Tensor product stabilization in Kac-Moody algebras

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Abstract

We consider a large class of series of symmetrizable Kac-Moody algebras (generically denoted X_n). This includes the classical series A_n as well as others like E_n whose members are of Indefinite type. The focus is to analyze the behavior of representations in the limit $n \to \infty$. Motivated by the classical theory of $A_n = sl_{n+1}\mathbb{C}$, we consider tensor product decompositions of irreducible highest weight representations of X_n and study how these vary with n. The notion of "double headed" dominant weights is introduced. For such weights, we show that tensor product decompositions in X_n do stabilize, generalizing the classical results for A_n. The main tool used is Littelmann's celebrated path model. One can also use the stable multiplicities as structure constants to define a multiplication operation on a suitable space. We define this so called *stable representation ring* and show that the multiplication operation is associative.

1 Introduction

In this article, we consider series of symmetrizable Kac-Moody algebras (generically denoted X_n). Our main objective is to prove that decompositions of tensor products of irreducible representations of X_n "stabilize," i.e, given an irreducible representation, its multiplicity in the tensor product decomposition becomes constant for sufficiently large n. To construct the X_n, let (X, ξ) be a marked Dynkin diagram with d nodes and a special node ξ. Assume that the generalized Cartan matrix of X is symmetrizable. We extend X by "attaching" the Dynkin diagram A_{n-d} (a linear string of $n-d$ nodes) to ξ. We denote this new diagram X_n.

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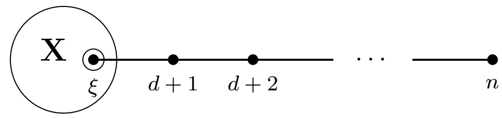

The four series of finite dimensional simple Lie algebras A_n, B_n, C_n, D_n are all of this form for suitable choices of (X, ξ). One can parametrize dominant integral weights of X_n by ordered pairs of partitions. The dominant weights thus obtained are "supported" on both ends of the Dynkin diagram of X_n. Such "double headed" weights have been previously considered in the literature [B, H, S1, S2, BKLS] in the context of A_n. Let $\mathcal{H}_2^+$ denote the set of ordered pairs of partitions (this definition will be slightly modified in the body of this paper). For $\lambda, \mu \in \mathcal{H}_2^+$ we consider the corresponding integrable highest weight (irreducible) representations $L(\lambda^{(n)})$ and $L(\mu^{(n)})$ of X_n and decompose their tensor product into irreducible components.

$$L(\lambda^{(n)}) \otimes L(\mu^{(n)}) = \bigoplus c_{\lambda\mu}^{\nu}(n)\, L(\nu^{(n)})$$

Here $c_{\lambda\mu}^{\nu}(n)$ denotes the multiplicity of the irreducible representation $L(\nu^{(n)})$ in the tensor product. For each fixed $\nu \in \mathcal{H}_2^+$ we prove that $c_{\lambda\mu}^{\nu}(n) = c_{\lambda\mu}^{\nu}(m)$ for all n, m sufficiently large. We refer to this as tensor product stabilization. The main tool used is Littelmann's path model [L2] for highest weight integrable representations of symmetrizable Kac-Moody algebras.

This result generalizes earlier work of R. Brylinski [B] on representations with double headed highest weights for the A_n case. The set of all partitions ($\mathcal{H}_1^+$) can be identified with the subset of $\mathcal{H}_2^+$ of ordered pairs whose second component is the zero partition. Our earlier association of double headed weights to elements of $\mathcal{H}_2^+$, when restricted to $\mathcal{H}_1^+$ gives the usual identification of partitions with dominant weights (irreducible representations) of A_n. So, as a special case of our result, one recovers the classical A_n situation, where tensor product stabilization is already implied by the Littlewood-Richardson rule.

Finally, we use the stable multiplicity values to define a new operation: the "stable tensor product" on a suitably defined $\mathbb{C}$ vector space Λ^X. We show that this operation is associative and captures tensor product decompositions in the limit $n \to \infty$. We call Λ^X the stable representation ring of type X. In the classical A_n case, Λ^A can be viewed as the tensor product of two copies of the ring of symmetric functions in infinitely many variables.

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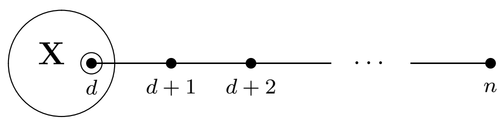
2 Formulation of the main Theorem

2.1 The X_n

We first define the series of symmetrizable Kac-Moody algebras that we will consider. Let X be a Dynkin diagram in which one of the vertices is distinguished; we call such an object a *marked Dynkin diagram.* We assume that the associated generalized Cartan matrix $C(X)$ is symmetrizable; see Kac [K, Chapter 4] for background. Let the number of nodes in X be d. For convenience we number the nodes of X as $1, 2, \cdots, d$ such that the distinguished vertex is numbered d. For $n \geq d$, we define X_n to be the Dynkin diagram obtained from X by attaching a tail of $n-d$ nodes to the marked vertex as shown in the figure below.



