Additive generators in interval-valued and intuitionistic fuzzy set theory

Glad Deschrijver
Fuzziness and Uncertainty Modeling Research Unit
Department of Mathematics and Computer Science, Ghent University
Krijgslaan 281 (S9), B-9000 Gent, Belgium
E-mail: Glad.Deschrijver@UGent.be
Homepage: http://www.fuzzy.UGent.be

Abstract

Intuitionistic fuzzy sets in the sense of Atanassov and interval-valued fuzzy sets can be seen as $\mathcal{L}$-fuzzy sets w.r.t. a special lattice $\mathcal{L}'$. Deschrijver [2] introduced additive and multiplicative generators on $\mathcal{L}'$ based on a special kind of addition introduced in [3]. Actually, many other additions can be introduced. In this paper we investigate additive generators on $\mathcal{L}'$ as far as possible independently of the addition. For some special additions we investigate which t-norms can be generated by continuous additive generators which are a natural extension of an additive generator on the unit interval.

Keywords: intuitionistic fuzzy set, interval-valued fuzzy set, additive generator, addition on $\mathcal{L}'$, representable

1 Introduction

Triangular norms on $([0, 1], \leq)$ were introduced in [18] and play an important role in fuzzy set theory (see e.g. [12, 14] for more details). Generators are very useful in the construction of t-norms: any generator on $([0, 1], \leq)$ can be used to generate a t-norm. Generators play also an important role in the representation of continuous Archimedean t-norms on $([0, 1], \leq)$. Moreover, some properties of t-norms which have a generator can be related to properties of their generator. See e.g. [9, 13, 14, 15, 16] for more information about generators on the unit interval.

Interval-valued fuzzy set theory [11, 17] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. In [6] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to $L$-fuzzy set theory in the sense of Goguen [10] w.r.t. a special lattice $\mathcal{L}'$. In [2] additive and multiplicative generators on $\mathcal{L}'$ are investigated based on a special kind of addition introduced in [3]. In [8] another addition was defined. In fact, many more additions can be introduced. Therefore, in this paper we will investigate additive generators on $\mathcal{L}'$ as far as possible independently of the addition. For some special additions we will investigate which t-norms can be generated
by continuous additive generators which are a natural extension of an additive generator on the unit interval.

2 The lattice $\mathcal{L}^I$

**Definition 2.1** We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where

$$ L^I = \{(x_1, x_2) \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\}, $$

$$ [x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2), \text{ for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I. $$

Similarly as Lemma 2.1 in [6] it can be shown that $\mathcal{L}^I$ is a complete lattice.

**Definition 2.2** [11, 17] An interval-valued fuzzy set on $U$ is a mapping $A : U \rightarrow \mathcal{L}^I$.

**Definition 2.3** [1] An intuitionistic fuzzy set on $U$ is a set

$$ A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}, $$

where $\mu_A(u) \in [0,1]$ denotes the membership degree and $\nu_A(u) \in [0,1]$ the non-membership degree of $u$ in $A$ and where for all $u \in U$, $\mu_A(u) + \nu_A(u) \leq 1$.

An intuitionistic fuzzy set $A$ on $U$ can be represented by the $\mathcal{L}^I$-fuzzy set $A$ given by

$$ A : U \rightarrow L^I : \quad u \mapsto [\mu_A(u), 1 - \nu_A(u)]. $$

In Figure 1 the set $L^I$ is shown. Note that to each element $x = [x_1, x_2]$ of $L^I$ corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

![Diagram of L^I](image)

