An efficient numerical procedure to compute stabilizing state feedback gains for linear time-varying periodic systems

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Abstract

This report presents a procedure to synthesize a stabilizing periodic static state-feedback controller that does not explicitly need the application of the Floquet theory and which can be easily computed numerically. In addition to the complete uniform controllability condition, which is a common requirement in this kind of problem, the open-loop system must also be uniformly observable with the synthesized controller in order to be stabilised. The periodic state feedback gain is obtained through the numerical integration of two differential matrix equations over two periods.

1 Introduction

The study of Linear Time-Varying (LTV) periodic systems is an important subject of research in control theory that has received a considerable attention over the last decades. As a matter of fact, many real systems can be modeled as LTV periodic systems, such as manipulators for repetitive tasks, helicopters, satellites [33, 11, 19] and other important practical applications. Moreover, the local uniform stability of periodic non-linear systems may be analyzed through the global uniform stability of the linearized time-varying periodic system [31, 39, 5, 40, 14].

The determination of a control law for an LTV system is not an easy task, being a challenge that still motivates many researches in the area [15, 16, 29, 1, 28]. Several important concepts were presented by Kalman in 1960 [23], such as the notions of controllability, uniform controllability and (by duality) observability. Particularly for periodic systems, in [9] it is proved that a LTV system of order n is controllable if and only if it is controllable over n periods. A more recent result, presented in [20], stated that the minimum number of periods necessary to determine the controllability of the system is equal to the controllability index of the pair state transition matrix and controllability Gramian [21]. Moreover, if the system is periodic then complete controllability is equivalent to uniform complete controllability [32], which is a strong result and of utmost importance for the synthesis of stabilizing time-varying control laws.

Another important result from [23] is the introduction of the Riccati Differential Equations (RDE) [8, 7, 17] as a tool for the computation of a stabilizing controller, which is optimal in a certain sense. However, the RDE should be integrated over the whole interval of time considered, which could be infinite. Other approaches based on the solution of an RDE were developed [26, 12, 13, 29], but the determination of such a solution is still a complicated problem, even for the periodic case [6, 35]. Another approach uses of the controllability Gramian matrix in the expression of the control law [28, 20] but, in general, the calculation of such a matrix is a hard task. When dealing with periodic LTV systems, the main approach for synthesizing a stabilizing control law consists in the application of the Floquet theory [38, 28]. Essentially, with the Floquet theory one can determine a linear transformation using the so-called Floquet matrices in such a way that the transformed system is a time-invariant equivalent one and, then, classical synthesis techniques may be applied [20, 24, 37, 10, 28]. This approach, however, presents some drawbacks. In [20], the calculation of the controllability Gramian is necessary and the controller may present discontinuities. The methods in [24, 10] perform pole-placement techniques on the transformed time-invariant system, but may be numerically unstable [28]. Finally, [28] proposes a complicated procedure that may lead to numerical instability, and the resultant periodic controller is not guaranteed to have the same period of the system. In fact, even the computation of the Floquet matrices can be an issue for the control design.

The procedure proposed in [34] could be an alternative, since it does not require the application of the Floquet theory, but the control synthesis is performed analytically, being thus useful only for a reduced class of systems. Even for systems that admit an analytical solution, some assumptions are required, making this method quite restrictive.

The aim of this report is to present a method to synthesize a stabilizing periodic static controller that does not depend explicitly on the Floquet theory and can be easily numerically computed. The only requirements are two assumptions on the open-loop system.

The report is organized as follows. Section 2 introduces some definitions and propositions that are used throughout the text. The main results and contributions are given in Section 3. Section 4 presents a set of numerical experiments and Section 5 concludes the report.

Notation. The notation used in the report is standard. In general, capital letters denote matrices. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A and the identity matrix is denoted by I. The norm of a vector is $\|x\|^2 = \langle x, x\rangle$, $\langle \cdot, \cdot \rangle$ being the inner product. The norm $\|A\|$ of a matrix A is defined as $\sup \|Ax\|$ over $\|x\| = 1$. Time-varying arguments imply on time-varying values for the corresponding norms.

2 Preliminaries

Consider the linear continuous time-varying system described by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ being the state and the input vectors, respectively, and $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are piecewise continuous on t. The solution of (1) exists for all t and is unique if $u(t)$ is Lebesgue integrable. In this case $x(t)$, $t \geq t_0$ can be expressed as

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

where $x(t_0)$ is an initial condition and $\Phi(t, \tau)$, defined for all t and all τ, is the state transition matrix of the free system (1) satisfying the additional condition

$$\Phi(t, t) = I.$$

Associated with (1), the important notion of controllability [22] plays a crucial role in control problems.

Definition 1 *A state $x \in \mathbb{R}^n$ is said to be controllable at time t_0 if, for any desired final state $x_f \in \mathbb{R}^n$, there exists a control $u(t)$ depending on x and t_0 and defined over the finite interval $[t_0, t_f]$ such that $x(t_f) = x_f$. If this is true for every t_0, we say that system (1), or the pair $\{A(t), B(t)\}$, is completely controllable.*

A now classical necessary and sufficient condition can be stated to check complete controllability.

Theorem 2 *The pair $\{A(t), B(t)\}$ in (1) is completely controllable at time t_0 if and only if the symmetric matrix*

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, t)B(t)B'(t)\Phi'(t_0, t)dt \tag{2}$$

is positive definite for some $t_f > t_0$.

Matrix $W(t_0, t_f)$, also known as controllability Gramian, is associated with the energy of the control driving the system from an initial state $x(t_0)$ to the origin in a finite length of time (see [23] for details). It is possible to define a controllability concept for the class of nonstationary systems which is of a particular interest.