We "extend" the numbering of the nodes of X to a numbering of the nodes of X_n as in figure. Let $\mathfrak{g}(X_n)$ be the Kac-Moody algebra (over $\mathbb{C}$) with Dynkin diagram X_n. It is clear that $\mathfrak{g}(X_n)$ is symmetrizable, with generalized Cartan matrix $C(X_n)$ given by:

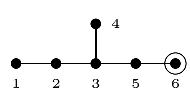
$$C(X_n) = \left[\begin{array}{c|ccccc} C(X) & & & & & \\ & -1 & & & & \\ \hline -1 & 2 & -1 & & & \\ & -1 & 2 & \ddots & & \\ & & \ddots & \ddots & -1 & \\ & & & -1 & 2 & \end{array}\right] \tag{2.1}$$

Example 2.1 In the following diagrams, the marked vertex is the one indicated by a circle.

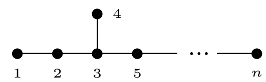
i. If X is the Dynkin diagram with a single vertex:  then X_n becomes

the Dynkin diagram A_n. The corresponding Lie algebra $\mathfrak{g}(X_n) \approx sl_{n+1}(\mathbb{C})$. We shall henceforth refer to this example as "Type A"

ii. Let X be the Dynkin diagram E_6:

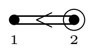


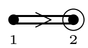
For $n \geq 6$, X_n is

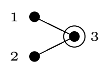


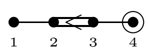
It is well known that $\mathfrak{g}(X_n)$ is a symmetrizable Kac-Moody algebra of Finite type for $n = 6, 7, 8$, of Affine type for $n = 9$ and of Indefinite type for $n \geq 10$. We shall refer to this example as "Type E"

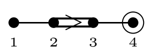
iii. We can also obtain the series B_n, C_n and D_n of finite dimensional simple Lie algebras by choosing X as follows

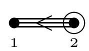
(a) Type B: 

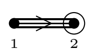
(b) Type C: 

(c) Type D: 

iv. Type $F^{(1)}$: 

v. Type $F^{(2)}$: 

vi. Type $G^{(1)}$: 

vii. Type $G^{(2)}$: 

2.2 Extensible families

For a Dynkin diagram Y, let $\det(Y)$ denote the determinant of the generalized Cartan matrix of Y. We allow Y to be empty, in which case $\det(Y) = 1$.

Lemma 2.2 *Let X be a marked Dynkin diagram. Then, the sequence $\{\det(X_n) : n \geq d\}$ is an arithmetic progression.*

Proof: Let $n \geq d + 2$. We can compute $\det(X_n)$ from Equation (2.1) by expanding along the last row of the matrix. This gives us

$$\det(X_n) = 2 \det(X_{n-1}) - \det(X_{n-2}) \quad \square$$

Remark 2.3 Let Δ denote the common difference of this arithmetic progression. The argument above also works for $n = d + 1$ and shows that $\Delta = \det(X) - \det(X_{d-1})$ where X_{d-1} denotes the Dynkin diagram obtained from X by deleting the distinguished vertex and all edges incident on it. We have, for $n \geq d$,

$$\det(X_n) = \det(X) + (n - d)\Delta \tag{2.2}$$

Type	A	B,C,D	E	$F^{(1)}$, $F^{(2)}$	$G^{(1)}$, $G^{(2)}$
Δ	1	0	-1	-1	-1

Table 1: Values of Δ

Definition 2.4 The marked Dynkin diagram X is said to be *extensible* if $\Delta \neq 0$, $\det(X) \neq 0$ and Δ is relatively prime to $\det(X)$.

This technical criterion will be an assumption for all our later results. If X is extensible then Equation (2.2) implies that Δ is relatively prime to $\det(X_n)$ for all $n \geq d$. From Table (1) we see that Types $A, E, F^{(i)}, G^{(i)}$ $(i = 1, 2)$ are extensible while Types B, C, D are not.

Remark 2.5 The condition $\det(X) \neq 0$ is not an essential part of the definition, but will be convenient for us. By Equation (2.2), $\det(X_n)$ can be zero for at most one value of n provided $\Delta \neq 0$. So if $\det(X) = 0$, then $\det(X_{d+1}) \neq 0$ and we can replace X with X_{d+1} without affecting anything in the rest of this paper.

2.3 Roundup of Notation

Most of our notation is that of Kac's book [K]. Let $\mathfrak{h}(X_n)$ denote the Cartan subalgebra of $\mathfrak{g}(X_n)$ and $\mathfrak{h}^*(X_n)$ denote its dual. The simple roots of $\mathfrak{g}(X_n)$ are denoted $\{\alpha_i^{(n)} : i = 1, \cdots, n\}$. Here $\alpha_i^{(n)}$ corresponds to the node i of X_n with respect to the node numbering mentioned in Section 2.1. Let $\check{\alpha}_i^{(n)} \in \mathfrak{h}(X_n)$ be the corresponding coroot. The $(i, j)^{th}$ element of the generalized Cartan matrix of X_n is thus given by $\alpha_j^{(n)}(\check{\alpha}_i^{(n)})$. The root lattice of $\mathfrak{g}(X_n)$ is

$$Q(X_n) := \mathbb{Z}\alpha_1^{(n)} \oplus \cdots \oplus \mathbb{Z}\alpha_n^{(n)} \subset \mathfrak{h}^*(X_n)$$

