Construction of Lyapunov functionals for coupled differential and continuous time difference equations

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Abstract—A new stability analysis technique for systems composed of a differential equation coupled with a continuous time difference equation is proposed. It is based on the explicit construction of Lyapunov functionals from the knowledge of Lyapunov functions for subsystems. Robustness results of iISS type are inferred from these functionals.

Key Words—stability, delay, difference equation, Lyapunov functional

I. INTRODUCTION

Systems composed of an ordinary differential equation coupled with a continuous time difference equation are of great relevance in engineering applications (see [16], [21], [22] and the references therein). Many motivations for the study of systems of this type are given in [7]. One of the most important ones is that the transformation of neutral systems into coupled (possibly delay) differential and continuous time difference equations with an auxiliary variable [17], [19] has been often fruitfully used for the stability analysis of neutral systems, even if many studies do not rely on this transformation. This is the case in particular of [12], where the stability of nonlinear neutral systems without the transformation into coupled differential and continuous time difference equations is analyzed and properties of integral-input-to-state stability (iISS) as well as ISS types are established through the construction of Lyapunov-type functionals. Many Lyapunov stability criteria have been recently established for large classes of coupled delay differential and continuous time difference equations (see, for instance, [2], [3], [9], [10], [20]) and references therein). In these papers the problem of the construction of the overall Lyapunov functional on the basis of the Lyapunov functions of the subsystems is not treated.

In the present paper, we consider systems composed of an ordinary differential equation coupled with a continuous time difference equation. That way, we complement [12]: we consider an alternative family of systems and, when applied to neutral systems, the stability results we propose cover a larger family of neutral systems than the one covered in the main result of [12]. We establish them through the construction of a new type of Lyapunov-type functionals, which enable us to prove robustness properties of iISS type with respect to continuous disturbances that are present in both the differential and the difference equations. In [7], small-gain arguments are exploited in order to prove the weighted input-to-output stability in a very general setting, which includes Lebesgue measurable and locally essentially bounded disturbances acting on both the subsystems (differential and difference ones). For the results here provided, we need to assume the continuity of the disturbance, in both the dynamics of the differentiated variable and in the one of the non-differentiated variable, because we need to assume some regularity of the solution (continuity is sufficient). On the other hand, with respect to [7], here the hypothesis on the Lyapunov function derivative, as far as the differentiated variable is concerned, is significantly weakened, since here the new perspective of an iISS-type notion is studied. It is worth mentioning also that the ISS small-gain-type result for coupled delay differential and difference equations is presented in [7] without explicitly constructing Lyapunov-type functionals for the overall system.

The construction of the overall Lyapunov functional relies here on the assumption that Lyapunov functions are available for subsystems that are a differential equation and a continuous time difference equation, which is realistic in the sense that, for these types of systems, there are classical techniques of constructions of strict Lyapunov functions (see, for instance, [11], [14], [13], [7]). Observe in particular that Lyapunov functions for continuous time difference equations with a single time-invariant delay can be immediately deduced from Lyapunov functions for discrete time systems. Moreover, it is worth pointing out that the proposed construction complements [17], [19] by composing a Lyapunov-Krasovskii functional explicitly for the overall system whereas the studies in [17], [19] are devoted to the development of a framework of ISS Lyapunov-Krasovskii functionals. The results in [17], [19] give ISS characterizations which are parallel to the ISS Lyapunov functions established for delay-free systems in [24], and they do not aim at developing a particular method of constructing such functionals. A special case of the coupled equations here studied is given by neutral functional differential equations in the Hale’s form [4], [8], [16] with a single discrete delay. Indeed, the difference operator involved in such equations correspond to the continuous time difference equation here dealt with. Input-to-state stability (ISS) of such a class of neutral systems (with multiple discrete and distributed delays) is studied in [17] for the case of linear difference
operators, and in [19] for the case of nonlinear difference operators. We point out that, for the class of neutral functional differential equations with discrete delays, and in particular for the one included in the class of systems here studied (i.e., with a single delay), the hypothesis of continuity of the disturbance, at least in the continuous time difference equation, is naturally satisfied. Indeed, when rewriting a single delay neutral functional differential equation, forced by an external disturbance, into a couple of differential equation and continuous time difference equation, then the external disturbance acting on the subsystem described by the continuous time difference equation is equal to zero. On the other hand, if the differentiated variable is considered as an input for the subsystem described by the continuous time difference equation (see [19], [7], [3], [2]), then such an input is clearly continuous.