Theorem 3 *The pair $\{A(t), B(t)\}$ in (1) is uniformly completely controllable if and only if the following conditions hold for all t*

$$\begin{aligned} 0 < \alpha_1(\delta_c)I \leq W(t, t + \delta_c) \leq \alpha_2(\delta_c)I \\ 0 < \alpha_3(\delta_c)I \leq \Phi(t + \delta_c, t)W(t, t + \delta_c)\Phi'(t + \delta_c, t) \leq \alpha_4(\delta_c)I \end{aligned} \tag{3}$$

where δ_c is a fixed positive scalar and, for a bounded δ_c, the values $\alpha_i(\delta_c)$, $i = 1, \ldots, 4$ are positive and bounded.

From the conditions above, it can be shown that [23]

$$\frac{\alpha_3(\delta_c)}{\alpha_2(\delta_c)} \leq \|\Phi(t + \delta_c, t)\|^2 \leq \frac{\alpha_4(\delta_c)}{\alpha_1(\delta_c)}, \qquad \frac{\alpha_1(\delta_c)}{\alpha_4(\delta_c)} \leq \|\Phi(t, t + \delta_c)\|^2 \leq \frac{\alpha_2(\delta_c)}{\alpha_3(\delta_c)}. \tag{4}$$

Combining the previous inequalities with the definition of $W(t, t + \delta_c)$, the inequalities of Definition 3 are also true for all $\hat{\delta} > \delta_c$ and then [23]

$$\|\Phi(t, \tau)\| \leq \alpha_5(|t - \tau|) \text{ for all } t, \tau,$$

being $\alpha_5(\cdot)$ a positive function that is bounded for bounded values of $|t - \tau|$. The above condition is satisfied if (this can be proved using the Gronwall-Bellman Lemma [4])

$$\int_{t_0}^{t_1} \|A(\tau)\| \, d\tau \leq \alpha_6(t_1 - t_0) \text{ for all } t_0, t_1, \quad t_1 > t_0,$$

being $\alpha_6(\cdot)$ a positive function that is bounded for bounded values of $t_1 - t_0$.

If system (1) has an output given by

$$y(t) = C(t)x(t), \tag{5}$$

then the observability of the pair $\{A(t), C(t)\}$ can be assessed through the following theorem.

Theorem 4 *The pair $\{A(t), C(t)\}$ is completely observable at time t_0 if and only if the symmetric matrix*

$$W_o(t_0, t_f) = \int_{t_0}^{t_f} \Phi'(t, t_0)C'(t)C(t)\Phi(t, t_0)dt \tag{6}$$

is positive definite for some $t_f > t_0$.

By duality [22], the definition of complete uniform observability is equivalent to the definition of complete uniform controllability presented in Theorem 3 with the replacement of the controllability Gramian by the observability Gramian $W_o(t_0, t_f)$.

2.1 Periodic systems

The analysis performed in this text is particularized for periodic systems, i.e., it is considered that there exists a positive real T such that the matrices of system (1) satisfy

$$A(t) = A(t+T) \text{ and } B(t) = B(t+T). \tag{7}$$

In addition, it is supposed that there exists a bounded non-decreasing positive function $M_B(\sigma),\ 0 < \sigma \leq T$, such that

$$\int_t^{t+\sigma} \|B(\tau)\|^2 \, d\tau \leq M_B^2(\sigma). \tag{8}$$

The following lemma presents the bounds for the transition matrix of periodic systems.

Lemma 1 *Consider system (1) periodic with period T. The following properties hold.*

1. *The transition matrix can always be written as*

$$\Phi(t,t_0) = G(t,t_0)e^{R(t-t_0)} \tag{9}$$

where $G(t,t_0) = G(t+T,t_0),\ G(t_0,t_0) = I$ and $R \in \mathbb{R}^{n\times n}$ is a constant matrix.

2. *Matrix $e^{R(t-t_0)}$ satisfies*

$$e^{\lambda_{\min}\left(\frac{R+R'}{2}\right)(t-t_0)} \leq \left\| e^{R(t-t_0)} \right\| \leq e^{\lambda_{\max}\left(\frac{R+R'}{2}\right)(t-t_0)}. \tag{10}$$

3. *The transition matrix $\Phi(t,\tau)$ satisfies*

$$G_m e^{\xi_m(t-\tau)} \leq \|\Phi(t,\tau)\| \leq G_M e^{\xi_M(t-\tau)} \text{ for all } t \geq \tau \tag{11}$$

$$\tfrac{1}{G_M} e^{\xi_M(t-\tau)} \leq \|\Phi(t,\tau)\| \leq \frac{1}{G_m} e^{\xi_m(t-\tau)} \text{ for all } t \leq \tau \tag{12}$$

where

$$\begin{array}{ll} 2\xi_m = \lambda_{\min}(R+R'), & 2\xi_M = \lambda_{\max}(R+R') \\ G_M = \max\limits_{t\in[t_1,t_1+T]} \|G(t,t_1)\|, & G_m = \min\limits_{t\in[t_1,t_1+T]} \|G(t,t_1)\|. \end{array}$$

Proof. The proof of the first item can be found in [38]. The proof of the second item follows from some properties of norms (see [18] for details). For the third item, remark that for $t \geq \tau$

$$\Phi(t,\tau)\Phi'(t,\tau) = G(t,\tau)e^{R(t-\tau)}e^{R'(t-\tau)}G'(t,\tau).$$

From (10), one has

$$\|G(t,\tau)\|^2 e^{\lambda_{\min}(R+R')(t-\tau)} \leq \left\|\Phi(t,\tau)\Phi'(t,\tau)\right\| \leq \|G(t,\tau)\|^2 e^{\lambda_{\max}(R+R')(t-\tau)}.$$

From the definitions of ξ_m and ξ_M, it follows that

$$\|G(t,\tau)\|^2 e^{2\xi_m(t-\tau)} \leq \left\|\Phi(t,\tau)\Phi'(t,\tau)\right\| \leq \|G(t,\tau)\|^2 e^{2\xi_M(t-\tau)}$$

which leads to

$$\|G(t,\tau)\| e^{\xi_m(t-\tau)} \leq \|\Phi(t,\tau)\| \leq \|G(t,\tau)\| e^{\xi_M(t-\tau)}.$$