The weight lattice is $P(X_n) := \{\lambda \in \mathfrak{h}^*(X_n) : \lambda(\check{\alpha}_i^{(n)}) \in \mathbb{Z}\ \forall i = 1, \cdots, n\}$ The fundamental weights $\omega_i^{(n)}$, $i = 1, \cdots, n$ of $\mathfrak{g}(X_n)$ are elements of $\mathfrak{h}^*(X_n)$ which satisfy $\omega_i^{(n)}(\check{\alpha}_j^{(n)}) = \delta_{ij}$. If $\det(X_n) = 0$ this does not determine the $\omega_i^{(n)}$ uniquely. In this case, we pick them arbitrarily such that they satisfy the above condition. We will also find it useful to index the fundamental weights "backwards". We let

$$\overline{\omega}_i^{(n)} := \omega_{n-i+1}^{(n)} \quad i = 1, \cdots, n$$

So for instance, $\omega_d^{(n)}$ is the fundamental weight corresponding to the distinguished vertex of X while $\overline{\omega}_1^{(n)}$ corresponds to the "end" vertex of the tail. The set of dominant weights is $P^+(X_n) := \{\lambda \in \mathfrak{h}^*(X_n) : \lambda(\check{\alpha}_i^{(n)}) \in \mathbb{Z}^{\geq 0} \quad \forall i = 1, \cdots, n\}$

When $\det(X_n) \neq 0$,

$$P(X_n) = \mathbb{Z}\omega_1^{(n)} \oplus \cdots \oplus \mathbb{Z}\omega_n^{(n)}$$

2.4 Double headed weights

In the representation theory of $sl_{n+1}(\mathbb{C})$ (Type A), dominant weights are often parametrized by partitions or equivalently by Young diagrams. The convention is that the coefficient of the i^{th} fundamental weight $\omega_i^{(n)}$ in a given dominant weight is the number of columns of height i in the corresponding Young diagram. A partition λ with r rows can thus be thought of as defining a dominant weight $\lambda^{(n)}$ of A_n for *each* $n \geq r$. We use this as motivation to similarly parametrize weights of X_n. Define:

$$\mathcal{H}_1 = \{(x_1, x_2, \cdots) : x_i \in \mathbb{Z}\, \forall i \textit{ and } x_i \neq 0 \text{ for only finitely many } i\}$$

Given $x = (x_1, x_2, \cdots) \in \mathcal{H}_1$ we define the *length* of x to be: $\ell(x) := \max\{i : x_i \neq 0\}$. The element $x \in \mathcal{H}_1$ can be used to define a weight of $\mathfrak{g}(X_n)$ for $n \geq \ell(x)$. We let x label the weight $x_1\omega_1^{(n)} + x_2\omega_2^{(n)} + \cdots + x_m\omega_m^{(n)}$ ($m = \ell(x)$) of $\mathfrak{g}(X_n)$ for $n \geq \ell(x)$. We also define

$$\mathcal{H}_1^+ = \{(x_1, x_2, \cdots) : x_i \in \mathbb{Z}^{\geq 0}\, \forall i \textit{ and } x_i \neq 0 \text{ for only finitely many } i\}$$

By the above prescription, elements of $\mathcal{H}_1^+$ define *dominant* weights of $\mathfrak{g}(X_n)$ for $n \geq \ell(x)$. The set $\mathcal{H}_1^+$ is also in bijection with the set of all partitions. One identifies $x = (x_1, x_2, \cdots) \in \mathcal{H}_1^+$ with the partition π with parts $(x_1 + x_2 + \cdots + x_m, x_2 + \cdots + x_m, \cdots, x_m)$. It is easy to see that the above prescriptions generalize that of the Type A situation.

There is also another approach to making dominant weights of different A_n's correspond to each other. Given an ordered pair of partitions (λ, μ), the convention now [B, BKLS] is to let the number of columns of height i in λ be the coefficient of $\omega_i^{(n)}$ and the number of columns of height i in μ be the coefficient of $\overline{\omega}_i^{(n)}$. Thus λ and μ encode information about the coefficients at the two ends of the Dynkin diagram of A_n. We term such dominant weights "double headed".

A straightforward generalization leads to the definitions: $\mathcal{H}_2 = \mathcal{H}_1 \times \mathcal{H}_1$ and $\mathcal{H}_2^+ = \mathcal{H}_1^+ \times \mathcal{H}_1^+$. Given $x, y \in \mathcal{H}_1$, say $x = (x_1, x_2, \cdots), y = (y_1, y_2, \cdots)$, let $\lambda = (x, y) \in \mathcal{H}_2$. One can use λ to define a weight of $\mathfrak{g}(X_n)$ for each $n \geq \ell(y) + \max(d, \ell(x))$ (recall $d =$ the number of nodes in X) as follows:

$$\lambda^{(n)} := \sum_{i=1}^{\ell(x)} x_i\omega_i^{(n)} + \sum_{i=1}^{\ell(y)} y_i\overline{\omega}_i^{(n)}$$

It is clear that elements of $\mathcal{H}_2^+$ define *dominant* weights of $\mathfrak{g}(X_n)$. We define the *length*: $\ell(\lambda, X) := \ell(y) + \max(d, \ell(x))$.