**Figure 1:** The grey area is $L^I$.

In the sequel, if $x \in L^I$, then we denote its bounds by $x_1$ and $x_2$, i.e. $x = [x_1, x_2]$. The smallest and the largest element of $\mathcal{L}^I$ are given by $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$. We define
the relation \( \ll_{L'} \) by \( x \ll_{L'} y \iff (x_1 < y_1 \text{ and } x_2 < y_2) \), for \( x, y \in L' \). We define for further usage the sets

\[
D = \{[x_1, x_1] \mid x_1 \in [0, 1]\}; \\
\tilde{L}' = \{[x_1, x_2] \mid (x_1, x_2) \in \mathbb{R}^2 \text{ and } x_1 \leq x_2\}; \\
\tilde{D} = \{[x_1, x_1] \mid x_1 \in \mathbb{R}\}; \\
\tilde{L}'_+ = \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty]^2 \text{ and } x_1 \leq x_2\}; \\
\tilde{D}_+ = \{[x_1, x_1] \mid x_1 \in [0, +\infty]\}; \\
\tilde{L}'_{\infty,+} = \{[x_1, x_2] \mid (x_1, x_2) \in [0, +\infty]^2 \text{ and } x_1 \leq x_2\}; \\
\tilde{D}_{\infty,+} = \{[x_1, x_1] \mid x_1 \in [0, +\infty]\}.
\]

Note that for any non-empty subset \( A \) of \( L' \) it holds that

\[
\sup A = \left\{ \sup \{x_1 \mid x_1 \in [0, 1] \text{ and } (\exists x_2 \in [0, 1]) ([x_1, x_2] \in A)\}, \right. \\
\left. \sup \{x_2 \mid x_2 \in [0, 1] \text{ and } (\exists x_1 \in [0, x_2]) ([x_1, x_2] \in A)\} \right\}.
\]

**Definition 2.4** [4] A \( t \)-norm on \( L' \) is a commutative, associative, increasing mapping \( T : (L')^2 \to L' \) which satisfies \( T(1_{L'}, x) = x \), for all \( x \in L' \). A \( t \)-conorm on \( L' \) is a commutative, associative, increasing mapping \( S : (L')^2 \to L' \) which satisfies \( S(0_{L'}, x) = x \), for all \( x \in L' \).

In [4, 5, 7] the following classes of \( t \)-norms on \( L' \) are introduced: let \( T \) and \( T' \) be \( t \)-norms on \(([0, 1], \leq)\), then the mappings \( T_{T,T'}, T_T, T_{T,T} \) and \( T_T' \) given by, for all \( x, y \in L' \),

\[
T_{T,T'}(x, y) = [T(x_1, y_1), T'(x_2, y_2)], \quad \text{(t-representable \( t \)-norms)} \\
T_T(x, y) = [T(x_1, y_1), \max(T(x_1, y_2), T(x_2, y_1))], \quad \text{(pseudo-t-representable \( t \)-norms)} \\
T_{T,T}(x, y) = [T(x_1, y_1), \max(T(t, T(x_2, y_2)), T(x_1, y_2), T(x_2, y_1))], \\
T_T'(x, y) = [\min(T(x_1, y_2), T(x_2, y_1)), T(x_2, y_2)],
\]

are \( t \)-norms on \( L' \). The corresponding classes of \( t \)-conorms are given by, for all \( x, y \in L' \),

\[
S_{S,S'}(x, y) = [S(x_1, y_1), S'(x_2, y_2)], \quad \text{(t-representable \( t \)-conorms)} \\
S_S(x, y) = [\min(S(x_1, y_2), S(x_2, y_1)), S(x_2, y_2)], \quad \text{(pseudo-t-representable \( t \)-conorms)} \\
S_{S,S}(x, y) = [\min(S(1 - t, S(x_1, y_1)), S(x_1, y_2), S(x_2, y_1)), S(x_2, y_2)], \\
S_S'(x, y) = [S(x_1, y_1), \max(S(x_1, y_2), S(x_2, y_1))].
\]

If for a mapping \( f \) on \([0, 1]\) and a mapping \( F \) on \( L' \) it holds that \( F(D) \subseteq D \), and \( F([a, a]) = [f(a), f(a)] \), for all \( a \in L' \), then we say that \( F \) is a natural extension of \( f \) to \( L' \). E.g. \( T_{T_T}, T_T, T_{T,T} \) and \( T_T' \) are all natural extensions of \( T \) to \( L' \).

