Real-valued Implication function based on Real-valued realization of Boolean algebra

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Abstract — Boolean algebra as algebra is value indifferent. Although it is very important, the classical two-valued realization of the Boolean algebra is only a special case. The real valued realization of the Boolean algebra is a frame for the Boolean consistent fuzzy logic in wider sense, such as the two valued realization is a frame for the classical case. This paper presents the brief review of the real-valued realization of the finite (atomic) Boolean algebra and its application. The real-valued implication is a Boolean consistent generalization of the classical binary implication. The real-valued implication plays important roles in the real-valued set theory as a generalization of the classical set theory, as well as, in many applications such as morphology in image processing, association rules in data mining and decision making generally.

I. INTRODUCTION

The fuzzy implications are of the special interest in the fuzzy logic in a wider sense from both practical and theoretical aspects. Since the conventional fuzzy implications are realized outside the Boolean frame, there are many problems considering that it is not possible to preserve the algebraic properties of the classical two-valued implication, on the whole. The desirable implication properties are defined by axioms [1].

A real-valued implication (R-implication) is introduced in this paper. R-implication is a consistent Boolean generalization of the classical binary implication and, as a consequence, it overcomes the shortage inherent to the conventional fuzzy implications. The R-implication is realized technically as a generalized Boolean polynomial.

In section 2, a review of the real-valued realization of the finite (atomic) Boolean algebra is given. The real-valued implications are described in section 3.

II. REAL-VALUED REALIZATION OF BOOLEAN ALGEBRA

Fuzzy logic realized in Boolean frame means that all Boolean axioms and theorems are valid in general case and/or in case of gradation. Since corresponding classical techniques are based on Boolean algebra, i.e. on its two-valued realization, consistent generalization or consistent fuzzy case should be based on real-valued realization of Boolean algebra. The real-valued realization of finite or atomic Boolean algebra [2, 3, 4, 5, 6] is briefly illustrated here.

The main problem of the conventional approaches is the fact that they are based on truth functionality principle, which is taken from multi-valued logics. This principle is adequate or valid, from the Boolean algebra point of view, only in classical two-valued case. The reason for this is very simple: Boolean function, in general case, has vector nature, but in classical case attention is reduced only to the one component (which is determined by the 0-1 values of free variables). In general case, such as treatment of graduation, it is necessary to include in computation more than one or even all components of vector immanent to Boolean function. To illustrate the main idea we use Boolean function of two free variables x and y, from the famous Boolean paper from the year 1848 [1]:

\[\phi(x, y) = \phi(1,1)x + \phi(1,0)x(1-y) + \phi(0,1)(1-x)y + \phi(0,0)(1-x)(1-y) \]  

This equation is actually special case of Boolean polynomial [2]:

\[\phi^\phi(x, y) = \phi(1,1)x \otimes y + \phi(1,0)(x - x \otimes y) + \phi(0,1)(y - x \otimes y) + \phi(0,0)(1-x - y + x \otimes y). \]  

Free variables in general case take the values from the unit interval \(x, y \in [0, 1]\). The generalized product \(\otimes\) is the elements from t-norm families with the following property [2]:

\[\max(x + y - 1, 0) \leq x \otimes y \leq \min(x, y).\]

The properties of variables determine which generalized product will be applied. In the case of same property min function is applied. For independent properties ordinary product is applied.

Boolean polynomials

Real valued realization of finite (atomic) Boolean algebra is based on Boolean polynomials. Any Boolean function can be uniquely transformed into corresponding Boolean polynomial.