In the classical Type A case, the usefulness of identifying dominant weights of different A_n's using partitions (or $\mathcal{H}_1^+$) is apparent when studying tensor products of representations. For instance the Littlewood-Richardson rule states that if $V_{\lambda^{(n)}}$ and $V_{\mu^{(n)}}$ are the irreducible highest weight representations correponding to partitions λ and μ, then for large enough n, the tensor product $V_{\lambda^{(n)}} \otimes V_{\mu^{(n)}}$ decomposes into a direct sum $\oplus c_{\lambda\mu}^{\nu} V_{\nu^{(n)}}$. The $c_{\lambda\mu}^{\nu}$ here are the Littlewood-Richardson coefficients and are independent of n. So the tensor product decomposition remains essentially the same for all large n. Special cases of tensor product decompositions for double-headed weights in Type A have been studied in [B] where again one gets such a stabilization behavior for large n. Double headed type A weights have also been considered by G. Benkart et al [BKLS] who study dimensions of correponding weight spaces as a function of n.

Analogously, given $\lambda, \mu, \nu \in \mathcal{H}_2^+$, we consider the irreducible representations $L(\lambda^{(n)})$, $L(\mu^{(n)})$ and $L(\nu^{(n)})$ of $\mathfrak{g}(X_n)$ with highest weights $\lambda^{(n)}$, $\mu^{(n)}$ and $\nu^{(n)}$ respectively. These are all defined provided n is larger than the lengths of each of λ, μ and ν. The tensor product $L(\lambda^{(n)}) \otimes L(\mu^{(n)})$ is an integrable representation of the symmetrizable Kac Moody algebra $\mathfrak{g}(X_n)$, in category $\mathcal{O}$. It thus decomposes into a direct sum of irreducible highest weight representations [K, Chapter 10]. We let $c_{\lambda\mu}^{\nu}(n)$ denote the multiplicity of occurrence of the representation $L(\nu^{(n)})$ in the decomposition of the tensor product $L(\lambda^{(n)}) \otimes L(\mu^{(n)})$.

Note that $c_{\lambda\mu}^{\nu}(n)$ is bounded above by the dimension of the weight space $\nu^{(n)}$ in $L(\lambda^{(n)}) \otimes L(\mu^{(n)})$. Since all weight spaces in this representation are finite dimensional, $c_{\lambda\mu}^{\nu}(n)$ is a finite number. However, if $\mathfrak{g}(X_n)$ is not of finite type, then there could in general be infinitely many ν for which $c_{\lambda\mu}^{\nu}(n) \neq 0$. Our main result is the following:

Theorem 2.6 *Let X be an extensible marked Dynkin diagram. Given $\lambda, \mu, \nu \in \mathcal{H}_2^+$, there exists a positive integer $N = N(\lambda, \mu, \nu)$ such that*

$$c_{\lambda\mu}^{\nu}(n) = c_{\lambda\mu}^{\nu}(m) \; \forall n, m \geq N$$

We denote this constant value by $c_{\lambda\mu}^{\nu}(\infty)$. In general, N will depend on λ, μ, ν and X. We shall prove this theorem over the course of the next two sections.

Example 2.7 We consider E_6, E_7, E_8 with nodes numbered as in Example (2.1), (ii). One has the following tensor product decompositions:

$$\begin{aligned} E_6 :\; & L(\omega_6^{(6)}) \otimes L(\omega_6^{(6)}) = L(2\omega_6^{(6)}) \oplus L(\omega_5^{(6)}) \oplus L(\omega_1^{(6)}) \\ E_7 :\; & L(\omega_7^{(7)}) \otimes L(\omega_7^{(7)}) = L(2\omega_7^{(7)}) \oplus L(\omega_6^{(7)}) \oplus L(\omega_1^{(7)}) \oplus L(\mathbf{0}^{(7)}) \\ E_8 :\; & L(\omega_8^{(8)}) \otimes L(\omega_8^{(8)}) = L(2\omega_8^{(8)}) \oplus L(\omega_7^{(8)}) \oplus L(\omega_1^{(8)}) \oplus L(\mathbf{0}^{(8)}) \oplus L(\omega_8^{(8)}) \end{aligned} \tag{2.3}$$

To re-express some of this information in terms of our notations, define the following elements of $\mathcal{H}_1^+$: $\mathbf{0} := (0, 0, 0, \cdots)$, $\epsilon_1 := (1, 0, 0, \cdots)$, $\epsilon_2 := (0, 1, 0, 0, \cdots)$.

ν	$\nu^{(n)}$	$c^{\nu}_{\lambda\mu}(6)$	$c^{\nu}_{\lambda\mu}(7)$	$c^{\nu}_{\lambda\mu}(8)$
$(\epsilon_1, \mathbf{0})$	$\omega_1^{(n)}$	1	1	1
$(\mathbf{0}, 2\epsilon_1)$	$2\overline{\omega}_1^{(n)}$	1	1	1
$(\mathbf{0}, \epsilon_2)$	$\overline{\omega}_2^{(n)}$	1	1	1
$(\mathbf{0}, \mathbf{0})$	$\mathbf{0}^{(n)}$	0	1	1

Table 2: Tensor product multiplicities in E_n, $n = 6, 7, 8$

Let $\lambda = \mu = (\mathbf{0}, \epsilon_1) \in \mathcal{H}_2^+$. Then $\lambda^{(n)} = \mu^{(n)} = \overline{\omega}_1^{(n)} = \omega_n^{(n)}$. For various choices of ν, the values of $c^{\nu}_{\lambda\mu}(n)$ for $n = 6, 7, 8$ can be read off from Equations (2.3) and are given in Table 2. Theorem (4.5) will give an explicit value of N for which $c^{\nu}_{\lambda\mu}(N) = c^{\nu}_{\lambda\mu}(\infty)$. Using this, it will be clear that $c^{\nu}_{\lambda\mu}(\infty) = 0$ for $\nu = (\mathbf{0}, \mathbf{0})$ and $c^{\nu}_{\lambda\mu}(\infty) = 1$ for the other three ν's in the table.