3 Additive generators on \( L' \)

In order to investigate additive generators on \( L' \), a suitable addition on \( \tilde{L}' \) is needed. We assume from now on that \( \oplus : (\tilde{L}' \cup \tilde{L}'_{\infty,+})^2 \to \tilde{L}' \) satisfies the following natural properties, for all \( a, b \) in \( \tilde{L}' \cup \tilde{L}'_{\infty,+} \),

(i) \( \oplus \) is commutative,
(ii) $\oplus$ is associative,

(iii) $\oplus$ is increasing,

(iv) $0_{L^I} \oplus a = a$,

Note that from (iii) and (iv) it follows that $a \oplus b \geq_{L^I} a$, if $b \geq_{L^I} 0_{L^I}$, for all $a, b$ in $L^I \cup L^I_{\infty,+}$.

**Definition 3.1** Let $f : L^I \to \bar{L}_{\infty,+}^I$ be a strictly decreasing function. The pseudo-inverse $f^{-1} : \bar{L}_{\infty,+}^I \to L^I$ of $f$ is defined by, for all $y \in \bar{L}_{\infty,+}^I$,

$$f^{-1}(y) = \begin{cases} 
\{ x \mid x \in L^I \text{ and } (f(x))_1 \leq \text{inf}_L y \}, & \text{if } y \ll_{L^I} f(0_{L^I}); \\
\{ 0_{L^I} \} \cup \{ x \mid x \in L^I \text{ and } (f(x))_1 > y_1 \text{ and } (f(x))_2 \geq (f(0_{L^I}))_2 \}, & \text{if } y_2 \geq (f(0_{L^I}))_2; \\
\{ 0_{L^I} \} \cup \{ x \mid x \in L^I \text{ and } (f(x))_2 > y_2 \text{ and } (f(x))_1 \geq (f(0_{L^I}))_1 \}, & \text{if } y_1 \geq (f(0_{L^I}))_1. 
\end{cases}$$

**Definition 3.2** A mapping $f : L^I \to \bar{L}_{\infty,+}^I$ satisfying the following conditions:

(AG.1) $f$ is strictly decreasing;

(AG.2) $f(1_{L^I}) = 0_{L^I};$

(AG.3) $f$ is right-continuous in $0_{L^I};$

(AG.4) $f(x) \oplus f(y) \in \mathcal{R}(f)$, for all $x, y$ in $L^I$, where 

$$\mathcal{R}(f) = \text{rng}(f) \cup \{ x \mid x \in \bar{L}_{\infty,+}^I \text{ and } [x_1, (f(0_{L^I}))_2] \in \text{rng}(f) \text{ and } x_1 \geq (f(0_{L^I}))_2 \}$$

$$\cup \{ x \mid x \in \bar{L}_{\infty,+}^I \text{ and } [(f(0_{L^I}))_1, x_2] \in \text{rng}(f) \text{ and } x_1 \geq (f(0_{L^I}))_1 \}$$

$$\cup \{ x \mid x \in \bar{L}_{\infty,+}^I \text{ and } x \geq_{L^I} f(0_{L^I}) \};$$

(AG.5) $f^{-1}(f(x)) = x$, for all $x \in L^I$;

is called an additive generator on $L^I$.

**Theorem 3.1** Let $f$ be an additive generator on $([0,1], \leq)$ and $f : L^I \to \bar{L}_{\infty,+}^I$ the mapping defined by, for all $x \in L^I$,

$$f(x) = [f(x_2), f(x_1)].$$

Then, for all $y \in \bar{L}_{\infty,+}^I$,

$$f^{-1}(y) = [f^{-1}(y_2), f^{-1}(y_1)]. \quad \text{(1)}$$

**Lemma 3.2** Let $f : L^I \to \bar{L}_{\infty,+}^I$ be a mapping satisfying (AG.1), (AG.2), (AG.3) and (AG.5). Then, for all $x \in L^I$ such that $x_1 > 0$, it holds that $(f(x))_2 < (f(0_{L^I}))_2$ and $(f(x))_1 < (f(0_{L^I}))_1$.