**Notation and definitions.** The notation will be simplified whenever no confusion can arise from the context. We let \(|\cdot|\) denote the Euclidean norm of matrices and vectors of any dimension. Given \(\phi: \mathbb{T} \to \mathbb{R}^p\) defined on an interval \(\mathbb{T}\), let \(|\phi|_{L^p}\) denote its (essential) supremum over \(\mathbb{T}\). For any integer \(p\), we let \(C_{in} = C([-\tau, 0], \mathbb{R}^p)\) denote the set of all continuous \(\mathbb{R}^p\)-valued functions defined on a given interval \([-\tau, 0]\). For a function \(x: [-\tau, +\infty) \to \mathbb{R}^k\), for all \(t \geq 0\), the function \(x_t\) is defined by \(x_t(\ell) = x(t + \ell)\) for all \(\ell \in [-\tau, 0]\).

II. MAIN RESULT

We consider the system
\[
\begin{align*}
\dot{x}(t) &= f(x(t), z(t), z(t-\tau), \delta(t)), \\
z(t) &= g(x(t), z(t-\tau), \delta(t)),
\end{align*}
\]
where \(x \in \mathbb{R}^n\), \(z \in \mathbb{R}^m\), \(\tau > 0\) is constant, \(f\) and \(g\) are locally Lipschitz nonlinear functions and where \(\delta: [0, +\infty) \to \mathbb{R}^p\) is a continuous unknown function, which represents a disturbance. This system is composed of an ordinary differential equation and a continuous time difference equation. We assume that the matching condition is satisfied, i.e. initial conditions \((\phi_x, \phi_z) \in C_{in}\) and disturbance \(\delta\) are such that
\[
\phi_z(0) = g(\phi_x(0), \phi_z(-\tau), \delta(0)).
\]

Due to the features of the system we consider, it seems natural to us to use the new notion of sup-integral-input-to-state-stable, which is similar, but slightly less restrictive than the usual iISS property:

**Definition.** The system (1) is siISS (sup-integral-input-to-state-stable) with respect to \(\delta\) if there is a function \(\Lambda\) of class \(KL\) and functions \(\rho_1, \rho_2, \rho_3\), of class \(K\) such that, for all \((x_0, z_0) \in C_{in}\) for any function \(\delta\), continuous over \([0, \infty)\) and for all \(t \geq 0\), the inequality
\[
|(x(t), z(t))| \leq \Lambda \left( \sup_{m \in [-\tau, 0]} |x_0(m), z_0(m)|, t \right) + \rho_1 \left( \int_0^t \rho_2(|\delta(m)|)dm \right) + \rho_3 \left( \sup_{m \in [t_0, t]} |\delta(m)| \right)
\]
is satisfied.

The above inequality with \(\rho_1 = 0\) (resp., \(\rho_3 = 0\)) is often called input-to-state stability or iISS for short (integral input-to-state stability, or iISS for short) [23], [24], [18]. Note that iISS implies siISS. In the same way, it is also true that ISS implies siISS.

We introduce assumptions.

**Assumption 1.** There exist two positive definite radially unbounded functions \(V_1: \mathbb{R}^n \to [0, +\infty), V_2: \mathbb{R}^m \to [0, +\infty)\) of class \(C^1\), a continuous positive definite function \(W_1: \mathbb{R}^n \to [0, +\infty), \) functions \(\kappa_1, \kappa_2\) of class \(K\) and real numbers \(r_1 \geq 0, r_2 \geq 0, \varepsilon \geq 0\), such that, for all \((a, b, c, d) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p\), the inequality
\[
\frac{\partial V_1}{\partial x}(a) + V_2(b) + r_1 V_2(c) + \kappa_1(|d|) \leq -W_1(a) + V_2(b) + r_1 V_2(c) + \kappa_1(|d|)
\]
holds, and \(b = g(a, c, d)\) implies that the inequality
\[
V_2(b) \leq \varepsilon V_2(c) + r_2 W_1(a) + \kappa_2(|d|)
\]
is satisfied.