Example 1: Using equation (2) relation of equivalence, exclusive disjunction and implication are, respectively:

\[\phi(x, y) = \begin{cases} x \leftrightarrow y & \text{def} \\ \phi(1,1) = 1; & \phi(1,0) = 0; & \phi(0,1) = 0; & \phi(0,0) = 1; \\ x \leftrightarrow y = 1 - x - y - 2x \otimes y. \end{cases} \]

\[\phi(x, y) = \begin{cases} x \vee y & \text{def} \\ \phi(1,1) = 0; & \phi(1,0) = 1; & \phi(0,1) = 1; & \phi(0,0) = 0; \\ x \vee y = x + y - 2x \otimes y. \end{cases} \]
c. \( \phi(x,y) =_{\text{def}} x \Rightarrow y \)

\[
\begin{align*}
\phi(1,1) &= 1; \\
\phi(1,0) &= 0; \\
\phi(0,1) &= 1; \\
\phi(0,0) &= 1;
\end{align*}
\]

Finite (atomic) Boolean algebra \( BA(\Omega) = P(\Omega) \), is generated by the set of free variables \( \Omega = \{x_1, \ldots, x_n\} \), where: \( P(\Omega) \) is power set of \( \Omega \) (set of all subsets of \( \Omega \)).

Atomic elements of analyzed Boolean algebra \( BA(\Omega) \) are:

\[
\alpha(S)(x_1, \ldots, x_n) = \bigwedge_{x_i \in S} x_i \bigwedge_{x_i \in \Omega \setminus S} \neg x_i, \quad S \in P(\Omega). \tag{3}
\]

Atomic Boolean polynomials: \( \alpha^o(S)(x_1, \ldots, x_n) \) uniquely correspond to atomic elements \( \alpha(S)(x_1, \ldots, x_n) \), and they are defined on the following equations:

\[
\begin{align*}
\alpha^o(\{x, y\}) &= x \otimes y; \\
\alpha^o(\{x\}) &= x - x \otimes y; \\
\alpha^o(\{y\}) &= y - y \otimes x; \\
\alpha^o(\emptyset) &= 1 - x - y + x \otimes y.
\end{align*}
\]

The values of atomic polynomials in real valued case are non-negative \( \alpha^o(S)(x_1, \ldots, x_n) \in [0,1] \) \( \forall S \in P(\Omega) \), but their sum is identically equal to 1. In example described by (2) for \( x, y \in [0,1] \):

\[
x \otimes y + (x - x \otimes y) + (y - y \otimes x) + (1 - x - y + x \otimes y) = 1.
\]

Classical two-valued case is only special case which satisfies this fundamental identity too, since value of only one atom is equal to 1 and all others are identical to 0.

Any Boolean function, element of analyzed Boolean algebra \( \phi(x_1, \ldots, x_n) \in BA(\Omega) \), can be uniquely represented in disjunctive canonical form (disjunction of relevant atomic elements):

\[
\phi(x_1, \ldots, x_n) = \bigvee_{S \in P(\Omega)} \sigma_g(S) \alpha(S)(x_1, \ldots, x_n). \tag{5}
\]

Where: \( \sigma_g(S) \), \( (S \in P(\Omega)) \) is a relation of inclusion of corresponding atom \( \alpha(S)(x_1, \ldots, x_n) \) in analyzed Boolean function \( \phi(x_1, \ldots, x_n) \), defined in the following way:

\[
\sigma_g(S) =_{\text{def}} \phi(x_5(x)) \vert \vec{x} = 1, \ldots, n), \\
\chi_S(x) =_{\text{def}} \begin{cases} 1, & x \in S, \\
0, & x \notin S. \end{cases} \quad (S \in P(\Omega)). \tag{6}
\]

Relations of inclusion define which atoms are included in analyzed Boolean function (values of corresponding relation of inclusion equal to 1) and which are not included (values of relation of inclusion equal to 0). To any Boolean function unequally corresponds Boolean polynomial, as a figure of (5):

\[
\phi^o(x_1, \ldots, x_n) = \sum_{S \in P(\Omega)} \sigma_g(S) \alpha^o(S)(x_1, \ldots, x_n),
\]

where: \( \phi^o(x_1, \ldots, x_n) \) is a relation of inclusion of corresponding atom \( \alpha^o(S)(x_1, \ldots, x_n) \) in analyzed Boolean function, actually, vector of relations of inclusion of atomic functions in it.