3 The *Number of Boxes* condition

The classical Littlewood-Richardson coefficients have the property that $c^{\nu}_{\lambda\mu} = 0$ unless $|\lambda| + |\mu| = |\nu|$, where $|\cdot|$ indicates the number of boxes in a Young diagram. In this section we give the analogous condition for double-headed weights.

Now, suppose X is an extensible marked Dynkin diagram, and let λ, μ, ν be elements of $\mathcal{H}_2^+$ then Theorem (2.6) is clearly true if $c^{\nu}_{\lambda\mu}(n) = 0$ for all large n. The interesting case is when $c^{\nu}_{\lambda\mu}(n) \neq 0$ for infinitely many values of n. This imposes a strong compatibility condition on λ, μ and ν. In Type A, this condition turns out precisely to be the number of boxes condition mentioned in the above paragraph.

3.1 Structure of $P(X_n)/Q(X_n)$

First, suppose n is such that $\det(X_n) \neq 0$, then it is well known that $P(X_n)/Q(X_n)$ is a finite abelian group of order $|\det(X_n)|$. For any $\eta \in P(X_n)$, we let $[\eta]$ denote its image in $P(X_n)/Q(X_n)$. The following lemma motivated the extensibility criterion.

Lemma 3.1 *Let X be an extensible marked Dynkin diagram with d nodes and take any $n \geq d$ such that* $\det(X_n) \neq 0$. *Then $P(X_n)/Q(X_n)$ is a cyclic group with generator* $[\overline{\omega}_1^{(n)}]$.

Proof: Since $\det(X_n) \neq 0$, $\mathfrak{h}^*(X_n)$ is spanned over $\mathbb{C}$ by the simple roots of $\mathfrak{g}(X_n)$. Consequently

$$\overline{\omega}_1^{(n)} = \sum_{i=1}^{n} k_i \alpha_i^{(n)}$$

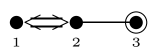
The k_i's can be determined as follows: the entries along the j^{th} column of $C(X_n)$ are the coefficients that one gets when expressing the j^{th} simple root of X_n in terms of the fundamental weights. To express the n^{th} fundamental weight in terms of the simple roots, we take the inverse of $C(X_n)$ - the k_i's are then just the entries along its n^{th} column. In particular

$$\begin{aligned} k_n &= \frac{cofactor\ of\ the\ (n,n)^{th}\ element\ of\ C(X_n)}{\det(X_n)} \\ &= \frac{\det(X_{n-1})}{\det(X_n)} \end{aligned}$$

The extensibility of X implies that $\det(X_n)$ and $\det(X_{n-1})$ are relatively prime. Hence, the smallest positive integer c such that $ck_n \in \mathbb{Z}$ is $c = |\det(X_n)|$. Thus, the order of the element $[\overline{\omega}_1^{(n)}]$ in $P(X_n)/Q(X_n)$ is at least $|\det(X_n)|$. Since $P(X_n)/Q(X_n)$ has exactly $|\det(X_n)|$ elements, it has to be a cyclic group generated by $[\overline{\omega}_1^{(n)}]$.

Remark 3.2 This lemma may be false if X is not extensible. For example if:

1. X is of Type D. Here $\Delta = 0$. The group $P(D_n)/Q(D_n)$ is of order 4 while its subgroup generated by $[\overline{\omega}_1^{(n)}]$ is only of order 2. In fact $P(D_n)/Q(D_n)$ fails to be a cyclic group when n is even.

2. Take X to be 

This is the Dynkin diagram of affine A_1, extended by one more vertex. The corresponding generalized Cartan matrix is

$$C(X) = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Here $\det(X) = \Delta = -2$ and hence they are not relatively prime. In this case, the group $P(X_n)/Q(X_n)$ has $2(n-2)$ elements while the subgroup generated by $[\overline{\omega}_1^{(n)}]$ has order $n-2$. Further $P(X_n)/Q(X_n)$ fails to be cyclic when n is even.

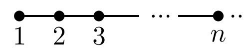
The next important proposition tells us more about the images of the fundamental weights in the groups $P(X_n)/Q(X_n)$.

Proposition 3.3 *Let X be an extensible marked Dynkin diagram with d nodes and let Δ be the common difference of $\{\det(X_n)\}_{n\geq d}$. Then, there exists a sequence of integers $(a_i)_{i\geq 1}$ (depending only on X and the node numbering chosen) such that in $P(X_n)$*

$$(-\Delta)\,\omega_i^{(n)} \equiv a_i\,\overline{\omega}_1^{(n)} \pmod{Q(X_n)} \tag{3.1}$$

for all $i = 1, \cdots, n$ and for all n such that $\det(X_n) \neq 0$. Further, the a_i's are unique integers with this property.

Example 3.4 Let X be of type A: Here $\Delta = 1$ and it can be easily checked that $a_i = i \; \forall i \geq 1$. We label the vertex i of the Dynkin diagram with the integer a_i as follows:



Recall from Section (2.4) that $\omega_i^{(n)}$ is represented by a Young diagram which is a single column of height i. Thus a_i "measures" the number of boxes in the Young diagram corresponding to $\omega_i^{(n)}$.