Introducing G_m and G_M, which exist and are bounded if $A(t)$ is piecewise continuous in t, inequality (11) follows. Inequality (12) is proved in a similar way, first noting

$$\Phi(\tau,t) = \Phi(t,\tau)^{-1} = e^{-R(t-\tau)}G(t,\tau)^{-1},$$

therefore

$$\Phi(\tau,t)'\Phi(\tau,t) = G'(t,\tau)^{-1}e^{-R'(t-\tau)}e^{-R(t-\tau)}G(t,\tau)^{-1}$$

and

$$\left\|G(t,\tau)^{-1}\right\|^2 e^{-2\xi_M(t-\tau)} \le \left\|\Phi'(\tau,t)\Phi(\tau,t)\right\| \le \left\|G(t,\tau)^{-1}\right\|^2 e^{-2\xi_m(t-\tau)}.$$

Remarking that

$$\left\|G(t,\tau)^{-1}\right\|^2 = \lambda_{max}(G(t,\tau)^{-1}G'(t,\tau)^{-1}) = \frac{1}{\lambda_{min}(G(t,\tau)G'(t,\tau))} \le \frac{1}{G_m^2}$$

and exchanging the role of t and τ, we have for $\tau \ge t$

$$\frac{1}{\|G(\tau,t)\|}e^{\xi_M(t-\tau)} \le \|\Phi(t,\tau)\| \le \frac{1}{\|G(\tau,t)\|}e^{\xi_m(t-\tau)}$$

and inequality (12) follows. ■

If the periodic system (1) is uniformly completely controllable, the bounds $\alpha_1,\ \alpha_2,\ \alpha_3$ and α_4 of Theorem 3 can be calculated using the following lemma.

Lemma 2 *If the periodic system (1) is uniformly completely controllable, then the bounds* $\alpha_1,\ \alpha_2,\ \alpha_3$ *and* α_4 *can be taken as*

$$\alpha_1(\delta_c) = \frac{M_B(\delta_c)}{G_M e^{2\xi_M\delta_c}}\sqrt{\frac{e^{2\xi_M\delta_c}-1}{2\xi_M}},\quad \alpha_2(\delta_c) = \frac{M_B(\delta_c)}{G_m}\sqrt{\frac{1-e^{-2\xi_m\delta_c}}{2\xi_m}}$$

$$\alpha_3(\delta_c) = M_B(\delta_c)G_m e^{2\xi_m\delta_c}\sqrt{\frac{1-e^{-2\xi_m\delta_c}}{2\xi_m}},\ \alpha_4(\delta_c) = M_B(\delta_c)G_M\sqrt{\frac{e^{2\xi_M\delta_c}-1}{2\xi_M}}$$

Proof. We have, using Schwarz inequality and Lemma 1

$$\begin{aligned}\|W(t,t+\delta_c)\| &\le \int_t^{t+\delta_c}\|\Phi(t,\tau)\|^2\|B(\tau)\|^2 d\tau \le \left(\int_t^{t+\delta_c}\|\Phi(t,\tau)\|^2 d\tau\right)^{1/2}\left(\int_t^{t+\delta_c}\|B(\tau)\|^2 d\tau\right)^{1/2} \\ &\le M_B(\delta_c)\left(\int_t^{t+\delta_c}\frac{1}{G_m^2}e^{2\xi_m(t-\tau)}d\tau\right)^{1/2} = \frac{M_B(\delta_c)}{G_m}\sqrt{\frac{1-e^{-2\xi_m\delta_c}}{2\xi_m}} = \alpha_2(\delta_c).\end{aligned}$$

For the last equality, note that G_m^2 is independent of τ and it suffices to solve the integral of the exponential. Similarly, one has

$$\begin{aligned}\left\|\Phi(t+\delta_c,t)W(t,t+\delta_c)\Phi'(t+\delta_c,t)\right\| &\le M_B(\delta_c)\left(\int_t^{t+\delta_c}\|\Phi(t+\delta_c,\tau)\|^2 d\tau\right)^{1/2} \\ \le M_B(\delta_c)\left(\int_t^{t+\delta_c}G_M^2 e^{2\xi_M(t+\delta_c-\tau)}d\tau\right)^{1/2} &= M_B(\delta_c)G_M\sqrt{\frac{e^{2\xi_M\delta_c}-1}{2\xi_M}} = \alpha_4(\delta_c).\end{aligned}$$

From item 3 of Lemma 1 and Definition 3, it follows

$$\|\Phi(t+\delta_c,t)\|^2 \le G_M^2 e^{2\xi_M\delta_c} = \frac{\alpha_4(\delta_c)}{\alpha_1(\delta_c)}.$$

Hence, it follows that $\alpha_1(\delta_c)$ is defined by

$$\alpha_1(\delta_c) = \frac{\alpha_4(\delta_c)}{G_M^2 e^{2\xi_M\delta_c}} = \frac{M_B(\delta_c)}{G_M e^{2\xi_M\delta_c}}\sqrt{\frac{e^{2\xi_M\delta_c}-1}{2\xi_M}}.$$

Finally, by using the same arguments one gets

$$\|\Phi(t+\delta_c,t)\|^2 \geq \frac{\alpha_3(\delta_c)}{\alpha_2(\delta_c)} = G_m^2 e^{2\xi_m\delta_c} \Rightarrow \alpha_3(\delta_c) = \alpha_2(\delta_c) G_m^2 e^{2\xi_m\delta_c} = M_B(\delta_c) G_m e^{2\xi_m\delta_c}\sqrt{\frac{1-e^{-2\xi_m\delta_c}}{2\xi_m}}.$$

The proof is completed by noticing that all the terms used to express the bounds $\alpha_i(\delta_c), i = 1, \ldots, 4$ are positive and bounded. ■

Although the uniform complete controllability is a stronger condition than the complete controllability, both concepts are equivalent for continuous periodic linear systems [32], as shown in the following theorem.