**Assumption 2.** The constants \(r_1 \geq 0, r_2 > 0, \varepsilon > 0\) satisfy
\[
\varepsilon < 1 - r_2 - r_1 r_2.
\]

We are ready to state and prove the following result:

**Theorem 1:** Let the system (1) satisfy Assumptions 1 and 2. Then this system is siISS with respect to \(\delta\).

**Remark 1:** Theorem 1 can be easily extended to systems of the type
\[
\begin{align*}
\dot{x}(t) &= f(x(t), z(t), z(t-\tau), \int_{t-\tau}^t z(m)dm, \delta(t)), \\
z(t) &= g \left( x(t), z(t-\tau), \int_{t-\tau}^t z(m)dm, \delta(t) \right).
\end{align*}
\]

However, for the sake of simplicity, we restrict ourselves to the systems (1).

**Remark 2:** Suppose that a neutral system \(\dot{x}(t) = h(x(t), \dot{x}(t-\tau))\) is studied. Suppose that the initial conditions are continuously differentiable and a sewing condition is satisfied so that the solution is continuously differentiable. Then, by introducing the variable \(z(t) = \dot{x}(t)\), we obtain
\[
\begin{align*}
\dot{x}(t) &= h(x(t), z(t-\tau)) \\
z(t) &= h(x(t), z(t-\tau)).
\end{align*}
\]

This system is of the from (1) and therefore its stability can be checked by using Theorem 1. From the stability properties of (7), the stability properties of the initial system can be deduced.

**Remark 3:** In the case of neutral systems described in Remark 2, the previous result in [12] uses \(|\dot{x}(t)|\) in the left hand side of (4), which requires information of all components of the vector \(\dot{x}(t)\). Instead, this paper uses \(V_2(z(t))\) on the left hand side of (4), which does not necessarily require all components of \(\dot{x}(t)\). As illustrated by Example (47) later on, the use of \(V_2(z(t))\) makes the assumption on the neutral term imposed by (4) less conservative, and it renders the functional term in the resulting Lyapunov-Krasovskii functional more flexible than the previous result in [12]. In fact, the second term in (8) is made of \(V_2(\phi_z(\cdot))\) which (along the trajectories) may contain only a part of \(\dot{x}_t\). It is worth...
noting that the characterization of ISS Lyapunov-Krasovskii functionals in [17], [19] also allows their functional terms to depend only on a part of $\dot{x}_t$ since their Lyapunov-Krasovskii functionals are formulated in terms of semi-norms. Besides the use of $V_2$ in (4), the assumption posed with (3) and (4) relaxes the one in [12] for the class of neutral systems. Note that the function $W_1$ in (3) is required to be merely positive definite, which contrasts with the study of ISS in [17], [19], [7].

**Remark 4:** Basically, the coupled delay differential and difference equations (1) provides an important extension of the system that can be tackled by [6]. In fact the system in the present paper differs from the system in [6] in that the difference equation in (1) admits an internal loop which allows us to cover neutral-type delays.

**Proof.** To begin with, we introduce the functionals

$$U_1(\phi_x, \phi_z) = V_1(\phi_x(0)) + k \int_{-\tau}^{0} e^{-l(t+\tau)} V_2(\phi_z(r)) dr, \quad (8)$$

where $k > 0$, $l \leq 0$ are real numbers to be selected later. Then, along the trajectories of (1),

$$U_1(x_t, z_t) = V_1(x(t)) + k \int_{-\tau}^{t} e^{(t-m+\tau)} V_2(z(m)) dm. \quad (9)$$

From the inequality (3), we deduce that the derivative of $V_1$ along the trajectories of (1) satisfies

$$\dot{V}_1(t) \leq -W_1(x(t)) + V_2(z(t)) + r_1 V_2(z(t-\tau)) + \kappa_1(\delta(t)), \quad (10)$$

which implies that the derivative of $U_1$ along the trajectories of (1) satisfies

$$\dot{U}_1(t) \leq -W_1(x(t)) + V_2(z(t)) + r_1 V_2(z(t-\tau)) + \kappa_1(\delta(t)). \quad (11)$$