3.2 Proof of Proposition (3.3)

To prove Proposition (3.3) in general, observe by Lemma (3.1) that for a fixed $n \geq d$ such that $\det(X_n) \neq 0$ we can find integers $a_1, \cdots, a_n$ such that Equation (3.1) holds for $i = 1, \cdots, n$. Each of these integers is determined up to a multiple of $\det(X_n)$. The trick is to find a single sequence $(a_i)_{i \geq 1}$ that makes Equation (3.1) hold for all n.

First, fix $n \geq d$ such that $\det(X_n) \neq 0$. Since $P(X_n)/Q(X_n)$ is cyclic with generator $[\varpi_1^{(n)}]$, there exist $b_1, \cdots, b_n \in \mathbb{Z}/(\det(X_n))\mathbb{Z}$ such that $(-\Delta)\, \omega_i^{(n)} \equiv b_i \, \varpi_1^{(n)} \pmod{Q(X_n)}$ for $i = 1, \cdots, n$. Set $R = \mathbb{Z}/(\det(X_n))\mathbb{Z}$. Let $b = (b_1 \; b_2 \; \cdots \; b_n)^T \in R^n$. We first obtain a simple characterization of the b_i.

Lemma 3.5 *i. $b \in R^n$ is a solution to $A^T b = 0 \in R^n$ where $A = C(X_n)$. Here we identify the elements of A with their images in R and treat A as an $n \times n$ matrix with entries in R.*

ii. If $x = (x_1 \; x_2 \; \cdots \; x_n)^T \in R^n$ is another solution to $A^T x = 0$, then x is a multiple of b .

iii. b is the unique element of R^n such that $A^T b = 0 \in R^n$ and $b_n = -\Delta + (\det(X_n))\mathbb{Z} \in R$.

Proof:

i. To prove that the i^{th} entry of $A^T b$ is 0 in R, it is enough to show that (i^{th} entry of $A^T b$) $\varpi_1^{(n)} \equiv 0 \pmod{Q(X_n)}$. This is because $P(X_n)/Q(X_n)$ is cyclic of order $|det(X_n)|$ with generator $[\overline{\omega}_1^{(n)}]$. We compute:

$$\begin{aligned}(i^{th} \text{ entry of } A^T b)\,\overline{\omega}_1^{(n)} &= (\sum_{j=1}^{n} (A^T)_{ij} b_j)\,\overline{\omega}_1^{(n)} \\ &= \sum_{j=1}^{n} \alpha_i^{(n)}(\check{\alpha}_j^{(n)})\, b_j\, \overline{\omega}_1^{(n)} \\ &\equiv (-\Delta) \sum_{j=1}^{n} \alpha_i^{(n)}(\check{\alpha}_j^{(n)})\, \omega_j^{(n)} \pmod{Q(X_n)}\end{aligned}$$

The last congruence just follows from the definition of the b_j. We observe now that the final expression is precisely $(-\Delta)\alpha_i^{(n)}$. This can be seen by expressing $\alpha_i^{(n)}$ as a linear combination of the $\omega_j^{(n)}$'s and using the "duality" relation $\omega_j^{(n)}(\check{\alpha}_k^{(n)}) = \delta_{jk}$. Clearly $\alpha_i^{(n)} \equiv 0 \pmod{Q(X_n)}$ □

ii. To show that any two solutions are multiples of each other, we will show that A has an $(n-1) \times (n-1)$ minor which is a unit in the ring R. More precisely, let B denote the principal submatrix of A comprising of the first $n-1$ rows and columns of A. Observe that $\det(B) = \det(X_{n-1})$ which is relatively prime to $\det(X_n)$ by the extensibility of X. Hence $\det(B)$ is a unit in $\mathbb{Z}/(\det(X_n))\mathbb{Z}$. Now

$$A^T = \begin{pmatrix} B^T & v \\ w^T & 2 \end{pmatrix}$$

where $v, w \in R^{n-1}$. Since $\det(B) = \det(B^T)$ is a unit in R, $(B^T)^{-1}$ exists with all its entries in R. Let C denote the $n \times n$ matrix $C = \begin{pmatrix} (B^T)^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Then $CA^T = \begin{pmatrix} \mathbf{I} & p \\ q^T & 2 \end{pmatrix}$ where $p, q \in R^{n-1}$ and $\mathbf{I}$ denotes the identity matrix of size $n-1$. We let $p = (p_1\ p_2\ \cdots\ p_n)^T$. If $x \in R^n$ such that $A^T x = 0 \in R^n$, then $CA^T x = 0$. This implies that $x_i + p_i x_n = 0$ for $1 \leq i \leq n-1$. For $x = b$, this gives $b_i = -p_i b_n = p_i \Delta$ since from its definition $b_n = -\Delta$. Here again, we identify all elements of $\mathbb{Z}$ with their images in R. Since Δ is a unit in R, $p_i = \Delta^{-1} b_i$. Substituting back , we get

$$x_i = (-x_n \Delta^{-1})\, b_i \ \ \forall i \ \ \square \tag{3.2}$$

iii. Follows from (i) and (ii). □

We will now explicitly define the a_i's. Armed with the simple characterization of the b_i's above, we will show that these a_i's satisfy Equation (3.1). To construct the a_i's, we recall the notion of the dual $\check{Y}$ of a Dynkin diagram Y. This is the Dynkin diagram which corresponds to the transpose of the generalized Cartan matrix of Y i.e, $C(\check{Y}) := C(Y)^T$. Let us now consider $\check{X}$ where X is our given Dynkin diagram. For $n \geq d$ we can form $\check{X}_n$ as before by stipulating that the distinguished node of $\check{X}$ be the same as that of X. Clearly $\check{X}_n$ is the dual of the Dynkin diagram X_n.

