaries of the elements K of \mathcal{T}_h , $\mathcal{E}_h^i = \mathcal{E}_h \setminus \Gamma$ the set of all interior edges, and $\mathcal{E}_h^0 = \mathcal{E}_h \setminus (\Gamma_F \cup \Gamma_C)$.

We introduce the following finite element spaces:

$$V_h = \{\mathbf{v}_h \in [L^2(\Omega)]^2 : v_{hi}|_K \in P_1(K) \, \forall K \in \mathcal{T}_h, i = 1, 2\},$$

$$W_h = \{\boldsymbol{\tau}_h \in [L^2(\Omega)]_s^{2 \times 2} : \tau_{hij}|_K \in P_l(K) \, \forall K \in \mathcal{T}_h, i, j = 1, 2\}, \quad l = 0 \text{ or } 1.$$

We define the finite element set K_h to approximate K as follows:

$$K_h = \{\mathbf{v}_h \in V_h : v_{h\nu}(\mathbf{x}) \leq 0 \text{ for all nodes } \mathbf{x} \in \overline{\Gamma_C}\}.$$

Because $v_{h\nu} \in K_h$ is a linear finite element function, $v_{h\nu} \leq 0$ at all nodes on $\overline{\Gamma_C}$ ensures $v_{h\nu} \leq 0$ on $\overline{\Gamma_C}$. For all vector-valued function \mathbf{v} and matrix-valued function $\boldsymbol{\tau}$, $\boldsymbol{\varepsilon}_h(\mathbf{v})$ and $\text{div}_h \boldsymbol{\tau}$ are defined by the relations $\boldsymbol{\varepsilon}_h(\mathbf{v}) = \boldsymbol{\varepsilon}(\mathbf{v})$ and $\text{div}_h \boldsymbol{\tau} = \text{div} \boldsymbol{\tau}$ on any element $K \in \mathcal{T}_h$.

Let e be an edge shared by two elements K^+ and K^- , and $\mathbf{n}^\pm = \mathbf{n}|_{\partial K^\pm}$ be the unit outward normal vector on ∂K^\pm . For a scalar function w , let $w^\pm = w|_{\partial K^\pm}$ and similarly, for a vector-valued function \mathbf{v} and a matrix-valued function $\boldsymbol{\tau}$, let $\mathbf{v}^\pm = \mathbf{v}|_{\partial K^\pm}$, $\boldsymbol{\tau}^\pm = \boldsymbol{\tau}|_{\partial K^\pm}$. Then define the averages $\{\cdot\}$ and the jumps $[\cdot]$, $\llbracket \cdot \rrbracket$ on $e \in \mathcal{E}_h^i$ by

$$\begin{aligned} \{w\} &= \frac{1}{2}(w^+ + w^-), & [w] &= w^+ \mathbf{n}^+ + w^- \mathbf{n}^-, \\ \{\mathbf{v}\} &= \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-), & \llbracket \mathbf{v} \rrbracket &= \frac{1}{2}(\mathbf{v}^+ \otimes \mathbf{n}^+ + \mathbf{n}^+ \otimes \mathbf{v}^+ + \mathbf{v}^- \otimes \mathbf{n}^- + \mathbf{n}^- \otimes \mathbf{v}^-), \\ \{\boldsymbol{\tau}\} &= \frac{1}{2}(\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-), & [\boldsymbol{\tau}] &= \boldsymbol{\tau}^+ \mathbf{n}^+ + \boldsymbol{\tau}^- \mathbf{n}^-. \end{aligned}$$

If e lies on the boundary Γ , we set

$$\begin{aligned} \{w\} &= w, & [w] &= w \boldsymbol{\nu}, \\ \{\mathbf{v}\} &= \mathbf{v}, & \llbracket \mathbf{v} \rrbracket &= \frac{1}{2}(\mathbf{v} \otimes \boldsymbol{\nu} + \boldsymbol{\nu} \otimes \mathbf{v}), \\ \{\boldsymbol{\tau}\} &= \boldsymbol{\tau}, & [\boldsymbol{\tau}] &= \boldsymbol{\tau} \boldsymbol{\nu}. \end{aligned}$$

Here $\mathbf{u} \otimes \mathbf{v}$ is a matrix with $u_i v_j$ as its (i, j) -th element.

For a vector-valued function \mathbf{v} and a matrix-valued function $\boldsymbol{\tau}$, after a direct manipulation, we have

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\boldsymbol{\tau} \mathbf{n}) \cdot \mathbf{v} \, ds = \sum_{e \in \mathcal{E}_h^i} \int_e [\boldsymbol{\tau}] \cdot \{\mathbf{v}\} \, ds + \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} : \llbracket \mathbf{v} \rrbracket \, ds. \quad (2.15)$$

To give the DG formulations, we need lifting operators $\mathbf{r}_0 : (L^2(\mathcal{E}_h^0))_s^{2 \times 2} \rightarrow W_h$, $\mathbf{r}_e : (L^2(e))_s^{2 \times 2} \rightarrow W_h$ defined by

$$\int_{\Omega} \mathbf{r}_0(\boldsymbol{\phi}) : \boldsymbol{\tau} \, dx = - \int_{\mathcal{E}_h^0} \boldsymbol{\phi} : \{\boldsymbol{\tau}\} \, ds \quad \forall \boldsymbol{\tau} \in W_h, \boldsymbol{\phi} \in (L^2(\mathcal{E}_h^0))_s^{2 \times 2}, \quad (2.16)$$

$$\int_{\Omega} \mathbf{r}_e(\boldsymbol{\phi}) : \boldsymbol{\tau} \, dx = - \int_e \boldsymbol{\phi} : \{\boldsymbol{\tau}\} \, ds \quad \forall \boldsymbol{\tau} \in W_h, \boldsymbol{\phi} \in (L^2(e))_s^{2 \times 2}. \quad (2.17)$$

It is easy to check that the following identity holds

$$\mathbf{r}_0(\boldsymbol{\phi}) = \sum_{e \in \mathcal{E}_h^0} \mathbf{r}_e(\boldsymbol{\phi}|_e) \quad \forall \boldsymbol{\phi} \in (L^2(\mathcal{E}_h^0))_s^{2 \times 2},$$

so we have

$$\|\mathbf{r}_0(\boldsymbol{\phi})\|^2 = \left\| \sum_{e \in \mathcal{E}_h^0} \mathbf{r}_e(\boldsymbol{\phi}|_e) \right\|^2 \leq 3 \sum_{e \in \mathcal{E}_h^0} \|\mathbf{r}_e(\boldsymbol{\phi}|_e)\|^2. \quad (2.18)$$

We now present some DG methods for the Signorini problem (2.2)–(2.6). We multiply the equations (2.2) and (2.3) by test functions $\boldsymbol{\tau}$ and \mathbf{v} , respectively, and integrate on a subset $D \subset \Omega$. We get by (2.7),

$$\int_D \left(\frac{1}{2\mu} \boldsymbol{\sigma} : \boldsymbol{\tau} - \frac{\lambda}{4\mu(\lambda + \mu)} \text{tr}(\boldsymbol{\sigma}) \text{tr}(\boldsymbol{\tau}) \right) dx = - \int_D \mathbf{u} \cdot \text{div} \boldsymbol{\tau} dx + \int_{\partial D} \mathbf{u} \cdot (\boldsymbol{\tau} \mathbf{n}) ds, \quad (2.19)$$

$$\int_D \mathbf{f} \cdot \mathbf{v} dx = \int_D \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{v}) dx - \int_{\partial D} (\boldsymbol{\sigma} \mathbf{n}) \cdot \mathbf{v} ds. \quad (2.20)$$