Theorem 5 *If system (1) of order n is T-periodic, then it is uniformly completely controllable if and only if $W(0, nT) > 0$.*

The proof of Theorem 5 is based on the results of [9], where it is shown that a continuous periodic system is completely controllable if and only if $W(0, nT) > 0$, and of [32], which states that, for continuous periodic systems, the complete controllability is equivalent to the uniform complete controllability. Actually, in [20] it is proved that the complete controllability of a periodic system can be verified for a number $k \leq n$ of periods, being k the controlalbility index of the pair transition matrix and controllability Gramian. By duality, similar results regarding the observability of the system can be obtained.

Concerning the stability for periodic systems, one has the following well known result [38].

Lemma 3 *System (1) periodic with period T and with $u(t) = 0$ is asymptotically stable if and only if the characteristic exponents of $A(t)$ have negative real parts, the characteristic exponents being defined as the eigenvalues of matrix R in (9).*

The determination of matrix R is in general a difficult task (see [28] for details). In [15], a simple numerical test to assess the stability of the system without calculating R is proposed. It needs the integration over a period of the following Lyapunov differential equation

$$\frac{\partial}{\partial t} X(t,t_0) = A(t)X(t,t_0) + X(t,t_0)A'(t), \quad X(t_0,t_0) = I. \tag{13}$$

The solution of the above Lyapunov differential equation is given by

$$X(t,t_0) = \Phi(t,t_0)\Phi'(t,t_0). \tag{14}$$

Lemma 4 *[15] System (1) periodic with period T and with $u(t) = 0$ is asymptotically stable if and only if*

$$\rho_T = \max_{t\in[t_0,t_0+T]} \lambda_{\max}^{1/2}\left(X(t,t_0)\right) < \infty$$

and

$$\overline{\rho}(t_0+T) < \overline{\rho}(t_0)$$

where

$$\overline{\rho}(t) = \lambda_{\max}^{1/2}\left(X(t,t_0)\right).$$

2.2 Problem statement

We are now in position to state the main problem addressed in this text.

Problem 1 ***(Stabilization problem)***. *Considering the periodic system (1), find a state feedback control law $u(t) = K(t)x(t)$ such that the closed-loop system is asymptotically stable.*

3 Main results

3.1 Stabilization problem

The following theorem gives a possible solution to Problem 1.

Theorem 6 *Consider the system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{15}$$

with $A(t) = A(t+T)$, $B(t) = B(t+T)$ *and* $X(t, t+\delta)^{-1}$ *as defined in (14). If the pair* $\{A(t), B(t)\}$ *is completely controllable and the pair* $\{A(t), B'(t)X(t, t+\delta)^{-1}\}$ *is completely observable for a certain value of* $\delta \geq 0$, *then the control*

$$u(t) = K_\delta(t)x(t)$$

with

$$K_\delta(t) = -\beta \; B'(t)X(t, t+\delta)^{-1},$$

for $\beta \in \mathbb{R}$ *sufficiently large, solves Problem 1. In addition, the control gain matrix* $K_\delta(t)$ *satisfies* $K_\delta(t) = K_\delta(t+T)$.

Proof.

The first step is to prove that the closed-loop dual system

$$\dot{\widetilde{x}}(t) = -\left(A(t) - \beta B(t)B'(t)X(t, t+\delta)^{-1}\right)' \widetilde{x}(t) \tag{16}$$

is unstable. Consider the function

$$V(t, \widetilde{x}) = \widetilde{x}(t)'X(t, t+\delta)\widetilde{x}(t)$$

Since $X(t, t+\delta)^{-1} = \Phi'(t, t+\delta)\Phi(t, t+\delta)$, it follows that $X(t, t+\delta) = \Phi(t+\delta, t)\Phi'(t+\delta, t)$. Thus, by using item 3 of Lemma 1, one can conclude that this function satisfies

$$\frac{1}{G_M}e^{-\xi_M \delta}\widetilde{x}(t)'\widetilde{x}(t) \leq V(t, \widetilde{x}) \leq \frac{1}{G_m}e^{-\xi_m \delta}\widetilde{x}(t)'\widetilde{x}(t).$$

The time-derivative of $V(t, \widetilde{x})$ along the trajectories of system (16) is given by

$$\begin{aligned}\dot{V}(t, \widetilde{x}) &= \widetilde{x}(t)' \left(-A(t)X(t, t+\delta) - X(t, t+\delta)A'(t) + 2\beta B(t)B'(t) + \dot{X}(t, t+\delta)(t)\right) \widetilde{x}(t) \\ &= \widetilde{x}(t)' \left(2\beta B(t)B'(t) - \Phi(t, t+\delta)\left(A(t+\delta) + A'(t+\delta)\right)\Phi'(t, t+\delta)\right) \widetilde{x}(t)\end{aligned}$$

due to

$$\begin{aligned}\dot{X}(t, t+\delta) &= \frac{d}{d\tau}X(\tau, t+\delta)\Big|_{\tau=t} + \frac{d}{d\sigma}X(t, \sigma)\Big|_{\sigma=t+\delta} \\ &= A(t)X(t, t+\delta) + X(t, t+\delta)A'(t) - \Phi(t, t+\delta)\left(A(t+\delta) + A'(t+\delta)\right)\Phi'(t, t+\delta). \end{aligned} \tag{17}$$

If the function $V(t, \widetilde{x})$ satisfies

$$\int_t^{t+nT} \dot{V}(\tau, \widetilde{x})d\tau = V(t+nT, \widetilde{x}) - V(t, \widetilde{x}) > 0 \tag{18}$$

then there exists a Lyapunov function $\hat{V}(t, \widetilde{x})$ which is positive definite, bounded and whose time-derivative is definite positive $\forall t$ [3], therefore the closed-loop dual system (16) is unstable. One has

$$\begin{aligned}\int_t^{t+nT} \dot{V}(\tau, \widetilde{x})d\tau = 2\int_t^{t+nT} &\widetilde{x}(\tau)'\beta B(\tau)B'(\tau)\widetilde{x}(\tau) \\ &- \int_t^{t+nT} \widetilde{x}(\tau)'\Phi(\tau, \tau+\delta)\left(A(\tau+\delta) + A'(\tau+\delta)\right)\Phi'(\tau, \tau+\delta)\widetilde{x}(\tau)d\tau. \end{aligned} \tag{19}$$