\[
\phi^o(x_1, \ldots, x_n) = \left[ \begin{array}{c} \sigma_g(S) \in P(\Omega) \end{array} \right] \text{ is a structure of } \phi(x_1, \ldots, x_n) \in BA(\Omega) \text{ analyzed Boolean function, actually, vector of relations of inclusion of atomic functions in it.}
\]

\[
\alpha^o(x_1, x_2) = \left[ \begin{array}{c} x_1 \otimes x_2 \\
x_1 - x_1 \otimes x_2 \\
x_2 - x_1 \otimes x_2 \\
1 - x_1 - x_2 + x_1 \otimes x_2 \end{array} \right].
\]

2.2 Structural functionality principle

Structural functionality principle: Structure of any combined Boolean function can be calculated directly on the basis of its component structures using following identities:

\[
\begin{align*}
\sigma_{x \otimes y} &= \sigma_x \wedge \sigma_y; \\
\sigma_{x \vee y} &= \sigma_x \vee \sigma_y; \\
\sigma_{\neg x} &= \neg \sigma_x; \\
\sigma = 1 - \sigma_x.
\end{align*}
\]

Famous truth functionality principle is figure of structural functionality on the value level only in the case of two-valued realization. In general case (multi-valued
and/or real-valued realization) truth functionality principle is not able to preserve all Boolean algebraic properties. This is reason way fuzzy approaches based on truth functionality principle cannot be in Boolean frame.

Structures of Boolean functions preserve all Boolean algebraic laws:

**Monotone laws**

**Associativity**

\[ \sigma_0 \lor (\sigma_y \lor \sigma_z) = (\sigma_0 \lor \sigma_y) \lor \sigma_z, \]

\[ \sigma_0 \land (\sigma_y \land \sigma_z) = (\sigma_0 \land \sigma_y) \land \sigma_z. \]  

(9.1)

**Commutativity**

\[ \sigma_0 \lor \sigma_y = \sigma_y \lor \sigma_0, \]

\[ \sigma_0 \land \sigma_y = \sigma_y \land \sigma_0; \]

(9.2)

**Distributivity**

\[ \sigma_0 \land (\sigma_y \lor \sigma_z) = (\sigma_0 \land \sigma_y) \lor (\sigma_0 \land \sigma_z) \]

(9.3)

**Identity**

\[ \sigma_0 \lor 0 = \sigma_0; \]

\[ \sigma_0 \land 1 = \sigma_0; \]

\[ \sigma_0 \lor 1 = 1; \]

\[ \sigma_0 \land 1 = 1. \]  

(9.4)

**Idempotence**

\[ \sigma_0 \lor \sigma_0 = \sigma_0, \]

\[ \sigma_0 \land \sigma_0 = \sigma_0. \]  

(9.5)

**Absorption**

\[ \sigma_0 \land (\sigma_y \lor \sigma_z) = \sigma_0, \]

\[ \sigma_0 \lor (\sigma_y \land \sigma_z) = \sigma_0. \]  

(9.6)

**Distributivity**

\[ \sigma_0 \lor (\sigma_y \land \sigma_z) = (\sigma_0 \lor \sigma_y) \land (\sigma_0 \lor \sigma_z) \]

(9.7)

**Non-monotone laws**

**Complementation**

\[ \overline{\sigma_0 \land \sigma_y} = 0; \]

\[ \overline{\sigma_0 \lor \sigma_y} = 1. \]  

(9.8)

**De Morgan laws**

\[ \overline{\sigma_0 \land \sigma_y} = \sigma_0 \lor \sigma_y; \]

\[ \overline{\sigma_0 \lor \sigma_y} = \sigma_0 \land \sigma_y. \]  

(9.9)

In this approach all Boolean algebraic laws are preserved in any valued realization (from classical two-valued until to the real-valued realizations) independently of chosen generalized products.