In above equations, we append subscript h on $\boldsymbol{\sigma}$, \mathbf{u} , div and $\boldsymbol{\varepsilon}$, add over all the elements, and use numerical traces $\widehat{\mathbf{u}}_h$ and $\widehat{\boldsymbol{\sigma}}_h$ to approximate \mathbf{u} and $\boldsymbol{\sigma}$ over element edges to obtain

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{2\mu} \boldsymbol{\sigma}_h : \boldsymbol{\tau}_h - \frac{\lambda}{4\mu(\lambda + \mu)} \text{tr}(\boldsymbol{\sigma}_h) \text{tr}(\boldsymbol{\tau}_h) \right) dx &= - \int_{\Omega} \mathbf{u}_h \cdot \text{div}_h \boldsymbol{\tau}_h dx \\ &+ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \widehat{\mathbf{u}}_h \cdot (\boldsymbol{\tau}_h \mathbf{n}_K) ds, \end{aligned} \quad (2.21)$$

$$\begin{aligned} \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h dx &= \int_{\Omega} \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) dx - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\widehat{\boldsymbol{\sigma}}_h \mathbf{n}_K) \cdot \mathbf{v}_h ds, \end{aligned} \quad (2.22)$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h) \in W_h \times V_h$ and all $K \in \mathcal{T}_h$. The numerical traces $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{\mathbf{u}}_h$ will be selected to guarantee consistency and stability of the above scheme.

To derive a new formulation which does not rely on $\boldsymbol{\sigma}_h$ explicitly, using (2.7) and (2.15), we have from (2.21) and (2.22) that

$$\begin{aligned} \int_{\Omega} \left(\frac{1}{2\mu} \boldsymbol{\sigma}_h : \boldsymbol{\tau}_h - \frac{\lambda}{4\mu(\lambda + \mu)} \text{tr}(\boldsymbol{\sigma}_h) \text{tr}(\boldsymbol{\tau}_h) \right) dx &= \int_{\Omega} \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\tau}_h dx + \int_{\mathcal{E}_h^i} \{\widehat{\mathbf{u}}_h - \mathbf{u}_h\} \cdot [\boldsymbol{\tau}_h] ds \\ &+ \int_{\mathcal{E}_h} \llbracket \widehat{\mathbf{u}}_h - \mathbf{u}_h \rrbracket : \{\boldsymbol{\tau}_h\} ds, \end{aligned} \quad (2.23)$$

$$\begin{aligned} \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h dx &= \int_{\Omega} \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) dx - \int_{\mathcal{E}_h^i} \llbracket \widehat{\boldsymbol{\sigma}}_h \rrbracket \cdot \{\mathbf{v}_h\} ds \\ &- \int_{\mathcal{E}_h} \llbracket \mathbf{v}_h \rrbracket : \{\widehat{\boldsymbol{\sigma}}_h\} ds. \end{aligned} \quad (2.24)$$

Choosing $\boldsymbol{\tau}_h = 2\mu \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h)) \mathbf{I}$ in (2.23), we get

$$\begin{aligned} \int_{\Omega} \boldsymbol{\sigma}_h : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) dx &= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) dx \\ &+ \int_{\mathcal{E}_h^i} \{\widehat{\mathbf{u}}_h - \mathbf{u}_h\} \cdot (2\mu \llbracket \boldsymbol{\varepsilon}_h(\mathbf{v}_h) \rrbracket + \lambda \llbracket \text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h)) \rrbracket) ds \\ &+ \int_{\mathcal{E}_h} \llbracket \widehat{\mathbf{u}}_h - \mathbf{u}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \text{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) ds. \end{aligned}$$

The combination of the last equation and (2.24) yields

$$\begin{aligned}
& \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) dx \\
& + \int_{\mathcal{E}_h^i} \{\widehat{\mathbf{u}}_h - \mathbf{u}_h\} \cdot (2\mu[\boldsymbol{\varepsilon}_h(\mathbf{v}_h)] + \lambda[\operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))]) ds \\
& + \int_{\mathcal{E}_h} \llbracket \widehat{\mathbf{u}}_h - \mathbf{u}_h \rrbracket : (2\mu\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) ds \\
& - \int_{\mathcal{E}_h^i} [\widehat{\boldsymbol{\sigma}}_h] \cdot \{\mathbf{v}_h\} ds - \int_{\mathcal{E}_h} \llbracket \mathbf{v}_h \rrbracket : \{\widehat{\boldsymbol{\sigma}}_h\} ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h dx. \tag{2.25}
\end{aligned}$$

We can get DG methods from (2.25) by proper choices of numerical traces $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{\mathbf{u}}_h$. There are three principles for choosing appropriate numerical traces. Conservation requires the numerical traces to be single-valued over all edges; consistency of the numerical traces requires $\widehat{\mathbf{u}}_h(\mathbf{u}) = \mathbf{u}|_{\mathcal{E}_h}$ and $\widehat{\boldsymbol{\sigma}}_h(\boldsymbol{\sigma}) = \boldsymbol{\sigma}|_{\mathcal{E}_h}$; stability is not easily ensured and it is usual to add a suitable penalty term (stability term) to guarantee it. We will introduce five consistent and stable DG methods. For example, take

$$\begin{cases} \widehat{\mathbf{u}}_h = \{\mathbf{u}_h\} & \text{on } \mathcal{E}_h \setminus \Gamma_D, & \widehat{\mathbf{u}}_h = \mathbf{0} & \text{on } \Gamma_D, \\ \widehat{\boldsymbol{\sigma}}_h = 2\mu\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I} - \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket & \text{on } \mathcal{E}_h^0, \\ \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = \mathbf{g} & \text{on } \Gamma_F, & \widehat{\boldsymbol{\sigma}}_{h\tau} = \mathbf{0}, & \widehat{\sigma}_{h\nu} \leq 0, & \widehat{\sigma}_{h\nu} u_{h\nu} = 0 & \text{on } \Gamma_C, \end{cases}$$

where the function η equals a constant η_e on each $e \in \mathcal{E}_h^0$, with $\{\eta_e\}_{e \in \mathcal{E}_h^0}$ having a uniform positive bound from above and below. We obtain from (2.25) that

$$B_{1,h}^{(1)}(\mathbf{u}_h, \mathbf{v}_h) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h dx + \int_{\Gamma_F} \mathbf{g} \cdot \mathbf{v}_h ds + \int_{\Gamma_C} \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} \cdot \mathbf{v}_h ds, \tag{2.26}$$

where

$$\begin{aligned}
B_{1,h}^{(1)}(\mathbf{u}_h, \mathbf{v}_h) & := \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) dx \\
& - \int_{\mathcal{E}_h^0} \llbracket \mathbf{u}_h \rrbracket : (2\mu\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) ds \\
& - \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h \rrbracket : (2\mu\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I}) ds + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds. \tag{2.27}
\end{aligned}$$