By grouping the terms, we obtain

$$\dot{U}_1(t) \leq -W_1(x(t)) + [1 + k e^{-l(t+\tau)}] V_2(z(t)) + r_2 W_1(x(t)) + \kappa_2(\delta(t)). \quad (12)$$

From (4), it follows that, along the trajectories of (1), the inequality

$$V_2(z(t)) \leq \varepsilon V_2(z(t-\tau)) + r_2 W_1(x(t)) + \kappa_2(\delta(t))$$

is satisfied. Since $1 + k e^{-l(t+\tau)} > 0$, it follows that

$$\dot{U}_1(t) \leq -W_1(x(t)) + \kappa_1(\delta(t)). \quad (13)$$

with $\kappa_3 = \kappa_1 + (1 + k e^{-l(t+\tau)}) \kappa_2$. By grouping the terms, we obtain

$$\dot{U}_1(t) \leq -W_1(x(t)) + \kappa_3(\delta(t)). \quad (14)$$

Since $k l \leq 0$, the right hand side of (13) is nonpositive in the absence of $\delta$ if

$$r_2 + k e^{-l(t)} r_2 - 1 \leq 0, \quad \varepsilon + r_1 + (e^{-l(t)} \varepsilon - 1) k \leq 0, \quad (15)$$

or, equivalently,

$$k \leq e^{\varepsilon} \left( \frac{1}{r_2} - 1 \right), \quad \varepsilon + r_1 \leq (1 - e^{-l(t)} \varepsilon) k. \quad (16)$$

Since $\varepsilon < 1$, one can choose $l \in \left( \frac{\ln(\varepsilon)}{-\varepsilon}, 0 \right)$ and then $1 - e^{-l(t)} \varepsilon > 0$. Then, one can find a couple $(k, l)$ such that (16) is satisfied if there is $l \in \left( \frac{\ln(\varepsilon)}{-\varepsilon}, 0 \right)$ such that the inequality

$$\varepsilon + r_1 < (e^{\varepsilon} - \varepsilon) \left( \frac{1}{r_2} - 1 \right) \quad (17)$$

holds. Assumption 2 ensures that $\ln \left( \frac{r_1 r_2 + \varepsilon}{1 + r_2} \right)$ is a negative number and if

$$l \in \left( \frac{1}{\tau} \ln(\varepsilon), \frac{1}{\tau} \ln \left( \frac{r_1 r_2 + \varepsilon}{1 + r_2} \right) \right), \quad (18)$$

then (16) is satisfied. Then a possible choice for $k$ is

$$k = \frac{1}{2} \left[ \frac{\varepsilon + r_1}{1 - e^{-l(t)} \varepsilon} + e^{\varepsilon} \left( \frac{1}{r_2} - 1 \right) \right]. \quad (19)$$

To summarize, we have established that one can choose parameters such that there are constants $r_3 > 0$, $r_4 > 0$, $r_5 > 0$ such that

$$\dot{U}_1(t) \leq -r_3 \int_{-\tau}^{t} e^{(t-m+\tau)} V_2(z(m)) dm -r_4 W_1(x(t)) - r_5 V_2(z(t-\tau))$$

$$+ \kappa_3(\delta(t)). \quad (20)$$

Since $W_1$ is positive definite and radially unbounded, we can adapt the proof of [11, Lemma A.7] to demonstrate that there exist two functions $\kappa_4$, $\kappa_5$ of class $K_\infty$ such that for all $x \in \mathbb{R}^m$,

$$r_4 W_1(x) \geq \frac{\kappa_4(V(x))}{1 + \kappa_5(V(x))},$$

with $\kappa_4$ such that, for all $m \geq 0$, $\kappa_4(m) \leq \frac{\varepsilon}{k} m$. We deduce successively that

$$\dot{U}_1(t) \leq -\kappa_4(V(x)) - r_3 \int_{-\tau}^{t} e^{(t-m+\tau)} V_2(z(m)) dm$$

$$-r_4 W_1(x(t)) - r_5 V_2(z(t-\tau))$$

$$+ \kappa_3(\delta(t)) \quad (21)$$

with $\kappa_3 = \kappa_1 + (1 + k e^{-l(t+\tau)}) \kappa_2$. By grouping the terms, we obtain