Introducing now the state

$$\overline{x}(t_2) = \Phi'(t_1,t_2)\widetilde{x}(t_1),$$

the right-hand term of equation (19) can be rewritten as

$$\begin{aligned}
2\overline{x}(t)'\beta\left(\int_t^{t+nT}\Phi(t,\tau)B(\tau)B'(\tau)\Phi'(t,\tau)d\tau\right)\overline{x}(t)-\overline{x}(t)'\int_t^{t+nT}\Phi(t,\tau+\delta)\left(A(\tau)+A'(\tau)\right)\Phi'(t,\tau+\delta)d\tau\overline{x}(t)\\
=2\overline{x}(t)'\beta W(t,t+nT)\overline{x}(t)+\overline{x}(t)'\int_t^{t+nT}\frac{d}{d\tau}(\Phi(t,\tau+\delta)\Phi'(t,\tau+\delta))d\tau\;\overline{x}(t)\\
=2\overline{x}(t)'\beta W(t,t+nT)\overline{x}(t)+\overline{x}(t)'\left(\Phi(t,t+nT+\delta)\Phi'(t,t+nT+\delta)-\Phi(t,t+\delta)\Phi'(t,t+\delta)\right)\overline{x}(t)
\end{aligned}$$

However by Lemma 1, item 3, one has

$$\begin{aligned}
&\overline{x}(t)'\left(\Phi(t,t+nT+\delta)\Phi'(t,t+nT+\delta)-\Phi(t,t+\delta)\Phi'(t,t+\delta)\right)\overline{x}(t)\\
&\geq\overline{x}(t)'\left(\frac{1}{G_M^2}e^{-2\xi_M(nT+\delta)}-\frac{1}{G_m^2}e^{-2\xi_m\delta}\right)\overline{x}(t)
\end{aligned}$$

Finally, if system (15) is completely controllable, it follows that

$$\int_t^{t+nT}\dot{V}(\tau,\widetilde{x})d\tau\geq\overline{x}(t)'\left(2\beta\lambda_{min}(W(t,t+nT))+\frac{1}{G_M^2}e^{-2\xi_M(nT+\delta)}-\frac{1}{G_m^2}e^{-2\xi_m\delta}\right)\overline{x}(t).$$

By invoking Lemma 2, one has

$$\lambda_{min}(W(t,t+nT))\geq\alpha_1(nT)=\frac{M_B(nT)}{G_Me^{2\xi_MnT}}\sqrt{\frac{e^{2\xi_MnT}-1}{2\xi_M}} \tag{20}$$

and there exists β satisfying

$$\beta>\beta_l=\left(\frac{1}{G_m^2}e^{-2\xi_m\delta}-\frac{1}{G_M^2}e^{-2\xi_M(nT+\delta)}\right)\left(\frac{G_Me^{2\xi_MnT}}{2M_B(nT)}\sqrt{\frac{2\xi_M}{e^{2\xi_MnT}-1}}\right) \tag{21}$$

such that

$$\int_t^{t+nT}\dot{V}(\tau,\widetilde{x})d\tau>0$$

proving the exponential unstability of the dual. The dual system is related to the primal system by the transformation

$$x(t)=X_{cl}(t,t_0)\tilde{x}(-t), \tag{22}$$

with $X_{cl}(t,t_0)=\Phi_{cl}(t,t_0)\Phi'_{cl}(t,t_0)$ the solution of the Lyapunov differential equation (13) for the closed-loop system and $\Phi_{cl}(t,t_0)$ the closed-loop transition matrix. Note that the direction of time in the dual system is the reverse of the direction of time in the primal system [23], and since $\tilde{x}(t)$ is demonstrated to be exponentially unstable, then $\tilde{x}(-t)$ is uniformly asymptotically stable. With (22), one has

$$||x(t)||\leq||X_{cl}(t,t_0)||\;||\tilde{x}(-t)||$$

and it suffices to show that $X_{cl}(t,t_0)$ is bounded to prove the uniform asymptotic stability of $x(t)$.

At this aim, consider the differential equation (13) for the closed-loop system, given by

$$\begin{aligned}
\frac{\partial}{\partial t}X_{cl}(t,t_0)=A(t)X_{cl}(t,t_0)+X_{cl}(t,t_0)A'(t)-\beta B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0)-\\
\beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)\;,\quad X_{cl}(t_0,t_0)=I.
\end{aligned}$$

It is easy to see that

$$\big(B(t)+\beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)\big)\big(B(t)+\beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)\big)' \geq 0$$

implies

$$\begin{aligned}&-\beta B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0)-\beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t) \leq \\ &B(t)B'(t)+\beta^2 X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0)\end{aligned}$$

and, as proved in the Appendix, one has that the matrix $P(t,t_0)$ solution of

$$\begin{aligned}\frac{\partial}{\partial t}P(t,t_0) = A(t)P(t,t_0)+P(t,t_0)A'(t)+B(t)B'(t)+& \\ \beta^2 P(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}P(t,t_0)\ ,\ & P(t_0,t_0)\geq I. \qquad (23)\end{aligned}$$

satisfies

$$X_{cl}(t,t_0) \leq P(t,t_0).$$

On the other hand, the solution $P(t,t_0)$ of (23) exists, is unique and has bounds

$$0 < \alpha_{p1} I \leq P(t,t_0) \leq \alpha_{p2} I$$

if and only if the pair $\{A(t),B(t)\}$ is uniformly completely controllable and the pair $\{A(t),B'(t)X(t,t+\delta)^{-1}\}$ is uniformly completely observable [6], which are conditions satisfied by hypothesis. Therefore, one has

$$X_{cl}(t,t_0) \leq P(t,t_0) \leq \alpha_{p2} I,$$

which means that $||X_{cl}(t,t_0)||$ has a finite upper-bound, and therefore the closed-loop system (15) is uniformly asymptotically stable.