Let $\mathbf{v}_h = \mathbf{w}_h - \mathbf{u}_h$ with $\mathbf{w}_h \in K_h$. Since $\widehat{\boldsymbol{\sigma}}_{h\tau} = \mathbf{0}$, $\widehat{\sigma}_{h\nu} \leq 0$, $\widehat{\sigma}_{h\nu} u_{h\nu} = 0$ on Γ_C , the equation (2.26) leads to the inequality

$$B_{1,h}^{(1)}(\mathbf{u}_h, \mathbf{w}_h - \mathbf{u}_h) \geq \int_{\Omega} \mathbf{f} \cdot (\mathbf{w}_h - \mathbf{u}_h) dx + \int_{\Gamma_F} \mathbf{g} \cdot (\mathbf{w}_h - \mathbf{u}_h) ds \quad \forall \mathbf{w}_h \in K_h.$$

The term $\int_{\mathcal{E}_h^0} \eta h_e^{-1} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds$ is the penalty term. This is the interior penalty (IP) formulation ([15]). With the lift operator \mathbf{r}_0 , we can rewrite $B_{1,h}^{(1)}$ as

$$\begin{aligned} B_{2,h}^{(1)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : (\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket))) dx \\ &\quad + \int_{\Omega} \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket) : (2\mu \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h)) \mathbf{I}) dx + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds. \end{aligned} \quad (2.28)$$

Note that (2.27) and (2.28) are equivalent on V_h , implying that either one can be used to define the numerical solution u_h . In this paper, we give a priori error estimate for the first formula (2.27). Because (2.27) and (2.28) are equivalent on V_h , we will prove the stability for the second formula $B_{2,h}^{(1)}$ on V_h , which guarantees the stability for the first formulation $B_{1,h}^{(1)}$ on V_h . This comment is valid for the other DG methods to be introduced later.

By changing the sign of the second term in the bilinear form $B_{1,h}^{(1)}$, we can give a non-symmetric interior penalty (NIPG) formulation (see [19]),

$$\begin{aligned} B_{1,h}^{(2)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) dx \\ &\quad + \int_{\mathcal{E}_h^0} \llbracket \mathbf{u}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) ds \\ &\quad - \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I}) ds + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds, \end{aligned}$$

or equivalently,

$$\begin{aligned} B_{2,h}^{(2)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : (\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket))) dx \\ &\quad - \int_{\Omega} \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket) : (2\mu \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h)) \mathbf{I}) dx + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds. \end{aligned}$$

Using the local lifting operator \mathbf{r}_e , we can give the third example. Taking

$$\begin{cases} \widehat{\mathbf{u}}_h = \{\mathbf{u}_h\} & \text{on } \mathcal{E}_h \setminus \Gamma_D, & \widehat{\mathbf{u}}_h = \mathbf{0} & \text{on } \Gamma_D, \\ \widehat{\boldsymbol{\sigma}}_h = 2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I} + 2\mu \{\mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)\} + \lambda \{\operatorname{tr}(\mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket))\} \mathbf{I} \\ \quad + 2\mu \{\eta \mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket)\} + \lambda \{\eta \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket))\} \mathbf{I} & \text{on } \mathcal{E}_h^0, \\ \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = \mathbf{g} & \text{on } \Gamma_F, & \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\tau} = \mathbf{0}, & \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} \leq 0, & \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} u_{h\nu} = 0 & \text{on } \Gamma_C, \end{cases}$$

from (2.25), we get

$$\begin{aligned}
B_{1,h}^{(3)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) \, dx \\
&\quad - \int_{\mathcal{E}_h^0} \llbracket \mathbf{u}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) \, ds \\
&\quad - \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I}) \, ds \\
&\quad + \int_{\Omega} \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket) : (2\mu \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket) + \lambda \operatorname{tr}(\mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)) \mathbf{I}) \, dx \\
&\quad + \sum_{e \in \mathcal{E}_h^0} \int_{\Omega} \eta (2\mu \mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket) : \mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket) + \lambda \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket)) \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket))) \, dx,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
B_{2,h}^{(3)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} 2\mu (\boldsymbol{\varepsilon}_h(\mathbf{u}_h) + \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)) : (\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) \, dx \\
&\quad + \int_{\Omega} \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h) + \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) \, dx \\
&\quad + \sum_{e \in \mathcal{E}_h^0} \int_{\Omega} \eta (2\mu \mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket) : \mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket) + \lambda \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket)) \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket))) \, dx,
\end{aligned}$$

which is an extension of the method of Brezzi et al. [9].

With the choice

$$\begin{cases} \widehat{\mathbf{u}}_h = \{\mathbf{u}_h\} & \text{on } \mathcal{E}_h \setminus \Gamma_D, & \widehat{\mathbf{u}}_h = \mathbf{0} & \text{on } \Gamma_D, \\ \widehat{\boldsymbol{\sigma}}_h = 2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I} + 2\mu \{\eta \mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket)\} + \lambda \{\eta \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket))\} \mathbf{I} & \text{on } \mathcal{E}_h^0, \\ \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = \mathbf{g} & \text{on } \Gamma_F, & \widehat{\boldsymbol{\sigma}}_{h\tau} = \mathbf{0}, & \widehat{\sigma}_{h\nu} \leq 0, & \widehat{\sigma}_{h\nu} u_{h\nu} = 0 & \text{on } \Gamma_C, \end{cases}$$

we obtain a DG formulation extended from the method of Bassi et al. [6],

$$\begin{aligned}
B_{1,h}^{(4)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) \, dx \\
&\quad - \int_{\mathcal{E}_h^0} \llbracket \mathbf{u}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) \, ds \\
&\quad - \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I}) \, ds \\
&\quad + \sum_{e \in \mathcal{E}_h^0} \int_{\Omega} \eta (2\mu \mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket) : \mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket) + \lambda \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket)) \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket))) \, dx,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
B_{2,h}^{(4)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : (\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket))) dx \\
&+ \int_{\Omega} \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket) : (2\mu \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h)) \mathbf{I}) dx \\
&+ \sum_{e \in \mathcal{E}_h^0} \int_{\Omega} \eta (2\mu \mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket) : \mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket) + \lambda \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{u}_h \rrbracket)) \operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v}_h \rrbracket))) dx.
\end{aligned}$$

If we choose

$$\left\{ \begin{array}{l}
\widehat{\mathbf{u}}_h = \{\mathbf{u}_h\} \quad \text{on } \mathcal{E}_h \setminus \Gamma_D, \quad \widehat{\mathbf{u}}_h = \mathbf{0} \quad \text{on } \Gamma_D, \\
\widehat{\boldsymbol{\sigma}}_h = 2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I} + 2\mu \{\mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)\} + \lambda \{\mathbf{r}_0(\operatorname{tr}(\llbracket \mathbf{u}_h \rrbracket) \mathbf{I})\} - \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket \quad \text{on } \mathcal{E}_h^0, \\
\widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} = \mathbf{g} \quad \text{on } \Gamma_F, \quad \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\tau} = \mathbf{0}, \quad \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} \leq 0, \quad \widehat{\boldsymbol{\sigma}}_h \boldsymbol{\nu} u_{h\nu} = 0 \quad \text{on } \Gamma_C,
\end{array} \right.$$

then

$$\begin{aligned}
B_{1,h}^{(5)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}_h(\mathbf{u}_h) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h))) dx \\
&- \int_{\mathcal{E}_h^0} \llbracket \mathbf{u}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{v}_h)\}) \mathbf{I}) ds \\
&- \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h \rrbracket : (2\mu \{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\} + \lambda \operatorname{tr}(\{\boldsymbol{\varepsilon}_h(\mathbf{u}_h)\}) \mathbf{I}) ds \\
&+ \int_{\Omega} \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket) : (2\mu \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket) + \lambda \operatorname{tr}(\mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)) \mathbf{I}) dx + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds,
\end{aligned}$$

or equivalently,

$$\begin{aligned}
B_{2,h}^{(5)}(\mathbf{u}_h, \mathbf{v}_h) &:= \int_{\Omega} 2\mu (\boldsymbol{\varepsilon}_h(\mathbf{u}_h) + \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)) : (\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) dx \\
&+ \int_{\Omega} \lambda \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_h) + \mathbf{r}_0(\llbracket \mathbf{u}_h \rrbracket)) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h) + \mathbf{r}_0(\llbracket \mathbf{v}_h \rrbracket)) dx + \int_{\mathcal{E}_h^0} \frac{\eta}{h_e} \llbracket \mathbf{u}_h \rrbracket : \llbracket \mathbf{v}_h \rrbracket ds,
\end{aligned}$$

which is an extension of LDG method [13].