Complementation or non-contradiction and excluded middle are also valid in general case! Classical definition of excluded middle and non-contradiction are correct only in classical two-valued case, but in general case in real-valued realization or Boolean consistent fuzzy logic, one proposition can simultaneously have both truth and untruth property but so that sum of intensity is identical to 1. In Boolean consistent fuzzy set theory one element can have the analyzed property with some intensity and then it must have complementary property with complementary intensity so that the sum of their intensity is identical to 1. In general case non-contradiction means that between the analyzed property and its complementary property there is nothing in common and excluded middle means that anything that is not contained in analyzed property is contained in its complementary property. Actually, for the arbitrary property excluded middle and non-contradiction uniquely define its complementary property, and as a consequence these laws are fundamental and unavoidable for cognition generally.

This can be illustrated in the simple example such as glass of water. In classical two-valued case a glass can be either full of water or empty. Empty is complement of full and vice versa. In general case glass can be partly full and then it is simultaneously empty with complementary intensity, so that the sum of intensities full and empty is identical to 1. It is clear that, besides the fact that properties empty and full do not have anything in common, they are simultaneously in the same glass.

A Boolean function of analyzed finite or atomic Boolean algebra contains relevant atoms or can be represented as a union of relevant atoms – disjunctive canonical form. Complementary Boolean function contains all atoms which are not contained in analyzed Boolean function – excluded middle, and there is no atom which is common to analyzed Boolean function and to its complementary function – non contradiction. In classical case only one atom has a value equal to 1 and all others equal to 0 and, so consequently, if an analyzed Boolean function has a value 1 then its complementary function is equal to 0 or vice versa. In general case of graduation all atoms can have non-negative values but so that their sum is equal to 1. Since intersection of analyzed Boolean function and its complement doesn’t have any atom, it is always identical to 0, and their union contains all atoms and as a consequence it is always identical to 1.

Introducing intensity of realization – graduation of Boolean variables and functions, the finite Boolean algebra is adequate for any real problems thanks to the descriptiveness of gradations. Any classical theory based on finite Boolean algebra using fuzzy logic in Boolean frame can be generalized immediately [5], [6]. This is very important for many interesting applications which are logically much more complex, such as: AI, mathematical cognition, theory of prototypes in psychology, theory of concepts generally, etc.

So, with real-valued realization of Boolean algebra it is possible immediately to generalize all classical results based on two-valued realization of finite Boolean algebra, and in Boolean frame there is enough space besides Aristotle for Zadeh’s ideas as well [5], [6].

**III. REAL-VALUED IMPLICATION**

Real-valued implication (R-implication) is a consistent Boolean generalization of classical binary implication. Contrary to conventional fuzzy implications [7], based on truth-functionality principle, R-implication is based on structural-functionality principle [8, 9, 10]. Technically, R-implication is realized on a Generalized Boolean Polynomial (GBP) [8, 9, 10]. GBP is a polynomial figure of analyzed element of finite Boolean algebra. Finite Boolean algebra \( BA(\Omega) = P(P(\Omega)) \) is generated by the
set of primary (basic, free) variables $\Omega = \{a_1, ..., a_n\}$.

Transformation of any Boolean function (element of Boolean algebra) into a corresponding GBP is given by the following procedure:

$$F(a_1, ..., a_n), G(a_1, ..., a_n) \in BA(\Omega)$$

$$(F \land G)^\oplus(a_1, ..., a_n) =_{df} F^\oplus(a_1, ..., a_n) \otimes G^\oplus(a_1, ..., a_n),$$

$$(F \lor G)^\oplus(a_1, ..., a_n) =_{df} F^\oplus(a_1, ..., a_n) + G^\oplus(a_1, ..., a_n) - (F \land G)^\oplus(a_1, ..., a_n),$$