Finally, from the properties of periodic systems,

$$\begin{aligned}K_\delta(t+T) &= -\beta\ B'(t+T)\Phi'(t+T+\delta,t+T)\Phi(t+T+\delta,t+T)\\ &= -\beta\ B'(t)\Phi'(t+\delta,t)\Phi(t+\delta,t) = K_\delta(t)\end{aligned}$$

and the proof is completed. ■

According to the proof of Theorem 6, the value of the constant β must be higher than a threshold to guarantee the stability of the closed-loop system. Note that such constant appears explicitly in the expression of the difference of the function $V(t,\tilde{x})$ shown in (18), meaning that variations of the value of β may have an influence on the convergence rate of the closed-loop trajectories [25, 36]. More precisely, from [27, Lemma 6] and adapting its results to the present notation, if the trajectories of the closed-loop system are bounded by

$$||x(t)|| \leq \kappa ||x(t_0)|| e^{-\gamma(t-t_0)},$$

then the convergence rate γ can be estimated by

$$\begin{aligned}\gamma &= \frac{1}{2}\max\left\{\frac{nT(c_3(\beta)+1)}{c_3(\beta)},\frac{c_2}{c_1}\right\}^{-2},\quad \text{with}\ \ c_1=\frac{1}{G_M}e^{-\xi_M\delta},\ c_2=\frac{1}{G_m}e^{-\xi_m\delta},\\ c_3(\beta) &= \lambda_{min}\left(2\beta\lambda_{min}(W(0,nT))+\frac{1}{G_M^2}e^{-2\xi_M(nT+\delta)}-\frac{1}{G_m^2}e^{-2\xi_m\delta}\right)\end{aligned}$$

Note that, since $\big(nT(c_3(\beta)+1)\big)/\big(c_3(\beta)\big)$ is a decreasing function of β, the convergence rate γ increases with β until a higher bound, defined by c_2/c_1. From this value, increasing β does not change the convergence of the trajectories of the closed-loop system.

3.2 Computational issues

This section discusses the computational issues and more particularly a way to compute the control gain $K_\delta(t)$. Because the control gain is periodic, it is only necessary to determine $K_\delta(t)$ for $t \in [t_0, t_0+T]$. This determination can be done as follows.

1. Choice of a $\delta \geq 0$ such that the pair $\{A(t), B'(t)X(t,t+\delta)^{-1}\}$ is completely observable;
2. Integration of the following differential matrix equations

$$\begin{aligned}
\frac{\partial}{\partial t}\Phi(t,t_0) &= A(t)\Phi(t,t_0) \text{ for } t \in [t_0, t_0+T+\delta] \\
\frac{\partial}{\partial t}\Phi(t_0,t) &= -\Phi(t_0,t)A(t) \text{ for } t \in [t_0, t_0+T] \\
\Phi(t_0,t_0) &= I
\end{aligned}$$

3. Computation of $\Phi(t+\delta, t)$ from the solutions to the above differential equations, with

$$\Phi(t+\delta,t) = \Phi(t+\delta,t_0)\Phi(t_0,t) \text{ for } t \in [t_0, t_0+T]$$

4. Computation of $K_\delta(t)$

$$K_\delta(t) = -\beta B'(t)\Phi'(t+\delta,t)\Phi(t+\delta,t) \quad \text{for } t \in [t_0, t_0+T]$$

From Theorem 6, it suffices to take β sufficiently large. Note that in the gain expression, one can change the values of β without recomputing the product $\Phi'(t+\delta,t)\Phi(t+\delta,t)$. Concerning the choice of δ, for the analyzed systems usually any $\delta \neq 0$ was able to generate a stabilizable gain. Closed-loop asymptotic stability can then be verified using Lemma 4.

4 Numerical examples

The routines were implemented in MATLAB, version 7.0.1 (R14), and the differential equations were solved using the procedure `ode45` with the standard parameters. The computer used was an Intel® Pentium® D (3.40 GHz), 1GB RAM, Linux Fedora 12.

4.1 Example 1

Consider the periodic LTV system given by

$$\dot{x}(t) = \begin{bmatrix} 2 & 0 \\ -10\sin(t) & -1 \end{bmatrix} x(t) + \begin{bmatrix} \sin(t) \\ -\sin(t)(\cos(t)+3\sin(t)) \end{bmatrix} u(t), \tag{24}$$

presented in [34], whose period is $T = 2\pi$. Figure 1 presents the function $\bar{\rho}(t)$, introduced in Lemma 4, showing that the open-loop system is indeed unstable. Previous calculations have shown that the hypotheses of Theorem 6 are satisfied for $\delta = T$.

An analytical way to calculate a stabilizing controller for LTV periodic systems, provided that some conditions are satisfied, is presented on [34]. Assuming that system (24) satisfies such conditions, the method presented on [34] yields the following stabilizing controller gain

$$\tilde{K}(t) = \begin{bmatrix} -8\sin(t) & 0 \end{bmatrix}. \tag{25}$$

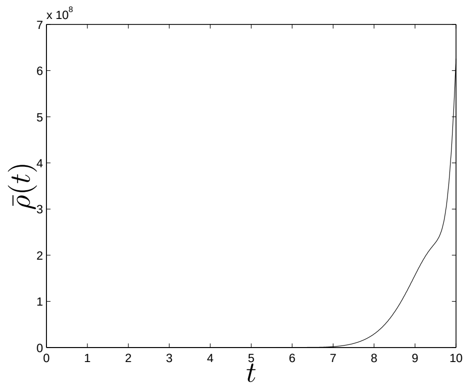

Figure 1: Function $\bar{\rho}(t)$ obtained from the open-loop system analyzed in Example 1.