$$\dot{U}_1(t) \leq -\kappa_4(V(x)) - \kappa_4(V(x) + k \int_{-\tau}^{t} e^{(t-m+\tau)} V_2(z(m)) dm)$$

$$+ \kappa_3(\delta(t)) \quad (22)$$

We deduce that there is a continuous positive definite function $\omega_3$ such that, for all $t \geq 0$,

$$\dot{U}_1(t) \leq -\omega_3(U_1(x_t, z_t)) + \kappa_3(\delta(t)). \quad (23)$$

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Our objective is now to prove that the solutions are defined over \( t_f = +\infty \). To this end, we let \([-\tau, t_f]\) be the largest interval of definition of a solution. To prove that \( t_f = +\infty \), we proceed by contradiction. Assume that \( t_f \) is a finite positive real number. Then

\[
U_1(x_t, z_t) \leq U_1(x_0, z_0) + \int_0^t \kappa_3(|\delta(m)|)dm, \quad \forall t \in [0, t_f).
\]

From the definition of \( U_1 \) and the fact that \( \delta \) is continuous over \([0, \infty)\), we deduce easily that there are two constants \( \mathfrak{B}_1 > 0, \mathfrak{B}_2 > 0 \), such that

\[
|x(t)| \leq \mathfrak{B}_1, \quad W_1(x(t)) \leq \mathfrak{B}_2, \quad \forall t \in [-\tau, t_f)
\]

and a constant \( \mathfrak{B}_3 > 0 \) such that

\[
V_2(z(t - \tau)) \leq \mathfrak{B}_3, \quad \forall t \in [0, t_f).
\]

Therefore from the inequality (4), for all \( t \in [0, t_f) \),

\[
V_2(z(t)) \leq e^{2\mathfrak{B}_3} \mathfrak{B}_2 + 2\mathfrak{B}_3 \kappa_2 \left( \sup_{m \in [0, t_f]} |\delta(m)| \right).
\]

Since \( V_2 \) is positive definite and radially unbounded, it follows from (23) and (24) that the solution is bounded in \([0, t_f)\). Therefore \( t_f \) cannot be a finite real number (see Remark 1 in [19]).

Next our objective if to establish that the system (1) is sILSS with respect to \( \delta \).

Since the functional \( U_1 \) is nonnegative, we deduce from (22) and [1, Coro. IV.3] that there exists a function \( \beta_\omega \) of class \( \mathcal{K}\mathcal{L} \) such that, for all \( t \geq 0 \),

\[
U_1(x_t, z_t) \leq \beta_\omega(U_1(x_0, z_0), t) + 2\int_0^t \pi(|\delta(m)|)dm.
\]

with \( \pi = 2\kappa_3 \). From the explicit expression of \( U_1 \), it follows that, for all \( t \geq 0 \),

\[
V_1(x(t)) \leq \beta_\omega(U_1(x_0, z_0), t) + 2\int_0^t \pi(|\delta(m)|)dm,
\]

\[
\int_{t-\tau}^t e^{\int_{t-m-\tau}^t V_2(z(m))dm} \leq \frac{1}{2} \left[ \beta_\omega(U_1(x_0, z_0), t) + 2\int_0^t \pi(|\delta(m)|)dm \right].
\]

Since \( V_1 \) is positive definite and radially unbounded, we easily deduce that there are a function \( \beta_\omega \) of class \( \mathcal{K}\mathcal{L} \) and a function \( \kappa_6 \) of class \( \mathcal{K} \) such that, for all \( t \geq 0 \),

\[
|x(t)| \leq \beta_\omega(U_1(x_0, z_0), t) + \kappa_6 \left( \int_0^t \pi(|\delta(m)|)dm \right).
\]

Now, let \( t > 0 \) and \( j \) be an integer such that \( t - (j + 1)\tau \in [-\tau, 0) \). Then, all the inequalities

\[
V_2(z(t)) \leq e^{j+1}V_2(z(t - (j + 1)\tau)) + r_2 \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|),
\]

\[
e\n\]

are satisfied. By adding all these inequalities, we obtain

\[
V_2(z(t)) \leq e^{j+1}V_2(z(t - (j + 1)\tau)) + r_2 \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|) + \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|).
\]