$$(-F)^\oplus(a_1, ..., a_n) =_{df} 1 - F^\oplus(a_1, ..., a_n).$$

:$$(a_i \land a_j)^\oplus = a_i \oplus a_j, \quad i \neq j, \quad (\oplus \in [Luk, min]) ;$$

$$(a_i \lor a_j)^\oplus = a_i + a_j - (a_i \land a_j)^\oplus ;$$

$$(a_i, a_j) = 1 - a_i;$$

$$(a_i, a_j) \in \Omega.$$"
4. DT: dominance of truth of consequent

\((\forall x \in [0,1]), (I^\otimes(x,1) = 1)\):
\((\forall \otimes \in [Luk, min])\)

Proof:

\[ I^\otimes(x,1) = 1 - x + x \otimes 1; \quad (x \otimes 1 = x) \]
\[ = 1, \quad (\forall \otimes \in [Luk, min]). \quad \Box \]

R-implication fulfills eight additional axioms [7] too:

1. NT: neutrality of truth

\((\forall x \in [0,1]) (I^\otimes(1,x) = x)\).
\((\forall \otimes \in [Luk, min])\).

Proof:

\[ I^\otimes(1,x) = 1 - 1 + x \otimes 1; \quad (x \otimes 1 = x) \]
\[ = x, \quad (\forall \otimes \in [Luk, min]). \quad \Box \]

2. EP: exchange principle

\((\forall (x,y,z) \in [0,1]^2) (I^\otimes(x,I(y,z)) = I^\otimes(y,I(x,z)))\).
\((\forall \otimes \in [Luk, min])\).

Proof:

\[ I^\otimes(x,I^\otimes(y,z)) = 1 - x + x \otimes (1 - y + y \otimes z), \]
\[ = 1 - x \otimes y + x \otimes y \otimes z; \]
\[ I^\otimes(y,I^\otimes(x,z)) = 1 - y \otimes y + x \otimes y \otimes z; \]
\[ (I^\otimes(x,I^\otimes(y,z))) = (I^\otimes(y,I^\otimes(x,z))); \]
\((\forall \otimes \in [Luk, min]). \quad \Box \)

3. OP: ordering principle

\((\forall (x,y) \in [0,1]^2) \quad I^{\min}(x,y) = 1 \iff x \leq y \quad (\otimes =_\text{def} \text{min})\)

Proof:

\[ I^\otimes(x,y) = 1 - x + x \otimes y \]
\[ y \geq x, \quad I^\otimes(x,y) = 1 \]
\[ x \otimes y = x \implies \otimes =_\text{def} \text{min} \]
\[ (y \geq x), (I^{\min}(x,y) = 1). \quad \Box \]

Note: Ordering principle is valid only for the case \(\otimes =_\text{def} \text{min}\), and/or for comparison two different objects \(x\) and \(y\) according to the same property \(A\)!

4. SN: defines a strong negation

\((\forall x \in [0,1]) \quad I^\otimes(x,0) = 1 - x \quad (\forall \otimes \in [Luk, min]).\)

Proof:

\[ I^\otimes(0,x) = 1 - x + x \otimes 0, \]
\[ = 1 - x. \quad (\forall \otimes \in [Luk, min]). \quad \Box \]

5. CB: consequent boundary

\((\forall (x,y) \in [0,1]^2) (I^\otimes(x,y) \geq y)\).
\((\forall \otimes \in [Luk, min])\).

Proof:

\[ I^\otimes(x,y) \geq y \]
\[ I^{\text{Luk}}(x,y) = 1 - x + \max(x + y - 1,0), \]
\[ = \begin{cases} 1 - x, & y \leq 1 - x \\ y & y > 1 - x \end{cases} \]
\[ \geq y. \quad \Box \]

Comment: Here Lukasiewitz T-norm is applied as a generalized product since its value is minimal possible, and as a consequence the above inequality is valid for all other possible generalized products.