Since the transition matrix of system (24) can be obtained analytically, it is possible to determine an expression for the gain $K_\delta(t)$ generated by the method presented in this text. Consider the transition matrix $\Phi(t, t_0)$ given by

$$\Phi(t,t_0) = \begin{bmatrix} e^{2(t-t_0)} & 0 \\ f(t,t_0) & e^{-(t-t_0)} \end{bmatrix}, \quad f(t,t_0) = -e^{-(t-t_0)} + e^{2(t-t_0)}(\cos(t) - 3\sin(t)). \tag{26}$$

Using $\delta = T$, the stabilizing gain $K_T(t)$ is then given from Theorem 6 by

$$K_T(t) = -\beta B'(t)\Phi'(t+T,t)\Phi(t+T,t) \tag{27}$$

$$= -\beta \begin{bmatrix} \sin(t)\big(-e^{4T} - f(t+T,t)^2 + f(t+T,t)e^{-T}(\cos(t) + 3\sin(t))\big) \\ \sin(t)\big(-f(t+T,t)e^{-T} + e^{-2T}(\cos(t) + 3\sin(t))\big) \end{bmatrix}'. \tag{28}$$

Figure 2 depicts the envelope $\bar{\rho}_{cl}(t)$ of the trajectories of the closed-loop system for the gain $K_T(t)$, generated numerically by the method presented in this report with $\beta = 10^{-11}$, showing that the system can be stabilized. Figure 3 shows the gain $K_T(t)$ (solid line) along with the gain $\tilde{K}(t)$ (dashed line) presented in Equation (25). Since the second element of $K_T(t)$ is almost equal to zero, as the second element of $\tilde{K}(t)$, such values are not shown in the figure. The main difference between $K_T(t)$ and $\tilde{K}(t)$ resides in the fact that $K_T(t)$ can be obtained numerically, whereas $\tilde{K}(t)$ is obtained by an analytical method.

A stabilizing gain can also be obtained by applying the receding horizon technique presented in [26], which consists in the resolution of a Riccati differential equation (RDE) for each time instant and on the utilization of the inverse of the solution on the control. One advantage of the present method is that the numerical complexity of calculating twice the transition matrix is considerably lower than the complexity of solving a RDE for each time instant, as can be seen by the execution times spent for each method: the present method calculated the gain in less than $1s$ while the method from [26] spent $39s$. Both times are related to the calculation of the control gain over only one period.

4.2 Example 2

Consider the periodic LTV system given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 + 3\sin(\pi t) & -2 + 3\cos(\pi t) \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t), \tag{29}$$

with period $T = 2$. Figure 4 represents the function $\bar{\rho}(t)$, showing that the open-loop system is indeed unstable. Previous calculations have shown that the hypotheses of Theorem 6 are satisfied for $\delta = T$.

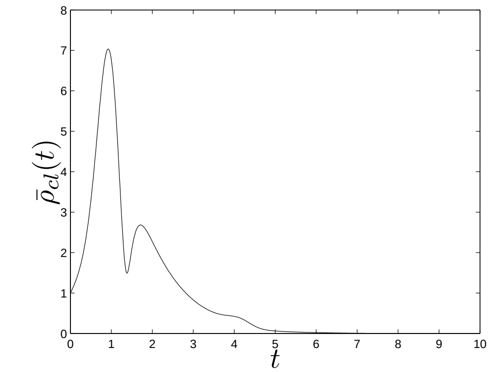

Figure 2: Function $\bar{\rho}_{cl}(t)$ obtained from the closed-loop system analyzed in Example 1.

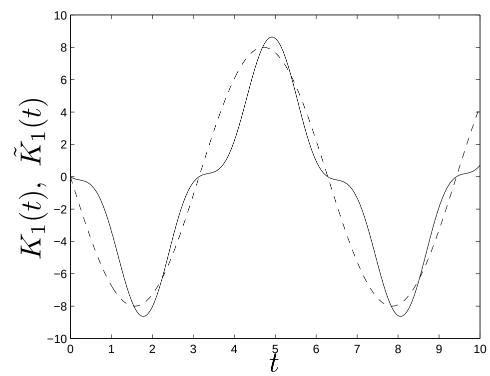

Figure 3: Gains $K_T(t) = [K_1(t)\ 0]$ (solid line) and $\tilde{K}(t) = [\tilde{K}_1(t)\ 0]$ (dashed line) for Example 1.

In order to apply the method presented on [34] yielding a stabilizing controller, all the characteristic multipliers of $A(t)$, which are the eigenvalues of the matrix $\Phi(t_0 + T, t_0)$, must be real and positive. In this example, however, the characteristic multipliers are all negative, therefore the method developed in [34] cannot be applied. Nevertheless, since the system satisfies all the hypotheses of Theorem 6, it is possible to calculate a stabilizing gain $K_\delta(t)$ using the method presented in this report. Figure 5 shows the gain $K_\delta(t) = K_T(t)$ obtained numerically for $\beta = 1$, and Figure 6 shows the function $\bar{\rho}_{cl}(t)$ for several values of β. Note that, as already discussed, greater values of β result on trajectories with higher decreasing rates, but there is a limit β_{max} such that considering $\beta \geq \beta_{max}$ yields the same result as for $\beta = \beta_{max}$. On the other hand, lower values of β result on trajectories with lower decreasing rates, with a limit β_{min} such that considering $\beta < \beta_{min}$ does not generate a stable behavior. The values of β considered in the example were determined through numerical analysis.

The numerical procedure to construct the stabilizing gain $K_\delta(t)$ spent again less than $1s$, while the procedure to synthesize the gain as proposed in [26] spent about $6s$, proving again that the method proposed in this report is considerably less complex than the ones that resort on the resolution of a RDE.

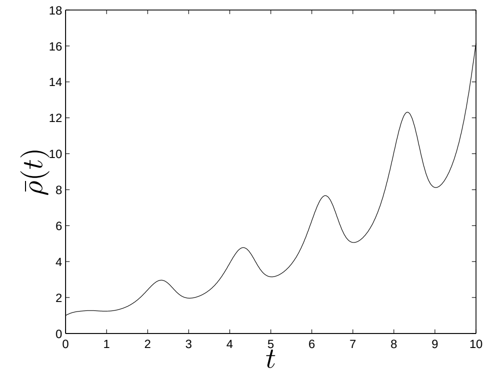

Figure 4: Function $\bar{\rho}(t)$ obtained from the open-loop system analyzed in Example 2.