Now, to ease the notation, we define a constant:

\[
q = -\frac{\ln(\varepsilon)}{\tau}.
\]

It is positive. Moreover, there is \( \kappa_7 \) of class \( \mathcal{K}\infty \) such that \( V_2(z) \leq \kappa_7(|z|) \) and \( W_1(x) \leq \kappa_7(|z|) \). It follows that

\[
V_2(z(t)) \leq e^{-q(t+1)\tau} \kappa_7(z(t - (j + 1)\tau)) + r_2 \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|) + \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|).
\]

Using (28), we deduce that

\[
V_2(z(t)) \leq e^{-q(t+1)\tau} \kappa_7(z(t - (j + 1)\tau)) + r_2 \sum_{i=0}^{j} e^{i} \kappa_7 \left( \varsigma((x_0, z_0), t - i\tau) + \kappa_6 \left( \int_0^{t-i\tau} \lambda(m)dm \right) \right) + \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|).
\]

with \( \varsigma((a, b), m) = \beta_\omega(U_1(a, b), m) \), \( \lambda(m) = 2\kappa_3(|\delta(m)|) \). Using the fact that the function \( \kappa_7 \) is of class \( \mathcal{K} \), we deduce that

\[
V_2(z(t)) \leq e^{-q(t+1)\tau} \kappa_7(-\tau, 0) + r_2 \sum_{i=0}^{j} e^{i} \kappa_7 \left( 2\kappa_6 \left( \int_0^{t-i\tau} \pi(|\delta(m)|)dm \right) \right) + \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|) \leq e^{-q(t+1)\tau} \kappa_7(-\tau, 0) + \sum_{i=0}^{j} e^{i} \beta_\omega(U_1(\phi_x, \phi_x), t - i\tau) + \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|) + \sum_{i=0}^{j} e^{i} \kappa_8 \left( \int_0^{t-i\tau} \kappa_3(|\delta(m)|)dm \right) + \sum_{i=0}^{j} e^{i} \kappa_2(|\delta(t - i\tau)|).
\]
with \( \beta_c = r_2\kappa_7(2\beta_b) \) and \( \kappa_8(m) = r_2\kappa_7(2\kappa_6(2m)) \). Therefore
\[
V_2(z(t)) \leq e^{-\eta t}V_2(z(t)) + \sum_{i=0}^{j-1} e^{-\eta i t} \beta_d(U_1(\phi_x, \phi_z), t)
\]
\[
+ \frac{1}{1-\varepsilon} \kappa_8 \left( \int_0^t \kappa_3(|\delta(m)|)dm \right)
\]
\[
+ \frac{1}{1-\varepsilon} \kappa_2 \left( \sup_{m \in [a, t]} |\delta(m)| \right).
\]

From Lemmas 1, we deduce that there exists a function \( \beta_d \) of class \( \mathcal{KL} \) such that
\[
\sqrt{\varepsilon \beta_d(a, t - \tau)} \leq \beta_d(a, t) \quad \forall a \geq 0, t \geq \tau.
\]

We can do backstepping based on Theorem 1 and Remark 5. Theorem 1 establishes via Theorem 1.

Example

We consider the two dimensional system
\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) + \frac{1}{2}z_2(t - \tau) \\
\dot{x}_2(t) &= -x_2(t) + \frac{1}{2}z_2(t - \tau).
\end{align*}
\]

Notice that this neutral system is nonlinear and not globally Lipschitz with respect to \( \delta \) and performing the change of \( \dot{g}(t) = y - y_s(x) \), we obtain
\[
\dot{g}(t) = H(z(t - \tau), x(t), y_s(x(t))) + \delta
\]
\[
\dot{y}(t) = u(t).
\]

Then, assuming that \( y_s(x) \) is such that \( \dot{g}(t) = H(z(t - \tau), x(t), y_s(x(t))) + \delta \) is ISS with respect to \( \delta \) and performing the change of \( x = z(t - \tau), x(t), y_s(x(t))) + \delta \) results in
\[
\dot{g}(t) = H(z(t - \tau), x(t), y_s(x(t))) + \delta
\]
\[
\dot{y}(t) = -g(t).
\]