Let $B_h(\mathbf{u}_h, \mathbf{v}_h)$ be one of the bilinear forms $B_{1,h}^{(j)}(\mathbf{u}_h, \mathbf{v}_h)$ with $j = 1, \dots, 5$. Then a DG method for the Signorini problem (2.12) is: Find $\mathbf{u}_h \in K_h$ such that

$$B_h(\mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) \geq \ell(\mathbf{v}_h - \mathbf{u}_h) \quad \forall \mathbf{v}_h \in K_h. \quad (2.29)$$

3 Consistency, boundedness and stability

We notice that if the solution of (2.12) has the regularity $\mathbf{u} \in [H^2(\Omega)]^2$, then \mathbf{u} is the solution of (2.2)–(2.6), and on any interior edge e , $\llbracket \mathbf{u} \rrbracket = \mathbf{0}$, $\{\mathbf{u}\} = \mathbf{u}$, $\{\boldsymbol{\varepsilon}(\mathbf{u})\} = \boldsymbol{\varepsilon}(\mathbf{u})$, $[\boldsymbol{\sigma}] = \mathbf{0}$, $\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}$. For all DG methods introduced in the previous section, we first show the consistency of the DG schemes.

Lemma 3.1 (Consistency) Assume $\mathbf{u} \in [H^2(\Omega)]^2$ is the solution of (2.12). Then for the DG methods $B_h(\mathbf{w}, \mathbf{v}) = B_{1,h}^{(j)}(\mathbf{w}, \mathbf{v})$ with $j = 1, \dots, 5$, we have

$$B_h(\mathbf{u}, \mathbf{v}_h - \mathbf{u}) \geq \ell(\mathbf{v}_h - \mathbf{u}) \quad \forall \mathbf{v}_h \in K_h. \quad (3.1)$$

Proof. Using (2.1), we obtain, for any $\mathbf{v}_h \in K_h$,

$$\begin{aligned} B_h(\mathbf{u}, \mathbf{v}_h - \mathbf{u}) &= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}_h(\mathbf{v}_h - \mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v}_h - \mathbf{u}))) dx \\ &\quad - \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h - \mathbf{u} \rrbracket : (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I}) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\mathbf{v}_h - \mathbf{u}) dx - \int_{\mathcal{E}_h^0} \llbracket \mathbf{v}_h - \mathbf{u} \rrbracket : \boldsymbol{\sigma} ds. \end{aligned}$$

By (2.7), (2.15) and noting $[\boldsymbol{\sigma}] = \mathbf{0}$ on \mathcal{E}_h^i , we get

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\mathbf{v}_h - \mathbf{u}) dx &= \sum_{K \in \mathcal{T}_h} \int_K -\operatorname{div} \boldsymbol{\sigma} \cdot (\mathbf{v}_h - \mathbf{u}) dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\boldsymbol{\sigma} \mathbf{n}_K) \cdot (\mathbf{v}_h - \mathbf{u}) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K -\operatorname{div} \boldsymbol{\sigma} \cdot (\mathbf{v}_h - \mathbf{u}) dx + \int_{\mathcal{E}_h} \llbracket \mathbf{v}_h - \mathbf{u} \rrbracket : \boldsymbol{\sigma} ds. \end{aligned}$$

Then

$$\begin{aligned} B_h(\mathbf{u}, \mathbf{v}_h - \mathbf{u}) &= \int_{\Omega} \mathbf{f} \cdot (\mathbf{v}_h - \mathbf{u}) dx + \int_{\Gamma_F} \mathbf{g} \cdot (\mathbf{v}_h - \mathbf{u}) dx + \int_{\Gamma_C} (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot (\mathbf{v}_h - \mathbf{u}) dx \\ &= \ell(\mathbf{v}_h - \mathbf{u}) + \int_{\Gamma_C} \sigma_{\nu} v_{h\nu} ds \geq \ell(\mathbf{v}_h - \mathbf{u}). \end{aligned}$$

The last inequality is obtained by (2.6) and $v_{h\nu} \leq 0$ for all $\mathbf{v}_h \in K_h$. Hence, (3.1) holds. \blacksquare

To consider the boundedness and stability of the bilinear form B_h , as in [20], let $V(h) = V_h + V \cap [H^2(\Omega)]^2$, and for $\mathbf{v} \in V(h)$, define seminorms as follows:

$$|\mathbf{v}|_K^2 := \int_K \boldsymbol{\varepsilon}(\mathbf{v}) : \boldsymbol{\varepsilon}(\mathbf{v}) dx, \quad |\mathbf{v}|_h^2 := \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_K^2, \quad |\mathbf{v}|_*^2 := \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2,$$

where

$$\|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2 = \int_e \llbracket \mathbf{v} \rrbracket : \llbracket \mathbf{v} \rrbracket ds.$$

Then define norms by

$$\|\mathbf{v}\|_*^2 := |\mathbf{v}|_h^2 + |\mathbf{v}|_*^2, \quad \|\mathbf{v}\|^2 := \|\mathbf{v}\|_*^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\mathbf{v}|_{2,K}^2. \quad (3.2)$$

The norm $\|\cdot\|_*$ defined in (3.2) is equivalent to the usual DG-norm $(|\cdot|_{1,h}^2 + |\cdot|_*^2)^{1/2}$, thanks to Korn's inequality (see [7]; see also Proposition 4.6 of [3]).

Before presenting the boundedness and stability of the bilinear forms, we give a useful estimate for the lifting operator \mathbf{r}_e . The following lemma is a trivial extension to vectors \mathbf{v} of Lemma 2 of [10], also restated in [2].