**Lemma 3.3** Let $f : L^I \to \bar{L}_{\infty,+}^I$ be a mapping satisfying (AG.1), (AG.2), (AG.3) and (AG.5). Then $(f([0,1]))_1 = (f(0_{L^I}))_1$ or $(f([0,1]))_2 = (f(0_{L^I}))_2$.

**Corollary 3.4** Let $f : L^I \to \bar{L}_{\infty,+}^I$ be a mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and $f(D) \subseteq D_{\infty,+}$. Then $(f([0,1]))_2 = (f(0_{L^I}))_2$. 

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Theorem 3.5 Let $f : L^I \to \bar{L}_\infty^I$ be a continuous mapping satisfying (AG.1), (AG.2), (AG.3), (AG.5) and $\{D\} \subseteq D_\infty$. Then there exists an additive generator $f$ on $([0,1], \leq)$ such that, for all $a \in \bar{L}_\infty^I$,

$$f^{-1}(a) = [f^{-1}(a_2), f^{-1}(a_1)].$$

The following theorem can be shown independently of the addition $\oplus$ used in (AG.4).

Theorem 3.6 A mapping $f : L^I \to \bar{L}_\infty^I$ is a continuous additive generator on $L^I$ such that $f(D) \subseteq D_\infty$ if and only if there exists a continuous additive generator $f$ on $([0,1], \leq)$ such that, for all $a \in L^I$,

$$f(a) = [f(a_2), f(a_1)].$$

4 Additive generators and t-norms on $L^I$

We will give a sufficient condition for $\oplus$ under which an additive generator associated to $\oplus$ generates a t-norm. First we give a lemma.

Lemma 4.1 Let $f$ be an additive generator on $L^I$ associated to $\oplus$. If, for all $x, y, a$ in $\bar{L}_+$ such that $x \leq_L a$ and $y \leq_L a \oplus a$,

$$y_2 \geq a_2 \implies ((x \oplus y)_1 = (x \oplus [y_1, a_2])_1 \text{ or } \min((x \oplus y)_1, (x \oplus [y_1, a_2])_1) \geq a_1) \quad (3)$$

and

$$y_1 \geq a_1 \implies ((x \oplus y)_2 = (x \oplus [a_1, y_2])_2 \text{ or } \min((x \oplus y)_2, (x \oplus [a_1, y_2])_2) \geq a_2) \quad (4)$$

then, for all $x \in L^I$ and $y \in R(f)$, we have that $f(x) \oplus f(f^{-1}(y)) \in R(f)$ and

$$f^{-1}(f(x) \oplus f(f^{-1}(y))) = f^{-1}(f(x) \oplus y).$$

Using Lemma 4.1, the following theorem can be shown.

Theorem 4.2 Let $f$ be an additive generator on $L^I$ associated to $\oplus$. If (3) and (4) hold, for all $x, y, a$ in $L^I$ such that $x \leq_L a$ and $y \leq_L a \oplus a$, then the mapping $T : (L^I)^2 \to L^I$ defined by, for all $x, y$ in $L^I$,

$$T(x, y) = f^{-1}(f(x) \oplus f(y)),$$

is a t-norm on $L^I$.

Theorem 4.3 Let $f$ be an additive generator on $([0,1], \leq)$. Then the mapping $f : L^I \to \bar{L}_\infty^I$ defined by, for all $x \in L^I$,

$$f(x) = [f(x_2), f(x_1)],$$

is an additive generator on $L^I$ associated to $\oplus$ if and only if, for all $x, y$ in $L^I$,

$$f(x) \oplus f(y) \in (\text{rng}(f) \cup [f(0), +\infty])^2.$$

Theorem 3.6 and Theorem 3.1 show that no matter which operation $\oplus$ is used in (AG.4), a continuous additive generator $f$ on $L^I$ satisfying $f(D) \subseteq D_\infty$ satisfies (2) and its pseudo-inverse satisfies (1). Therefore it depends on the operation $\oplus$ which classes of t-norms on $L^I$ can have continuous additive generators that are a natural extension of additive generators on $([0,1], \leq)$. 