6. ID: identity

\((\forall x \in [0,1]) \quad I^{\min}(x,x) = 1 \quad (\otimes =_\text{def} \text{min})\)

Proof:

\[ I^\otimes(x,x) = 1 - x + x \otimes x, \]
\[ = 1 \quad (\otimes =_\text{def} \text{min}) \]
\[ I^{\min}(x,x) = 1. \quad \Box \]

7. CP: contra positive principle

\((\forall (x,y) \in [0,1]^2). \)
\[ (I^\otimes(x,y) = I^\otimes(N(y),N(x))) (N(x) =_\text{def} 1 - x). \]
\((\forall \otimes \in [Luk, min]). \quad \Box \)

Proof:

\[ I^\otimes(x,y) = I^\otimes(N(y),N(x)), \forall \otimes \in [Luk, min]. \quad \Box \]

8. CO: continuity
\[ I^\otimes(x, y) = 1 - x + x \otimes y \]
\[ I^\otimes(N(y), N(x)) = 1 - (1 - y) + (1 - x) \otimes (1 - y) \]
\[ = y + 1 - x - y + x \otimes y \]
\[ = 1 - x + x \otimes y \]
\[ I^\otimes(A(x), B(y)) = 1 - x + x \otimes y, \]
\( \forall \otimes \in \{ Luk, min \} \).

**Proof:** It follows directly from the definition of real-valued implication for all possible generalized products \( \forall \otimes \in \{ Luk, min \} \).

Axioms of ordering OR and identity ID are valid only in the case when generalized product is \( min \) function. It means that one can order objects and/or makes identity between objects only on the base of the same property. Implication is relation of order (fulfill reflexivity, transitivity and antisymmetry) only in the case when generalized product is \( min \) function and/or one can ordered different objects only according the same property.

**Interesting property of R-implication**

In classical logic, \( p \in \{0, 1\} \), the following identities are valid:

\[ (p \Rightarrow \overline{p}) = \overline{p}, \]
\[ (\overline{p} \Rightarrow p) = p. \]

Proof is trivial:

\[
\begin{array}{cccc}
  & p & \overline{p} & p \Rightarrow \overline{p} & \overline{p} \Rightarrow p \\
 0 & 1 & 1 & 0 & 1 \\
 1 & 0 & 0 & 1 & 1
\end{array}
\]

In the real-valued implications case the following equations are valid too.

\[ (p \Rightarrow \overline{p}) = \overline{p}, \]
\[ (\overline{p} \Rightarrow p) = p. \]

\[ p \in [0, 1], \quad \overline{p} = 1 - p. \]

**Proof:**

\[
\begin{align*}
p \Rightarrow \overline{p} &= 1 - p + p \otimes \overline{p}, \\
&= 1 - p + p \otimes (1 - p), \\
&= 1 - p + p - p, \\
&= 1 - p. \\
\overline{p} \Rightarrow p &= 1 - \overline{p} + \overline{p} \otimes \overline{p}, \\
&= 1 - \overline{p} + 1 - p, \\
&= p.
\end{align*}
\]

This result is Boolean consistent answer on very actual discussion of leading fuzzy scientists gathered in the famous Berkeley BISC-group.

IV. **CONCLUSION**

The conventional fuzzy implications, as other conventional fuzzy functions, are based on the truth functional principle, whereas the component values are enough for calculations of the analyzed fuzzy functions. Since the truth functional principle in a general case (multi-valued and real-valued) is not Boolean, it is not possible to preserve all Boolean properties based on this principle. In this paper, we describe fuzzy and/or real-valued implications based on the real-valued realization of the Boolean algebra. As the real-valued realization of the Boolean algebra preserves all Boolean axioms and theorems, the real-valued implication preserves all properties of the classical implications with richer interpretation. The real-valued implication based on the real-valued realization of the Boolean algebra on the real problems in many interesting fields will be a subject of the following papers.

**REFERENCES**