Lemma 3.2 *For any $\mathbf{v} \in V(h)$ and $e \in \mathcal{E}_h^0$,*

$$C_1 h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2 \leq \|\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 \leq C_2 h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2. \quad (3.3)$$

From (3.3) and (2.18), we have

$$\|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 = \left\| \sum_{e \in \mathcal{E}_h^0} \mathbf{r}_e(\llbracket \mathbf{v} \rrbracket) \right\|_{0,h}^2 \leq 3C_2 \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2 = 3C_2 |\mathbf{v}|_*^2.$$

To consider the boundedness of the primal forms B_h , noting $\|\text{tr}(\boldsymbol{\tau})\| \leq \|\boldsymbol{\tau}\|$ for a matrix-valued function $\boldsymbol{\tau}$, we use the Cauchy-Schwarz inequality to bound them term by term,

$$\int_{\Omega} \boldsymbol{\varepsilon}_h(\mathbf{w}) : \boldsymbol{\varepsilon}_h(\mathbf{v}) \, dx \leq |\mathbf{w}|_h |\mathbf{v}|_h, \quad (3.4)$$

$$\int_{\Omega} \mathbf{r}_0(\llbracket \mathbf{w} \rrbracket) : \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket) \, dx \lesssim |\mathbf{w}|_* |\mathbf{v}|_*, \quad (3.5)$$

$$\int_{\mathcal{E}_h^0} \eta h_e^{-1} \llbracket \mathbf{w} \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds \leq \sup_{e \in \mathcal{E}_h^0} \eta_e |\mathbf{w}|_* |\mathbf{v}|_*, \quad (3.6)$$

$$\sum_{e \in \mathcal{E}_h^0} \int_{\Omega} \eta \mathbf{r}_e(\llbracket \mathbf{w} \rrbracket) : \mathbf{r}_e(\llbracket \mathbf{v} \rrbracket) \, dx \lesssim \sup_{e \in \mathcal{E}_h^0} \eta_e |\mathbf{w}|_* |\mathbf{v}|_*. \quad (3.7)$$

Here “ $\lesssim \dots$ ” stands for “ $\leq C \dots$ ”, where C is a positive generic constant independent of h and other parameters, which may take different values at different appearances. Using the trace inequality $\|\boldsymbol{\varepsilon}_h(\mathbf{v})\|_{0,e}^2 \lesssim h_e^{-1} |v|_{1,K}^2 + h_e |v|_{2,K}^2$, we have

$$\begin{aligned} \int_{\mathcal{E}_h^0} \llbracket \mathbf{w} \rrbracket : \{\boldsymbol{\varepsilon}_h(\mathbf{v})\} \, ds &\leq \left(\sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\llbracket \mathbf{w} \rrbracket\|_{0,e}^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h^0} h_e \|\{\boldsymbol{\varepsilon}_h(\mathbf{v})\}\|_{0,e}^2 \right)^{1/2} \\ &\lesssim |\mathbf{w}|_* \left(\sum_{K \in \mathcal{T}_h} (|\mathbf{v}|_{1,K}^2 + h_e^2 |\mathbf{v}|_{2,K}^2) \right)^{1/2} \leq |\mathbf{w}|_* \|\mathbf{v}\|. \end{aligned} \quad (3.8)$$

The inequalities (3.4) and (3.8) are needed by all bilinear forms. For the DG methods with the bilinear form $B_{1,h}^{(j)}$, $j = 1, 2, 5$, the inequality (3.6) is needed. The inequality (3.5) is needed by the formulas $B_{1,h}^{(j)}$ with $j = 3, 5$. For the methods with the bilinear forms $B_{1,h}^{(j)}$, $j = 3, 4$, the inequality (3.7) is needed. So we have the following results about the boundedness of B_h .

Lemma 3.3 (Boundedness) For $1 \leq j \leq 5$, $B_h = B_{1,h}^{(j)}$ satisfies

$$B_h(\mathbf{w}, \mathbf{v}) \lesssim \|\mathbf{w}\| \|\mathbf{v}\| \quad \forall \mathbf{w}, \mathbf{v} \in V(h). \quad (3.9)$$

For the stability, note that $\|\mathbf{v}\| = \|\mathbf{v}\|_*$ for any $\mathbf{v} \in V_h$. Since $B_{1,h}^{(j)}$ and $B_{2,h}^{(j)}$ coincide on V_h , once we have proved the stability for $B_{2,h}^{(j)}$ on V_h , the stability of $B_{1,h}^{(j)}$ on V_h follows. We use the Cauchy-Schwarz inequality and Lemma 3.2 to get

$$\begin{aligned} B_{2,h}^{(1)}(\mathbf{v}, \mathbf{v}) &= 2\mu \int_{\Omega} \boldsymbol{\varepsilon}_h(\mathbf{v}) : \boldsymbol{\varepsilon}_h(\mathbf{v}) \, dx + \lambda \int_{\Omega} (\text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v})))^2 \, dx + 4\mu \int_{\Omega} \boldsymbol{\varepsilon}_h(\mathbf{v}) : \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket) \, dx \\ &\quad + 2\lambda \int_{\Omega} \text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v})) \text{tr}(\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)) \, dx + \int_{\mathcal{E}_h^0} \eta h_e^{-1} \llbracket \mathbf{v} \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds \\ &\geq 2\mu |\mathbf{v}|_h^2 + \lambda \|\text{div}_h \mathbf{v}\|_{0,h}^2 - 2\mu \left(\epsilon |\mathbf{v}|_h^2 + \frac{1}{\epsilon} \|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 \right) \\ &\quad - \lambda (\|\text{div}_h \mathbf{v}\|_{0,h}^2 + \|\text{tr}(\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket))\|_{0,h}^2) + \eta_0 \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2 \\ &\geq 2\mu(1 - \epsilon) |\mathbf{v}|_h^2 + \left(\eta_0 - \frac{6\mu C_2}{\epsilon} - 6C_2\lambda \right) |\mathbf{v}|_*^2. \end{aligned}$$

Here $\|\mathbf{v}\|_{0,h}^2 = \sum_{K \in \mathcal{T}_h} \|\mathbf{v}\|_{0,K}^2$, $0 < \epsilon < 1$ is a constant and C_2 is the positive constant in (3.3). Therefore, stability is valid for the IP method when $\min_{e \in \mathcal{E}_h^0} \eta_e = \eta_0 > 6C_2(\mu + \lambda)$.

$$\begin{aligned} B_{2,h}^{(2)}(\mathbf{v}, \mathbf{v}) &= 2\mu \int_{\Omega} \boldsymbol{\varepsilon}_h(\mathbf{v}) : \boldsymbol{\varepsilon}_h(\mathbf{v}) \, dx + \lambda \int_{\Omega} (\text{tr}(\boldsymbol{\varepsilon}_h(\mathbf{v})))^2 \, dx + \int_{\mathcal{E}_h^0} \eta h_e^{-1} \llbracket \mathbf{v} \rrbracket : \llbracket \mathbf{v} \rrbracket \, ds \\ &\geq 2\mu |\mathbf{v}|_h^2 + \eta_0 |\mathbf{v}|_*^2. \end{aligned}$$

So stability holds for the NIPG method for any $\eta_0 > 0$.

$$\begin{aligned} B_{2,h}^{(3)}(\mathbf{v}, \mathbf{v}) &\geq 2\mu \|\boldsymbol{\varepsilon}_h(\mathbf{v}) + \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 + \eta_0 \sum_{e \in \mathcal{E}_h^0} (2\mu \|\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 + \lambda \|\text{tr}(\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket))\|_{0,h}^2) \\ &\geq 2\mu \left(|\mathbf{v}|_h^2 + \|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 + 2 \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\varepsilon}_h(\mathbf{v}) : \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket) \, dx \right) + 2\mu C_1 \eta_0 |\mathbf{v}|_*^2 \\ &\geq 2\mu \left((1 - \epsilon) |\mathbf{v}|_h^2 + (1 - \frac{1}{\epsilon}) \|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 \right) + 2\mu C_1 \eta_0 |\mathbf{v}|_*^2 \\ &\geq 2\mu(1 - \epsilon) |\mathbf{v}|_h^2 + 2\mu \left(3C_2(1 - \frac{1}{\epsilon}) + C_1\eta_0 \right) |\mathbf{v}|_*^2. \end{aligned}$$