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4.1 Additive generators based on $\oplus_{\mathcal{L}^I}$

Starting from the observation that the Łukasiewicz t-conorm $S_W$ is given by $S_W(x, y) = \min(1, x + y)$, for all $x, y$ in $[0, 1]$, and that the pseudo-t-representable t-conorm $S_{S_W}$ is given by $S_{S_W}(x, y) = [\min(1, x_1 + y_2, x_2 + y_1), \min(1, x_1 + y_2)]$, for all $x, y$ in $L^I$, the following definition of addition on $\bar{L}^I$ is introduced in such a way that $S_{S_W}(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus_{\mathcal{L}^I} y)$, for all $x, y$ in $L^I$.

**Definition 4.1** [3] We define the addition on $\bar{L}^I \cup \bar{L}^I_{\infty,+}$ by, for all $x, y$ in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$,

$$x \oplus_{\mathcal{L}^I} y = [\min(x_1 + y_2, x_2 + y_1), x_2 + y_2],$$

where, for all $x \in \mathbb{R}$, $x + \infty = +\infty$ and $+\infty + \infty = +\infty$.

The other arithmetic operations introduced in [3] allow to write also some other important operations on $\mathcal{L}^I$, such as the Łukasiewicz t-norm, the product t-norm on $\mathcal{L}^I$ and their residual implications, using a similar algebraic formula as their counterparts on $([0, 1], \leq)$.

**Theorem 4.4** [2] Let $f$ be any generator on $\mathcal{L}^I$ associated to $\oplus_{\mathcal{L}^I}$. Then the mapping $T : (L^I)^2 \rightarrow L^I$ defined by, for all $x, y$ in $L^I$,

$$T(x, y) = f^{-1}((f(x) \oplus_{\mathcal{L}^I} f(y))),$$

is a t-norm on $\mathcal{L}^I$.

**Theorem 4.5** [2] Let $f$ be a continuous additive generator on $\mathcal{L}^I$ associated to $\oplus_{\mathcal{L}^I}$ for which $f(D) \subseteq D_{\infty,+}$. Then there exists a t-norm $T$ on $([0, 1], \leq)$ such that, for all $x, y$ in $L^I$,

$$f^{-1}((f(x) \oplus_{\mathcal{L}^I} f(y))) = T_T(x, y).$$

Thus, using $\oplus_{\mathcal{L}^I}$, only pseudo-t-representable t-norms on $\mathcal{L}^I$ can have continuous additive generators $f$ for which $f(D) \subseteq D_{\infty,+}$. Other natural extensions of t-norms on $([0, 1], \leq)$ which have a continuous generator $f$ cannot have a continuous additive generator on $\mathcal{L}^I$ that is a natural extension of $f$.

4.2 Additive generators based on $\oplus^t_{\mathcal{L}^I}$

Now we discuss a second type of addition on $\bar{L}^I$ which was introduced in [8]. Similarly as for $\oplus_{\mathcal{L}^I}$, we have that $S_{S_W,t}(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus^t_{\mathcal{L}^I} y)$, for all $x, y$ in $L^I$.

**Definition 4.2** [8] Let $t \in [0, 1]$. Then we define the t-addition on $\bar{L}^I \cup \bar{L}^I_{\infty,+}$ by, for all $x, y$ in $\bar{L}^I \cup \bar{L}^I_{\infty,+}$,

$$x \oplus^t_{\mathcal{L}^I} y = [\min(1 - t + x_1 + y_1, x_1 + y_2, x_2 + y_1), x_2 + y_2].$$

**Theorem 4.6** Let $t \in [0, 1]$ and $f$ be any generator on $\mathcal{L}^I$ associated to $\oplus^t_{\mathcal{L}^I}$. Then the mapping $T : (L^I)^2 \rightarrow L^I$ defined by, for all $x, y$ in $L^I$,

$$T(x, y) = f^{-1}((f(x) \oplus^t_{\mathcal{L}^I} f(y))),$$

is a t-norm on $\mathcal{L}^I$.
**Theorem 4.7** Let \( t \in [0, 1] \) and \( \triangledown \) be a continuous additive generator on \( \mathcal{L}^I \) associated to \( \oplus_{\mathcal{L}^I} \) for which \( \triangledown(D) \subseteq D_{\infty,+} \). Then there exists a t-norm \( T \) on \( ([0,1], \leq) \) such that, for all \( x, y \) in \( \mathcal{L}^I \),

\[
\triangledown^{-1}(\triangledown(x) \oplus_{\mathcal{L}^I} \triangledown(y)) = T_{t^{-1}}(x, y).
\]

