Design of an Optimal and Robust Controller for a Free-Electron Laser Exploiting Symmetries of the RF-System

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Abstract—This paper shows a mixed sensitivity $H_{\infty}$ controller design, which uses the symmetric radio frequency system structure of a free electron laser. The controller design includes plant decoupling, which is needed for additional feedbacks. Furthermore unwanted additional resonant modes, so-called passband modes are suppressed. This paper shows a strategy by rewriting the multi-input multi-output model as a single-input single-output model applying the two dimensional special orthogonal group symmetry of the plant. The controller design is separated into two steps. An analytical controller design, which is calculated from the single-input single-output representation is mapped back to a multi-input multi-output controller and optimized by discrete-time $H_{\infty}$ fixed-order optimization to guarantee optimality and robustness.

I. INTRODUCTION

The European X-Ray Free Electron Laser (X-FEL) of the German Electron Synchrotron (DESY) will supply laser light with a tunable wavelength in X-ray range ($10^{-10}$ m) [1]. A smaller facility, namely the Free electron LASer in Hamburg (FLASH) is used to develop and test new hard- and software. Those developments include high precision beam related measurements. E.g. the beam arrival time, which is proportional to the energy of electrons or the beam compression, which is a measurement of the propagation of electrons within an electron bunch. A radio frequency (RF) driven resonator, the so-called cavity, increases the speed and energy of the electrons. The amplitude of the RF field is related to the energy of each electron, whereas the phase is related to the energy spread of electrons along the longitudinal bunch distribution. So each bunch is simply an electron cloud with thousands of electrons depending on the bunch charge. The overall goal is to control the beam properties instead of the RF field, but these properties are unfortunately not always available. Nevertheless, perfect field control is the first step for optimal beam control. In 2008 the first implementation of a multi-input multi-output (MIMO) field controller was done [2] and set up in order to increase the field performance. Together with an iterative learning control (ILC) algorithm [3] the performance could be increased towards the desired amplitude and phase specifications ($\frac{\Delta A}{A} < 0.01$ % and $\Delta \phi < 0.01$ deg.) by using the $H_{\infty}$ fixed-order optimization (HIFOO, [4]). But the reliability of those optimization tools are often depending on the initial conditions of the minimization algorithm, which is mostly arbitrarily chosen. In this paper an extended approach by using the discrete-time version of HIFOO [5] with initial controller conditions and in addition a fixed controller structure is used to achieve the reliability. The special orthogonal group (SO(2)) structure of the identified grey box model helps to fix the initial controller structure such that a desired decoupling of the input-output-relation can be found.

The paper is organized as follows: Section II starts with an overview of the plant and planned hardware upgrades. Section III focuses on the system identification, especially the symmetric SO(2) structure of the used grey box model. The general controller design is briefly explained in Section IV and ends with simplifications by using the SO(2) symmetry of the model. Based on this simplification, a mixed-sensitivity design of a discrete $H_{\infty}$ controller is done to guarantee an optimal and robust closed-loop behavior. Finally in Section V this procedure is applied at the free electron laser FLASH. Conclusions are given in the last Section.

II. FREE ELECTRON LASER FLASH

FLASH is a facility for research with tunable laser light in the X-ray range down to 4.2 nm [1]. Within high quality cavities particles are accelerated to a desired energy and accordingly used to generate the laser light. Free electrons interact with an electromagnetic radio frequency (RF), where each cavity is operated with a voltage field gradient up to 30 MV/m at a resonance frequency of 1.3 GHz. This frequency is the so-called fundamental or $\pi$-mode in that the field vector changes the sign from cell to cell. A cavity is a 9 cell resonator and therefore it houses also additional other modes, so-called passband modes. Fig. 1 shows how the electrical field in direction of acceleration is present for the individual modes. The passband modes are unwanted and
only the fundamental mode is used to accelerate the electrons. A klystron is used to amplify a modulated reference frequency which is transmitted to the acceleration modules. To reach this high RF power the klystron is driven in a pulsed mode, where a pulse is enabled for about 2 ms. One pulse can be divided into three parts, see Fig. 2. First the filling phase, forcing the field as fast as possible to the operating set point. During the flattop phase the amplitude and phase is maintained at a desired value. Lastly the decay phase where the RF power is turned off. A usual control amplitude signal for one channel is sketched by the black dashed line. The arrival time, which is one important beam property, will be controlled mainly by an amplitude change. As also couplings act on the system the phase would also change, such that one looses beam performance. Assume the set point of the phase is around zero, known in accelerator physics as oncrest phase, the arrival time is mostly related to the I-channel, whereas the beam compression is related to the phase and hence mostly to the Q-channel. Decoupling avoids dynamic transient amplitude and phase errors which are caused by a coupled system. However, the final goal is to optimize the beam performance and therefore to decouple the system. The currently used system is known as VME system, where the name is defined by the industrial standard of the system. Further important plant upgrades for the X-FEL project which are relevant to this paper are a higher sampling rate (9 MHz instead of 1 MHz) and a higher resolution (16 bit instead of 14 bit), known as uTCA system.