Stability holds for $\eta_0 > 0$.

$$\begin{aligned}
B_{2,h}^{(4)}(\mathbf{v}, \mathbf{v}) &\geq 2\mu|\mathbf{v}|_h^2 + \lambda\|\operatorname{div}_h \mathbf{v}\|_{0,h}^2 + 4\mu \int_{\Omega} \boldsymbol{\varepsilon}_h(\mathbf{v}) : \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket) dx + 2\lambda \int_{\Omega} \operatorname{div}_h \mathbf{v} \operatorname{tr}(\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)) dx \\
&\quad + \eta_0 \sum_{e \in \mathcal{E}_h^0} (2\mu\|\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 + \lambda\|\operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket))\|_{0,h}^2) \\
&\geq 2\mu|\mathbf{v}|_h^2 + \lambda\|\operatorname{div}_h \mathbf{v}\|_{0,h}^2 - 2\mu \left(\epsilon|\mathbf{v}|_h^2 + \frac{1}{\epsilon}\|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 \right) - \lambda\|\operatorname{div}_h \mathbf{v}\|_{0,h}^2 \\
&\quad - \lambda\|\operatorname{tr}(\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket))\|_{0,h}^2 + \eta_0 C_1 2\mu \sum_{e \in \mathcal{E}_h^0} h_e^{-1} \|\llbracket \mathbf{v} \rrbracket\|_{0,e}^2 + \eta_0 \lambda \sum_{e \in \mathcal{E}_h^0} \|\operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket))\|_{0,h}^2 \\
&\geq 2(1-\epsilon)\mu|\mathbf{v}|_h^2 + 2\mu \left(\eta_0 C_1 - \frac{3C_2}{\epsilon} \right) |\mathbf{v}|_*^2 + \lambda(\eta_0 - 3) \sum_{e \in \mathcal{E}_h^0} \|\operatorname{tr}(\mathbf{r}_e(\llbracket \mathbf{v} \rrbracket))\|_{0,h}^2.
\end{aligned}$$

Since $C_2 > C_1$, $\eta_0 > 3$ is guaranteed from $\eta_0 > 3C_2/C_1$. So stability is valid for this DG formulation when $\eta_0 > 3C_2/C_1$.

$$\begin{aligned}
B_{2,h}^{(5)}(v, v) &\geq 2\mu \|\boldsymbol{\varepsilon}_h(\mathbf{v}) + \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 + \int_{\mathcal{E}_h^0} \frac{\eta_e}{h_e} \llbracket \mathbf{v} \rrbracket : \llbracket \mathbf{v} \rrbracket ds \\
&\geq 2\mu \left(|\mathbf{v}|_h^2 + \|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 + 2 \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\varepsilon}_h(\mathbf{v}) : \mathbf{r}_0(\llbracket \mathbf{v} \rrbracket) dx \right) + \eta_0 |\mathbf{v}|_*^2 \\
&\geq 2\mu \left((1-\epsilon)|\mathbf{v}|_h^2 + \left(1 - \frac{1}{\epsilon}\right) \|\mathbf{r}_0(\llbracket \mathbf{v} \rrbracket)\|_{0,h}^2 \right) + \eta_0 |\mathbf{v}|_*^2 \\
&\geq 2\mu(1-\epsilon)|\mathbf{v}|_h^2 + \left(\eta_0 + 6\mu C_2 - \frac{6\mu C_2}{\epsilon} \right) |\mathbf{v}|_*^2.
\end{aligned}$$

It is clear that stability holds for the LDG method when $\eta_0 > 0$. Summarizing the above argument, we have the next result.

Lemma 3.4 (Stability) For $1 \leq j \leq 5$, $B_h = B_{1,h}^{(j)}$ and $B_{2,h}^{(j)}$ satisfy

$$B_h(\mathbf{v}, \mathbf{v}) \gtrsim \|\mathbf{v}\|_*^2 \quad \forall \mathbf{v} \in V_h, \quad (3.10)$$

if $\eta_0 = \min_{e \in \mathcal{E}_h^0} \eta_e > 0$ for the methods with $j = 2, 3, 5$, and η_0 is large enough for the methods with $j = 1, 4$.

4 Approximation and error estimates

Considering the error estimation for the DG methods, we first write the error as

$$\mathbf{e} = \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{u}_I) + (\mathbf{u}_I - \mathbf{u}_h),$$

where $\mathbf{u}_I \in V_h$ is the usual continuous piecewise linear interpolant of the exact solution \mathbf{u} . Then $[[\mathbf{u} - \mathbf{u}_I]] = \mathbf{0}$ on the interelement boundaries. By the definition of norm (3.2), we have the approximation property, assuming $\mathbf{u} \in [H^2(\Omega)]^2$,

$$\|\|\mathbf{u} - \mathbf{u}_I\|\|^2 = |\mathbf{u} - \mathbf{u}_I|_h^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |\mathbf{u} - \mathbf{u}_I|_{2,K}^2 \lesssim h^2 |\mathbf{u}|_{2,\Omega}^2. \quad (4.1)$$

In the next result, we need some additional solution regularity assumption. Assume that both tangential and normal derivatives of \mathbf{u} on Γ_C are piecewise in $[L^\infty]^2$, i.e., on each line segment piece of Γ_C , both tangential and normal derivatives of \mathbf{u} belongs to the space $[L^\infty]^2$ (a similar assumption is made in [8] in the proof of Lemma 6.1 there). As in [8], we also assume that the number of changes from $u_\nu < 0$ to $u_\nu = 0$ on Γ_C is finite.

Theorem 4.1 *Let \mathbf{u} and \mathbf{u}_h be the solutions of (2.12) and (2.29), respectively. Assume $\mathbf{u} \in [H^2(\Omega)]^2$, both tangential and normal derivatives of \mathbf{u} on Γ_C are piecewise in $[L^\infty]^2$, and the number of changes from $u_\nu < 0$ to $u_\nu = 0$ on Γ_C is finite. Then for the DG methods with $j = 1, \dots, 5$, we have the error bound*

$$\|\|\mathbf{u} - \mathbf{u}_h\|\| \lesssim h. \quad (4.2)$$

Proof. By the stability of the bilinear form B_h , we have

$$\|\|\mathbf{u}_I - \mathbf{u}_h\|\|^2 \lesssim B_h(\mathbf{u}_I - \mathbf{u}_h, \mathbf{u}_I - \mathbf{u}_h) \equiv T_1 + T_2, \quad (4.3)$$

where

$$\begin{aligned} T_1 &= B_h(\mathbf{u}_I - \mathbf{u}, \mathbf{u}_I - \mathbf{u}_h), \\ T_2 &= B_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_I - \mathbf{u}_h). \end{aligned}$$

We bound T_1 as follows, using the boundedness of B_h ,

$$T_1 \lesssim \|\|\mathbf{u}_I - \mathbf{u}\|\| \|\|\mathbf{u}_I - \mathbf{u}_h\|\| \lesssim \epsilon \|\|\mathbf{u}_I - \mathbf{u}_h\|\|^2 + \frac{1}{4\epsilon} \|\|\mathbf{u}_I - \mathbf{u}\|\|^2, \quad (4.4)$$

where $\epsilon > 0$ is an arbitrarily small number.