The Cartan subalgebra $\mathfrak{h}(\check{X}_n)$ can be identified with $\mathfrak{h}^*(X_n)$. The simple roots of $\check{X}_n$ are just the simple coroots $\check{\alpha}_i^{(n)}$ of X_n and the simple coroots of $\check{X}_n$ are $\alpha_i^{(n)}$. Let $\check{\omega}_i^{(n)} \in \mathfrak{h}(X_n)$ denote the fundamental weights of $\check{X}_n$ i.e, $\alpha_j^{(n)}(\check{\omega}_i^{(n)}) = \delta_{ij}$. The extensibility of X implies $\det(\check{X}) = \det(X) \neq 0$. Hence $\check{\alpha}_i^{(n)}\,(1 \leq i \leq n)$ span $\mathfrak{h}^*(\check{X}_n) = \mathfrak{h}(X_n)$. The group $P(\check{X})/Q(\check{X})$ has order $|\det(\check{X})|$. So $\det(\check{X})\,\lambda \in Q(\check{X})$ for all $\lambda \in P(\check{X})$. We define the $a_i\,(1 \leq i \leq d)$ by setting:

$$\det(\check{X})\,\check{\omega}_d^{(d)} = \sum_{i=1}^{d} a_i \check{\alpha}_i^{(d)} \tag{3.3}$$

The argument of Lemma (3.1) shows that $a_d = \det(\check{X}_{d-1}) = \det(X_{d-1})$. We define $a_i := \det(X_{i-1})$ for all $i > d$. Since $\{\det(X_i) : i \geq d\}$ forms an arithmetic progression, the preceding definition of a_i for $i > d$ and Equation (3.3) imply the following important relation:

$$\det(\check{X}_n)\,\check{\omega}_n^{(n)} = \sum_{i=1}^{n} a_i \check{\alpha}_i^{(n)} \quad \forall n \geq d \tag{3.4}$$

We claim that these a_i's do our job i.e, if we fix $n \geq d$ such that $\det(X_n) \neq 0$, then

$$(-\Delta)\,\omega_i^{(n)} \equiv a_i\,\overline{\omega}_1^{(n)} \pmod{Q(X_n)}\ \forall i = 1, \cdots, n$$

It is now enough to show that the $a_i\,(1 \leq i \leq n)$ satisfy the condition of part (3) of Lemma (3.5). This is the content of the next

Lemma 3.6 *1. Let $a = (a_1\ a_2\ \cdots\ a_n)^T \in R^n$ (usual identification). Then $A^T a = 0 \in R^n$.*

2. $a_n \equiv -\Delta \pmod{\det(X_n)}$.

Proof: (2) is obvious from the definition : $a_n := \det(X_{n-1}) = \det(X_n) - \Delta$. To prove (1), we calculate the i^{th} entry of $A^T a$. This is equal to $\sum_{j=1}^n (A^T)_{ij} a_j = \sum_{j=1}^n a_j \alpha_i^{(n)}(\check{\alpha}_j^{(n)}) = \alpha_i^{(n)}(\sum_{j=1}^n a_j \check{\alpha}_j^{(n)}) = \alpha_i^{(n)}(\det(X_n)\check{\omega}_n^{(n)})$, where the last equality uses Equation (3.4). This final expression is clearly 0 unless $i = n$ in which case it is $\det(X_n)$. But $\det(X_n) = 0$ in R and we're done. $\square$

For the uniqueness of the a_i's observe that if a_i', $i \geq 1$ is another such sequence for which Equation (3.1) holds, then for each i, $a_i - a_i'$ must be divisible by $\det(X_n)$ for all $n \geq i$ (for which $\det(X_n) \neq 0$). Since X is extensible, $\Delta \neq 0$ and Equation (2.2) implies $|\det(X_n)| \to \infty$ as $n \to \infty$. Hence $a_i = a_i'$.

This finally proves Proposition (3.3). We in fact get an explicit method for computing the a_i as well.

Equation (3.4) leads to the following additional interpretation of the a_i, which we shall use later.

Lemma 3.7 *Let X be an extensible marked Dynkin diagram with d nodes and let $n \geq d$ such that* $\det(X_n) \neq 0$. *Fix i, $1 \leq i \leq n$ and suppose*

$$\omega_i^{(n)} = \sum_{k=1}^{n} c_k \alpha_k^{(n)}$$

Then $a_i = \det(X_n)\, c_n$.

Proof: We have $c_n = \omega_i^{(n)}(\check{\omega}_n^{(n)})$. Using Equation (3.4), we get

$$\begin{aligned} \det(X_n)\, c_n &= \omega_i^{(n)}(\det(X_n)\, \check{\omega}_n^{(n)}) \\ &= \omega_i^{(n)}(\sum_{j=1}^{n} a_j \check{\alpha}_j^{(n)}) \ = a_i \qquad \square \end{aligned}$$

The next lemma and its corollary re-express the a_i for $i > d$ in a more convenient form.