III. SYSTEM IDENTIFICATION BY GREY-BOX-MODEL

The application at FLASH is approximated by a grey box model based on [8], where the identification is separated into a two-step identification. Fig. 4 shows the main parts of the plant behavior. During the first step the bandwidth of about 200 Hz is identified. The second step is to identify the first passband mode at about 800 kHz, which is aliased down to about 200 kHz for the VME system and suppressed by a factor of about four, by the Nyquist-Shannon sampling theorem. The advantage of higher resolution and a faster sampling rate for the uTCA system comes at the price of investing more effort on the suppression of the remaining passband modes. The discrete-time model with low pass character and one resonance frequency can be described by

\[
x_{k+1} = \begin{bmatrix} \Phi_r & 0 \\ 0 & \Phi_c \end{bmatrix} x_k + \begin{bmatrix} \Gamma_r \\ \Gamma_c \end{bmatrix} u_k,
\]

\[
y_k = \begin{bmatrix} C_r \\ C_c \end{bmatrix} x_k.
\]

The index \( r \) represents the bandwidth and corresponds to real eigenvalues, whereas the index \( c \) corresponds to complex-conjugated pairs of eigenvalues and therefore one passband
mode. The extension of the model by additional blocks of \( \Phi_c, \Gamma_c \) and \( C_c \) leads to the possibility to include additional passband modes. The model is structured such that the model obeys the SO(2) symmetry.

The bode diagrams of the currently used VME system and the advanced uTCA system are shown in Fig. 4. The light lines represent the identified model of the uTCA system, whereas the normal lines is identified with the VME system. It is clearly visible that the \( \frac{8}{9} \pi \)-mode is mapped to the left of the Nyquist frequency, which is marked by the black line. The identification of both systems is based on a SO(2) structure [10] and results in a model structure as

\[
\begin{bmatrix}
G(z) \\
Y_Q
\end{bmatrix} =
\begin{bmatrix}
G_1(z) & -G_2(z) \\
G_2(z) & G_1(z)
\end{bmatrix}
\begin{bmatrix}
U_1 \\
U_Q
\end{bmatrix}.
\]

The model has a time delay either of \( T_d = 2 \mu s \) (uTCA) or \( T_d = 4 \mu s \) (VME). The resulting model is shown in Fig. 5, where \( \tilde{G}(z) \) is identified by shifting the output table with respect to the input table. Afterwards the model is extended by the known time delay. A cross validation compares the low frequency behavior, see Fig. 6, with the high frequency behavior which includes also the \( \frac{8}{9} \pi \)-mode, see Fig. 7. This is done to check whether the identified model is usable for the controller design. Especially the \( \frac{8}{9} \pi \)-mode, which has the most effect on the system stability must be well identified. First a consideration of the separation of an amplitude and phase part is done and performed in the model, where the plant \( G(z) \) can be rewritten as

\[
G(z) = \begin{bmatrix}
G_1(z) & -G_2(z) \\
G_2(z) & G_1(z)
\end{bmatrix},
\]

\[
\tilde{G}_1(z) = \frac{G_1(z)}{\cos(\phi_G) \cdot \sqrt{\det(G(z))}},
\]

\[
\Phi_G = \begin{bmatrix}
\cos(\phi_G) & -\sin(\phi_G) \\
\sin(\phi_G) & \cos(\phi_G)
\end{bmatrix}.
\]

This simplification holds only for frequencies below the \( \frac{8}{9} \pi \)-mode, so only for \( \omega \ll \frac{8}{9} \pi \). Each passband mode destroys the SO(2) symmetry, therefore the new system representation holds only for frequencies smaller then these modes. Hereby \( \tilde{G}_1(z) \) is used to fulfill the determinant is one for each rotation matrix. Within the controller calculation, this scaling and rotation part will be used.