Then we bound T_2 . Note that on an interior edge, $[[\mathbf{u}]] = \mathbf{0}$, $\{\mathbf{u}\} = \mathbf{u}$, $\{\boldsymbol{\sigma}\} = \boldsymbol{\sigma}$, and on Γ_D , $[[\mathbf{u}]] = \mathbf{0}$. Then

$$\begin{aligned} B_h(\mathbf{u}, \mathbf{u}_I - \mathbf{u}_h) &= \int_{\Omega} (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}_h(\mathbf{u}_I - \mathbf{u}_h) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) \operatorname{tr}(\boldsymbol{\varepsilon}_h(\mathbf{u}_I - \mathbf{u}_h))) dx \\ &\quad - \int_{\mathcal{E}_h^0} [[\mathbf{u}_I - \mathbf{u}_h]] : (2\mu \boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) \mathbf{I}) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\mathbf{u}_I - \mathbf{u}_h) dx - \int_{\mathcal{E}_h^0} [[\mathbf{u}_I - \mathbf{u}_h]] : \boldsymbol{\sigma} ds. \end{aligned}$$

Since $[\boldsymbol{\sigma}] = \mathbf{0}$ on an interior edge and remembering (2.3), we have

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \boldsymbol{\sigma} : \boldsymbol{\varepsilon}_h(\mathbf{u}_I - \mathbf{u}_h) dx &= \sum_{K \in \mathcal{T}_h} \int_K -\operatorname{div} \boldsymbol{\sigma} \cdot (\mathbf{u}_I - \mathbf{u}_h) dx + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\boldsymbol{\sigma} \mathbf{n}_K) \cdot (\mathbf{u}_I - \mathbf{u}_h) ds \\ &= \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f} \cdot (\mathbf{u}_I - \mathbf{u}_h) dx + \int_{\mathcal{E}_h} \llbracket \mathbf{u}_I - \mathbf{u}_h \rrbracket : \boldsymbol{\sigma} ds. \end{aligned}$$

Then

$$B_h(\mathbf{u}, \mathbf{u}_I - \mathbf{u}_h) = \int_{\Omega} \mathbf{f} \cdot (\mathbf{u}_I - \mathbf{u}_h) dx + \int_{\Gamma_F} \mathbf{g} \cdot (\mathbf{u}_I - \mathbf{u}_h) ds + \int_{\Gamma_C} (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot (\mathbf{u}_I - \mathbf{u}_h) ds. \quad (4.5)$$

Choosing $\mathbf{v}_h = \mathbf{u}_I$ in (2.29), we have

$$B_h(\mathbf{u}_h, \mathbf{u}_I - \mathbf{u}_h) \geq \ell(\mathbf{u}_I - \mathbf{u}_h). \quad (4.6)$$

Let Γ_T and Γ_N denote the sets of edges $\subset \Gamma_C$ where $u_\nu = 0$ and $u_\nu < 0$ respectively. Combining (4.6) and (4.5), we obtain

$$\begin{aligned} T_2 = B_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}_I - \mathbf{u}_h) &\leq \int_{\Gamma_C} (\boldsymbol{\sigma} \boldsymbol{\nu}) \cdot (\mathbf{u}_I - \mathbf{u}_h) ds = \int_{\Gamma_C} \sigma_\nu (u_{I\nu} - u_{h\nu}) ds \\ &\leq \int_{\Gamma_C} \sigma_\nu u_{I\nu} ds = T_3 + T_4 + T_5, \end{aligned} \quad (4.7)$$

where

$$T_3 = \int_{e \in \Gamma_T} \sigma_\nu u_{I\nu} ds, \quad T_4 = \int_{e \in \Gamma_N} \sigma_\nu u_{I\nu} ds, \quad T_5 = \int_{e \in \Gamma_C \setminus (\Gamma_T \cup \Gamma_N)} \sigma_\nu u_{I\nu} ds.$$

From $\sigma_\nu u_\nu = 0$ on Γ_C , it is easy to know that $T_3 = 0$ and $T_4 = 0$. Consider the term T_5 . If $e \subset \Gamma_C \setminus (\Gamma_T \cup \Gamma_N)$, then there exists a point $P \in e$ satisfying $u_\nu(P) = 0$. By the regularity assumption over Γ_C , we have $u_{I\nu} = O(h)$ on $e \subset \Gamma_C \setminus (\Gamma_T \cup \Gamma_N)$ and $\sigma_\nu \in L^\infty(\Gamma_C)$. Hence,

$$T_5 = \int_{e \in \Gamma_C \setminus (\Gamma_T \cup \Gamma_N)} \sigma_\nu u_{I\nu} ds \lesssim h^2.$$

Thus, under the stated regularity assumptions, $T_2 \lesssim h^2$ and the proof is completed. \blacksquare

Remark 4.2 *For the scalar unilateral variational inequality*

$$u \in K, \quad \int_{\Omega} [\nabla u \cdot \nabla (v - u) + u(v - u)] dx \geq \int_{\Omega} f(v - u) dx \quad \forall v \in K,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded, convex polygonal domain, and

$$K = \{v \in H^1(\Omega) : v \geq 0 \text{ a.e. on } \partial\Omega\},$$

the optimal linear convergence for the linear element solution is proved in [14] without the additional assumption that the number of switches between $u > 0$ and $u = 0$ on Γ_C is finite. For the Signorini problem we study in this paper, it appears not possible to adapt the arguments in [14] to show the optimal error bound (4.2) without the additional assumption that the number of changes from $u_\nu < 0$ to $u_\nu = 0$ on Γ_C is finite.

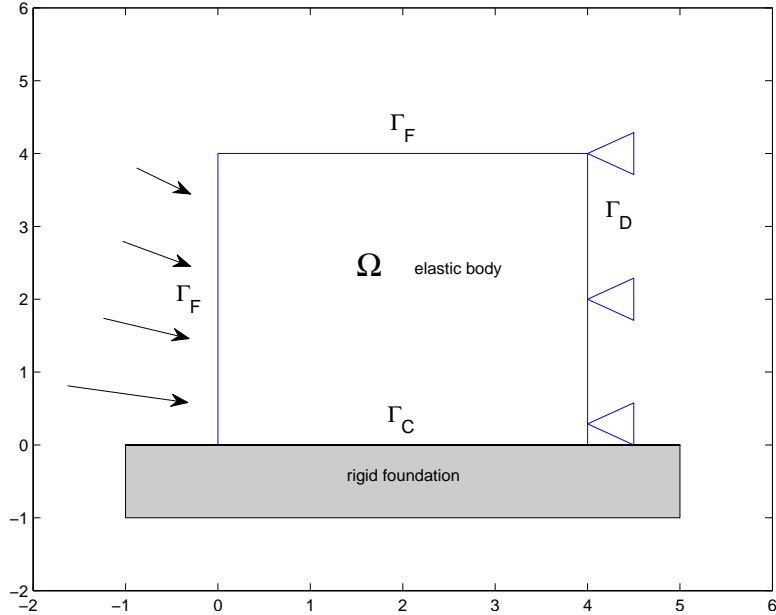


Figure 1: An elastic body on a frictionless rigid foundation

5 Numerical example

We report some numerical results on a two dimensional test problem solved by the LDG method. The physical setting is given as in Figure 1. The domain $\Omega = (0, 4) \times (0, 4)$ can be viewed as the cross section of a linearized elastic body. On the boundary $\Gamma_D = \{4\} \times (0, 4)$, the body is clamped and hence the displacement field vanishes there. The traction acts on the boundary $\{0\} \times (0, 4)$ and the boundary of $(0, 4) \times \{4\}$ is traction free. Therefore $\Gamma_F = \{0\} \times (0, 4) \cup (0, 4) \times \{4\}$. On the boundary $\Gamma_C = (0, 4) \times \{0\}$, the body contacts with a frictionless rigid foundation. No volume force is assumed to act on the body Ω . Let E be the Young's modulus and κ be the Poisson's ratio of the material. Then the Lamé coefficients are

$$\lambda = \frac{E\kappa}{(1 + \kappa)(1 - 2\kappa)}, \quad \mu = \frac{E}{2(1 + \kappa)}.$$