Then, the global asymptotic stability of this system can be establish via Theorem 1.

where \( u \) is the control input to be designed. Then, introducing \( z = \dot{x}(t) \), we obtain the cascade of a system of the form (1) and an integrator as follows
\[
\dot{z}(t) = H(z(t - \tau), x(t), y(t))
\]
\[
\dot{y}(t) = u(t).
\]

Then, assuming that \( y_s(x) \) is such that \( \dot{g}(t) = H(z(t - \tau), x(t), y_s(x(t))) + \delta \) is ISS with respect to \( \delta \) and performing the change of \( g = y - y_s(x) \), we obtain
\[
\dot{g}(t) = H(z(t - \tau), x(t), y_s(x(t))) + \delta
\]
\[
\dot{y}(t) = u(t) - \frac{\partial y_s}{\partial x}(x)H(z(t - \tau), x(t), y(t)).
\]

Therefore, the choice \( u(t) = -g(t) + \frac{\partial y_s}{\partial x}(x)H(z(t - \tau), x(t), y(t)) \), results in
\[
\dot{g}(t) = H(z(t - \tau), x(t), y_s(x(t))) + \delta
\]
\[
\dot{y}(t) = -g(t).
\]

Then, the global asymptotic stability of this system can be establish via Theorem 1.
It follows that
\[ V_2(z(t)) \leq \frac{1}{576} x_2(t)^2 + \frac{1}{1152} \left( x_2(t)^2 \right)^2 + \left(\frac{1}{8} + \alpha \right) \frac{253}{243} \left( x_2(t)^2 \right)^4 + \frac{1}{288} \left[ z(t-\tau)^2 + \left(3 + 2a\right) \frac{253}{243} \left( x_2(t)^4 \right)^2 \right]. \] (55)

Then
\[ V_2(z(t)) \leq \frac{1}{576} x_2(t)^2 + \left( \frac{1}{8} + \alpha \right) \frac{253}{243} x_2(t)^4 + \frac{1}{288} \left[ z(t-\tau)^2 + \left(3 + 2a\right) \frac{253}{243} \left( x_2(t)^4 \right)^2 \right]. \] (56)

It follows that
\[ V_2(z(t)) \leq \frac{3}{288} x_3(t)^2 + \frac{1}{288} \left[ z(t-\tau)^2 + \left(3 + 2a\right) \frac{253}{243} \left( x_2(t)^4 \right)^2 \right]. \] (57)

Thus, we have
\[ V_1(t) \leq -W_1(x(t)) + V_2(z(t-\tau)) \]
\[ V_2(z(t)) \leq \frac{1}{288} \left[ z(t-\tau)^2 + \left(3 + 2a\right) \frac{253}{243} \left( x_2(t)^4 \right)^2 \right]. \] (58)

We deduce that Assumption 2 is satisfied if
\[ \frac{1}{288} \left[ z(t-\tau)^2 + \left(3 + 2a\right) \frac{253}{243} \left( x_2(t)^4 \right)^2 \right] < 1 - \frac{9}{768} \left( \frac{253}{243} \right)^3. \]

This inequality is satisfied with \( \alpha = 1 \).

IV. CONCLUSION

For systems composed of delayed differential and difference equations, we developed a new stability analysis technique. It is based on the knowledge of Lyapunov functions for the differential and the difference equations which are used to construct a Lyapunov functional for the overall system. It would not be difficult to incorporate distributed delays and time-varying components into systems.

The relation to the characterization of ISS Lyapunov-Krasovskii functionals developed in [17], [19] is an interesting topic to be addressed in future works.

REFERENCES


APPENDIX

Lemma 1: Let \( \chi \) be a function of class \( K_C \). Then there are a function \( \varphi \) of class \( K \) and a positive and decreasing continuous function \( \mu : [0, +\infty) \rightarrow (0, +\infty) \) such that
\[ \lim_{t \to +\infty} \mu(t) = 0 \] and for all \( (s, t) \in [0, +\infty) \times [0, +\infty) \), the inequality
\[ \chi(s, t) \leq \varphi(s)\mu(t) \] (59)
is satisfied.