IV. CONTROLLER DESIGN

In this section the focus is on controller design for decoupling the plant within the controller. As is known, the optimal plant decoupling would be a multiplication by the inverse plant. This leads to a decoupled plant, where the controller design can be done for each channel independently. The decoupled controller would be a multiplication of the inverse plant and the designed controller. The combination of the inverse plant and the controller optimization leads in this application to a higher controller order than the plant order, which is 6 for one passband mode. But the computed controller will be implemented on a digital processor, here
a field programmable gate array (FPGA), where an upper bound on the controller order is given. The following section shows how one can use the SO(2) symmetry of the plant to compute a suboptimal second order controller. Finally, this controller is optimized by a mixed sensitivity design with the discrete-time $H\infty$ fixed-order optimization (HIFOOd) such that optimality and robustness are fulfilled.

### A. Controller Design with SO(2) Symmetry

Corresponding to the plant decoupling, the analytically calculated controller is separated into a scaling and rotation part. The controller design is separated into several steps. First the MIMO plant is divided in an amplitude and phase representation, where the amplitude part contains the dynamics of the system. The phase part is just a mapping of the static gains. Therefore the controller design is done on the SISO system $\tilde{G}_1(z)$, where a notch filter is implemented to suppress the $\frac{8}{3}\pi$-mode. Afterwards the SISO controller is mapped back into a MIMO controller, this procedure can be seen in Fig. 8. Within the dashed box a SISO computation is done. Fig. 9 shows the effects of the $\frac{8}{3}\pi$-mode for different additional time delays, e.g. due to different cable length. The color code represents the standard deviation of the RF amplitude error. The red area is unstable while the blue one is the desired stable operating point. One important point in the controller design is to suppress the $\frac{8}{3}\pi$-mode and therefore the resulting positive feedback. Finally, all needed parts can be expressed in a second order controller, in which a decoupling and a notch is realized.

Fig. 8. Plant transformation and SISO controller design

Fig. 9. Gain scan for diagonal proportional controller with uTCA System

First, the controller calculation of $\tilde{G}_1(z)$, Eqn. (4), is done, which is in this case a SISO system and mapped back to a MIMO system by a multiplication of the resulting controller with $\Phi_G^{-1}$, see Eqn. (5).

$$
\Phi_G^{-1} = \begin{bmatrix}
\cos(\phi_G) & \sin(\phi_G) \\
-\sin(\phi_G) & \cos(\phi_G)
\end{bmatrix}
$$

The $\frac{8}{3}\pi$-mode can be read out directly as eigenvalue of $\tilde{G}_1(z)$ and a notch filter around this mode can be calculated and placed as a second order transfer function in $\tilde{C}_1(z)$. Afterwards this filter is mapped back from SISO to MIMO with $C(z) = \tilde{C}_1(z) \cdot \Phi_G^2$. For a plant $G(z)$, given in Eqn. (1), which is of symmetry SO(2) it is obvious that the optimal controller is also of symmetry SO(2), but with changed sign of the off-diagonal elements, see Eqn. (7).

$$
C(z) = \begin{bmatrix}
C_1(z) & C_2(z) \\
-C_2(z) & C_1(z)
\end{bmatrix}
$$

Fig. 10 shows that especially for high frequencies decoupling is not achieved. This is due to the fact that the controller order is fixed to a second order transfer function for each input/output connection. The fast beam based feedback will also act close to this frequency region and therefore a higher suppression of the $\frac{8}{3}\pi$-mode is needed. Fig. 11 shows the improvements over a standard P-controller with the analytically computed controller for the VME system. Due to the aliased and suppressed passband mode, the performance is not so much effected by the $\frac{8}{3}\pi$-mode. Next the LLRF hardware will be exchanged by the new hardware with many improvements, e.g. higher sampling rate, higher resolution. But that means that the controller calculation must be optimized.
around high frequencies, where the performance improvements depend strictly on the suppression of the passband modes. Nevertheless, this analytical approach works fine for the controller design especially for low frequencies. A similar gain scan, as it was introduced before, is done for this analytically calculated MIMO controller and shown in Fig. 12. It establishes that the closed-loop plant is always stable, with more or less stable regions, where the color code is again the standard deviation of the RF amplitude error. Hereby close to the $\frac{\pi}{2}$-mode an optimization is necessary bandwidth, especially around the $\frac{\pi}{2}$-mode there is the possibility to shape such behavior by $W_{KS}(z)$. Therefore the goal is to find a controller which fulfills the $H_\infty$ norm condition from the input $r(k)$ to the fictitious output $z(k)$.

The analytical controller is used as initialization for the HIFOOd design. In addition the controller structure is fixed to fulfill the SO(2) symmetry based on Eqn. (7).