In this example, we use the following data

$$E = 200 \text{ daN/mm}^2, \quad \kappa = 0.3, \quad \mathbf{f} = \mathbf{0} \text{ daN/mm}^2, \\ \mathbf{g}(x_1, x_2) = (0.02(5 - x_2), -0.01) \text{ daN/mm}^2,$$

where the unit daN/mm^2 denotes decanewtons per square millimeter. We solve the discretized problem on uniform triangle meshes, see Figure 2.

To consider the convergence order, we solve the problem on a family of uniform meshes. We start with $h = 2\sqrt{2}$ which decreases by half, and adopt the numerical solution on the mesh $h = \sqrt{2}/16$

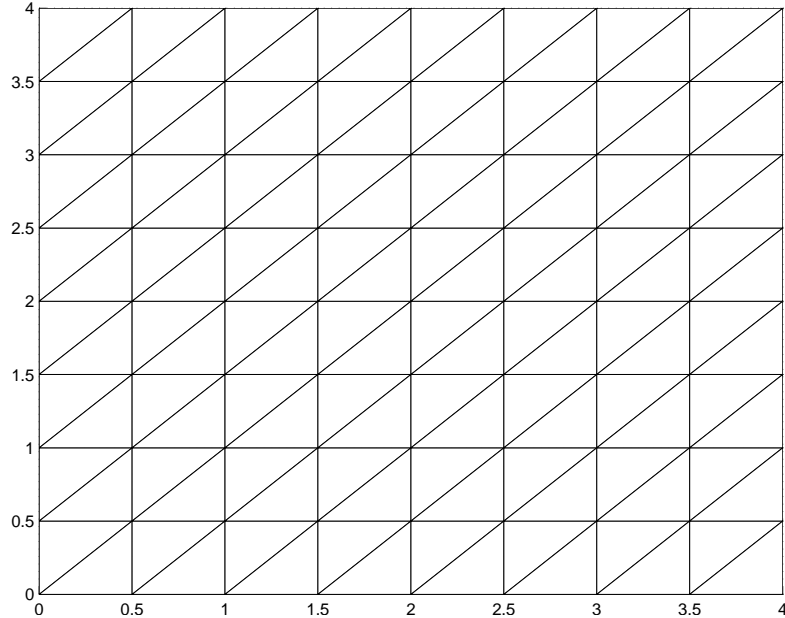


Figure 2: A uniform triangle mesh of the domain

as “exact” solution in computing errors of numerical solutions on other meshes. In Figure 3, we show the deformed mesh (amplified by 200) for $h = \sqrt{2}/16$. We observe from Figure 4 that the numerical convergence orders in both norms $||| \cdot |||$ and $|\cdot|_{Q,h}$ are approximately 1, matching well the theoretical prediction.

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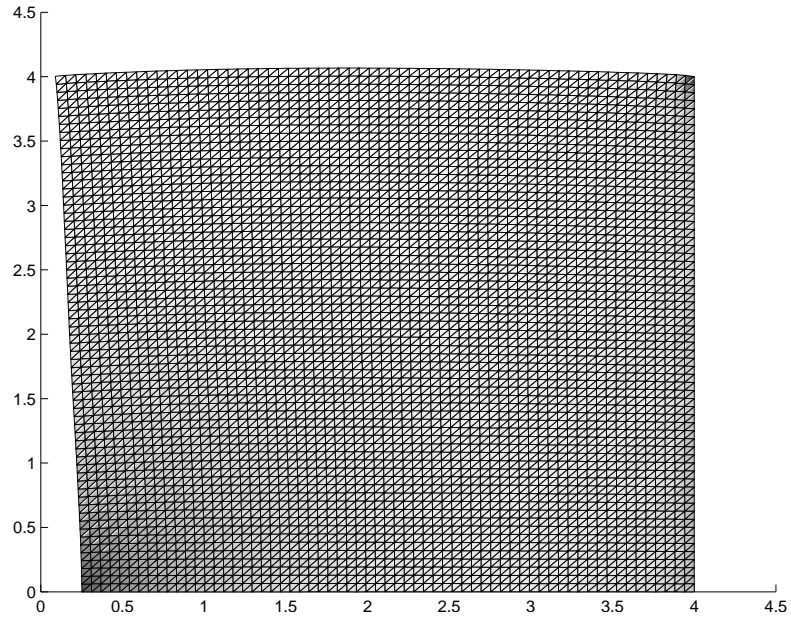


Figure 3: Deformed mesh (amplified by 200) for $h = \sqrt{2}/16$

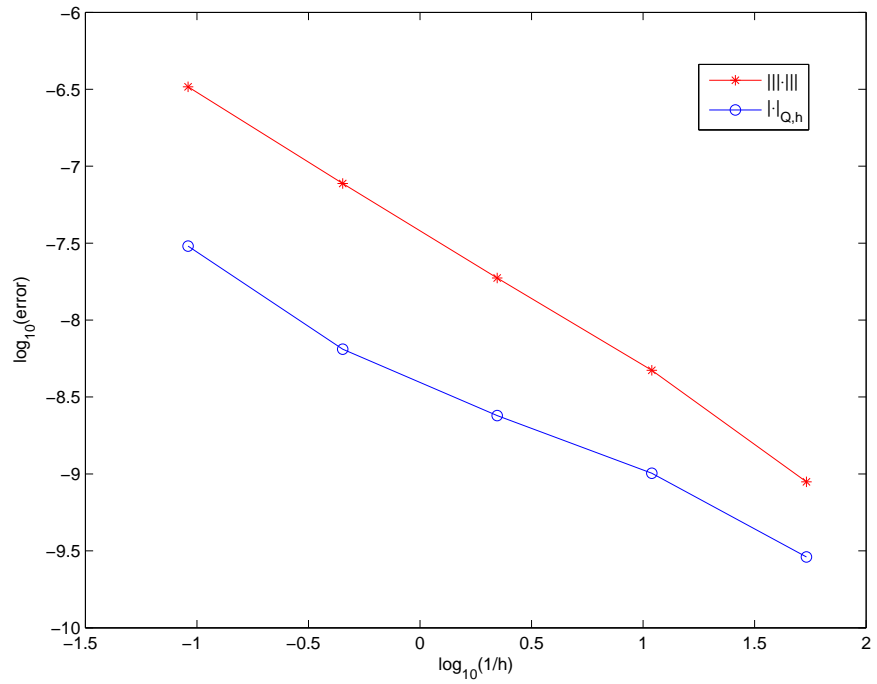


Figure 4: Numerical errors

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