Similarly as for \( \oplus_{\mathcal{L}^I} \), from Theorem 4.7 it follows that a t-norm \( T \) on \( \mathcal{L}^I \) which is a natural extension of a t-norm on \( ([0,1], \leq) \) generated by a continuous additive generator \( f \) can only have a continuous additive generator associated to \( \oplus_{\mathcal{L}^I} \) which is a natural extension of \( f \), if \( T \) belongs to the class of t-norms \( T_{T,t} \).

### 4.3 Additive generators based on \( \oplus_{\mathcal{L}^I} \)

Finally, we introduce the following addition on \( \bar{L}^I \).

**Definition 4.3** We define the addition on \( \bar{L}^I \cup \bar{L}^I_{\infty,+} \) by, for all \( x, y \) in \( \bar{L}^I \cup \bar{L}^I_{\infty,+} \),

\[
x \oplus_{\mathcal{L}^I} y = [x_1 + y_1, \max(x_1 + y_2, x_2 + y_1)].
\]

This addition is closely related to the t-conorm \( S'_{SW} \): for all \( x, y \) in \( \mathcal{L}^I \), \( S'_{SW}(x, y) = \inf(1_{\mathcal{L}^I}, x \oplus_{\mathcal{L}^I} y) \).

**Theorem 4.8** Let \( f \) be any generator on \( \mathcal{L}^I \) associated to \( \oplus_{\mathcal{L}^I} \) such that \( f(D) \in D_{\infty,+} \). Then the mapping \( T : (\mathcal{L}^I)^2 \to \mathcal{L}^I \) defined by, for all \( x, y \) in \( \mathcal{L}^I \),

\[
T(x, y) = f^{-1}(f(x) \oplus_{\mathcal{L}^I} f(y)),
\]

is a t-norm on \( \mathcal{L}^I \).

**Theorem 4.9** Let \( \triangledown \) be a continuous additive generator on \( \mathcal{L}^I \) associated to \( \oplus_{\mathcal{L}^I} \) for which \( \triangledown(D) \subseteq D_{\infty,+} \). Then there exists a t-norm \( T \) on \( ([0,1], \leq) \) such that, for all \( x, y \) in \( \mathcal{L}^I \),

\[
\triangledown^{-1}(\triangledown(x) \oplus_{\mathcal{L}^I} \triangledown(y)) = T_{T}(x, y).
\]

Similarly as for the two other additions, only t-norms on \( \mathcal{L}^I \) belonging to the class of t-norms \( T_{T} \) can have continuous additive generators \( \triangledown \) associated to \( \oplus_{\mathcal{L}^I} \) which are a natural extension of a continuous additive generator on \( ([0,1], \leq) \).

### 5 Conclusion

In [3, 8] two kinds of arithmetic operations on \( \mathcal{L}^I \) are introduced. In [2] one of these kinds of operations is used to construct additive generators on \( \mathcal{L}^I \). Since these are not the only possible ways to define addition, subtraction, multiplication and division on \( \mathcal{L}^I \), we developed a new theory of additive generators on \( \mathcal{L}^I \) as much as possible independently of the addition needed. We found a sufficient condition for \( \oplus \) such that additive generators associated to \( \oplus \) generate t-norms on \( \mathcal{L}^I \). We showed that continuous additive generators on \( \mathcal{L}^I \) which are a natural extension to \( \mathcal{L}^I \) of a generator on \( ([0,1], \leq) \) can be represented in a unique way by the generator on \( ([0,1], \leq) \). As a consequence, the choice of the operation \( \oplus \) determines which classes of t-norms on \( \mathcal{L}^I \) can have continuous additive generators which form a natural extension of a generator on the unit interval.
References