V. CONTROLLER AT FLASH

The implemented controller on the FPGA is given as MIMO controller with

$$\begin{bmatrix} \tilde{u}_I(k) \\ \tilde{u}_Q(k) \end{bmatrix} = \begin{bmatrix} C_{II}(z) & C_{IQ}(z) \\ C_{QI}(z) & C_{QQ}(z) \end{bmatrix} \begin{bmatrix} e_I(k) \\ e_Q(k) \end{bmatrix},$$

where each controller transfer function is implemented as

$$C_{ij}(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$ 

For the general HIFOOd procedure within this application, one has to identify 20 parameter, 5 for each controller transfer function. By using the INIT function of HIFOOd. To fix the SO(2) structure it is necessary to define the controller structure, see Eqn. (8), such that the state space model fulfills the symmetric SO(2) structure. The parameter block $\Phi_c$ defines a complex pole pair, $\Gamma$ the input matrix, $C_i$ the output matrix and $D_i$ the direct feedthrough for $i = 1 \ldots 2$ respectively.

$$x_{k+1} = \begin{bmatrix} \Phi_c & 0 \\ 0 & \Phi_c \end{bmatrix} x_k + \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma \end{bmatrix} e_k,$$

$$u_k = \begin{bmatrix} C_1 \\ -C_2 \end{bmatrix} x_k + \begin{bmatrix} D_1 & D_2 \\ -D_2 & D_1 \end{bmatrix} e_k$$

For this example the shaping filters are shown in Fig. 14, where the main focus is on the second subplot. The controller

![Fig. 12. Gain scan with second order MIMO Controller for uTCA System and will be introduced in the next section.](image)

B. Controller Optimization with HIFOOd

The analytical controller design leads to sufficient results. But the optimality and robustness of the closed-loop behavior is not checked for the MIMO system. Therefore a mixed-sensitivity design, together with the analytical calculated controller is done by using the discrete-time $H_\infty$ fixed-order optimization tool (HIFOOd) [5]. The goal is to achieve $\|T_{sr}(z)\|_\infty < 1$, where the shaping filters $W_S(z)$, $W_T(z)$ and $W_{KS}(z)$ are used to define upper bounds on the closed-loop sensitivities, see Fig. 13.

$$\|T_{sr}(z)\|_\infty = \left\| \begin{bmatrix} W_S(z) & S(z) \\ W_T(z) & T(z) \\ W_{KS}(z) & KS(z) \end{bmatrix} \right\|_\infty < 1$$

Each shaping filter can be used for different design requirements [11], e.g to shape the desired tracking and disturbance rejection by $W_S(z)$ and the desired noise attenuation by $W_T(z)$. Furthermore to ensure K is small outside the system

![Fig. 13. $H_\infty$ controller design](image)

![Fig. 14. Shaping filter for HIFOOd Design](image)
sensitivity is taken from the analytical controller computation. That guarantees that the optimized controller will have a notch filter. The closed-loop behavior in Fig. 15 is shown by the light lines for the initial controller settings. The controller optimization by HIFOOd is indicated by the normal lines. The $\frac{\pi}{2}$-mode is further suppressed by about 25 dB. With a flattop length of about 1 ms, the considered frequency range of this application will be from 1 kHz upwards. The low frequency behavior is almost unchanged, whereas HIFOOd optimizes mostly the high frequency behavior around the $\frac{\pi}{2}$-mode. The first test was performed at the FLASH facility using the analytical controller design and the resulting histogram is shown in Fig. 16. The suboptimal controller is based on the design in Section IV-A, where the time domain plot contains still some oscillations caused by the $\frac{\pi}{2}$-mode. This leads to a plant performance which can be further improved by the HIFOOd design. Nevertheless, the field performance in comparison of the VME and uTCA systems is improved by a factor of about 3.

VI. CONCLUSION AND OUTLOOK

System identification seems to be the key to reach high controller performance. By using well known symmetries of this model, the plant can be separated into a SISO like scaling and MIMO like rotation part. The calculated controller is designed as SISO controller by using the scaling part, which contains the dynamics of the system. The rotation part with the static behavior maps the controller back to a MIMO controller. But this holds only for frequencies below the passband modes. Therefore the control problem is extended to an optimization problem, which is solved by the discrete-time HIFOO tool. The analytical controller is used as initial controller setting, furthermore the controller structure is fixed to a SO(2) structure to decouple the system. This results in an optimal and robust controller, which is calculated in a two-step method. First controller results show that the analytical solution is sufficient, but beam based feedback together with the optimized controller will further improve the total beam performance.

REFERENCES