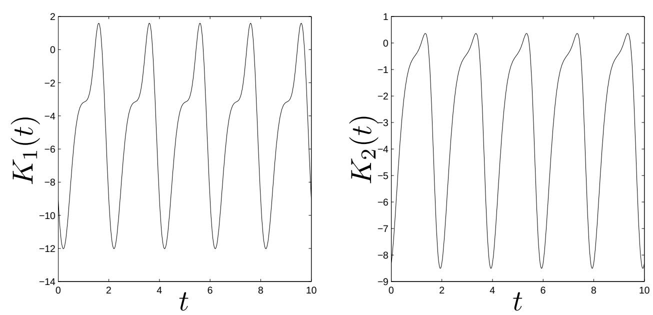

Figure 5: Gain $K_T(t) = [K_1(t)\ K_2(t)]$ for Example 2.

5 Conclusion

A new stabilization procedure for linear continuous time-varying periodic systems is presented in this report. It is shown that the proposed controller, which can be constructed by following a computationally simple numerical method, is capable of stabilizing asymptotically a wide class of periodic LTV systems. The proposed synthesis procedure is numerically stable and, since it is based on the calculation of the transition matrix and not on the resolution of a Riccati differential equation, it is simpler than other methods in the literature. The advantages of the proposed technique are illustrated through a series of examples.

6 Appendix

On the proof of Theorem 6 it is stated that, given

$$\frac{\partial}{\partial t}X_{cl}(t,t_0) = A(t)X_{cl}(t,t_0) + X_{cl}(t,t_0)A'(t) - \beta B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0) - \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t),\ X_{cl}(t_0,t_0) = I, \quad (30)$$

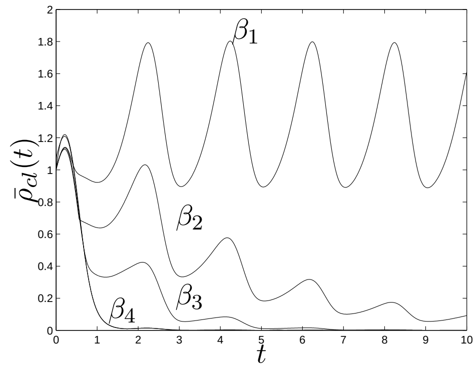

Figure 6: Function $\bar{\rho}_{cl}(t)$ obtained from the closed-loop system analyzed in Example 2. The values $\beta_1 = 0.0222, \beta_2 = 0.05, \beta_3 = 0.1$ and $\beta_4 = 1$ are the different values of β used to calculate $K(t)$.

then one has that the solution $P(t,t_0)$ of

$$\frac{\partial}{\partial t}P(t,t_0) = A(t)P(t,t_0) + P(t,t_0)A'(t) + B(t)B'(t)+ \beta^2 P(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}P(t,t_0)\ ,\ P(t_0,t_0) \geq I \quad (31)$$

satisfies

$$X_{cl}(t,t_0) \leq P(t,t_0).$$

To prove the validity of the latter statement, it suffices to verify if the variable $\Delta(t,t_0)$ defined as

$$\Delta(t,t_0) = X_{cl}(t,t_0) - P(t,t_0)$$

is semi-definite negative for all $t \geq t_0$. Using equations (30) and (31), one has

$$\frac{\partial}{\partial t}\Delta(t,t_0) = A(t)\Delta(t,t_0) + \Delta(t,t_0)A'(t) - \beta B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0) - \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t) - B(t)B'(t) - \beta^2 P(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}P(t,t_0). \quad (32)$$

Replacing $P(t,t_0) = X_{cl}(t,t_0) - \Delta(t,t_0)$ and

$$G(t,t_0,\delta) = -\beta B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0) - \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t) - B(t)B'(t),$$

one has

$$\frac{\partial}{\partial t}\Delta(t,t_0) = A(t)\Delta(t,t_0)+\Delta(t,t_0)A'(t)+G(t,t_0,\delta)-\beta^2 X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0) +\beta^2\Delta(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0)+\beta^2 X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}\Delta(t,t_0) - \beta^2\Delta(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}\Delta(t,t_0) \quad (33)$$

Noting that

$$G(t,t_0,\delta) - \beta^2 X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}X_{cl}(t,t_0) = -\big(B(t) + \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)\big)\big(B(t) + \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)\big)' \quad (34)$$

equation (33) can be rewritten as

$$\frac{\partial}{\partial t}\Delta(t,t_0) = M(t,t_0,\delta)\Delta(t,t_0) + \Delta(t,t_0)M'(t,t_0,\delta) + W(t,t_0,\delta), \tag{35}$$

being

$$\begin{array}{rl} M(t,t_0,\delta) = & A(t) + \beta^2 X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1} \\ W(t,t_0,\delta) = & -\beta^2\Delta(t,t_0)X(t,t+\delta)^{-1}B(t)B'(t)X(t,t+\delta)^{-1}\Delta(t,t_0) \\ & -\big(B(t) + \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)\big)\big(B(t) + \beta X_{cl}(t,t_0)X(t,t+\delta)^{-1}B(t)\big)' \end{array} \tag{36}$$

According to [30, Lemma 3], the solution $\Delta(t,t_0)$ is given by

$$\Delta(t,t_0) = \int_{t_0}^{t} \Phi_M(t,\tau)W(\tau)\Phi_M'(t,\tau)d\tau + \Phi_M(t,t_0)\Delta(t_0,t_0)\Phi'(t,t_0),$$

being $\Phi_M(t,t_0)$ the transition matrix of the system whose dynamic matrix is $M(t,t_0,\delta)$. Since $W(t,t_0,\delta) \leq 0$, the matrix $\Delta(t,t_0)$ is semi-definite negative if and only if

$$\Delta(t_0,t_0) = X_{cl}(t_0,t_0) - P(t_0,t_0) = I - P(t_0,t_0) \leq 0.$$

Therefore, the validity of the inequality (31) is demonstrated.

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