Lemma 3.8 *Let X be any marked Dynkin diagram (not necessarily extensible) with d nodes. Let $n \geq d$ be such that* $\det(X_n) \neq 0$. *Then in* $P(X_n)$,

$$\overline{\omega}_i^{(n)} \equiv i\, \overline{\omega}_1^{(n)} \pmod{Q(X_n)}$$

for $1 \leq i \leq (n - d + 1)$.

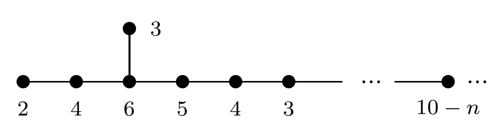
Proof: We only need to observe that if $1 \leq i \leq (n - d + 1)$,

$$i\, \overline{\omega}_1^{(n)} - \overline{\omega}_i^{(n)} = \sum_{j=1}^{i-1} j \alpha_{n-i+1+j}^{(n)} \ \in Q(X_n) \ \square \tag{3.5}$$

Corollary 3.9 *If $1 \leq i \leq (n - d + 1)$, then $a_{n-i+1} \equiv -i\Delta \pmod{\det(X_n)}$.*

Remark 3.10 The above corollary is also obvious from the definition of the a_i. We have $a_{n-i+1} = \det(X_{n-i}) = \det(X_n) - i\Delta$.

Example 3.11 Type E. We indicate the a_i's as labels on the Dynkin diagram.



3.3 The $|\lambda|_X + |\mu|_X = |\nu|_X$ criterion

Definition 3.12 If $\lambda = (x, y) \in \mathcal{H}_2$, we define our *number of boxes* function $|\lambda|_X$ to be

$$|\lambda|_X := \sum_{i=1}^{\ell(x)} a_i x_i - \Delta \sum_{i=1}^{\ell(y)} i y_i \tag{3.6}$$

For instance, in our Type A example (3.4) above, $|\lambda|_A = \sum_{i=1}^{\ell(x)} i x_i - \sum_{i=1}^{\ell(y)} i y_i$. If we assume further that $y = (0, 0, 0, \cdots)$, then $|\lambda|_A = \sum_{i=1}^{\ell(x)} i x_i$. If the dominant weight $\lambda^{(n)}$ (for $n \geq \ell(\lambda)$) is represented as a Young diagram (as in Section (2.4)), then $|\lambda|_A$ is precisely the number of boxes in this Young diagram. For general y, $|\lambda|_A$ measures the difference between the numbers of boxes in the Young diagrams of x and y.

Now, let $\lambda = (x, y) \in \mathcal{H}_2$ and fix $n \geq \ell(\lambda, X)$ such that $\det(X_n) \neq 0$. Consider the following element of $P(X_n)$: $(-\Delta)\lambda^{(n)} - |\lambda|_X \,\overline{\omega}_1^{(n)}$.

$$(-\Delta)\lambda^{(n)} - |\lambda|_X \,\overline{\omega}_1^{(n)} = \sum_{i=1}^{\ell(x)} x_i((-\Delta)\,\omega_i^{(n)} - a_i \overline{\omega}_1^{(n)}) + \sum_{i=1}^{\ell(y)} (-\Delta)\, y_i \,(\overline{\omega}_i^{(n)} - i\,\overline{\omega}_1^{(n)}) \tag{3.7}$$

The right hand side clearly lies in $Q(X_n)$ by Proposition (3.3) and Lemma (3.8). We have thus proved that

$$(-\Delta)\lambda^{(n)} \equiv |\lambda|_X \,\overline{\omega}_1^{(n)} \pmod{Q(X_n)} \tag{3.8}$$

Hence $|\lambda|_X$ identifies the coset of $Q(X_n)$ in $P(X_n)$ to which $\lambda^{(n)}$ belongs.

Proposition 3.13 *Let X be extensible and $\lambda, \mu, \nu \in \mathcal{H}_2^+$. Suppose $c_{\lambda\mu}^{\nu}(n) > 0$ for infinitely many values of n greater than than the lengths of each of λ, μ, ν. Then*

$$|\lambda|_X + |\mu|_X = |\nu|_X$$

Proof: Let $S = \{n : c_{\lambda\mu}^{\nu}(n) > 0\}$. If $n \in S$, then the representation $L(\nu^{(n)})$ of $\mathfrak{g}(X_n)$ occurs in the decomposition of the tensor product $L(\lambda^{(n)}) \otimes L(\mu^{(n)})$. In particular $\nu^{(n)}$ is a weight of this tensor product. All weights of $L(\lambda^{(n)}) \otimes L(\mu^{(n)})$ are congruent modulo the root lattice $Q(X_n)$ to the weight $\lambda^{(n)} + \mu^{(n)}$. So, we must have $\nu^{(n)} \equiv \lambda^{(n)} + \mu^{(n)} \pmod{Q(X_n)}$. Thus $(-\Delta)\,\nu^{(n)} \equiv (-\Delta)(\lambda^{(n)} + \mu^{(n)}) \pmod{Q(X_n)}$. Equation (3.8) then implies that

$$(|\lambda|_X + |\mu|_X - |\nu|_X)\,\overline{\omega}_1^{(n)} \equiv 0 \pmod{Q(X_n)}$$

Finally, we use Lemma (3.1) to conclude that $|\det(X_n)|$ divides $|\lambda|_X + |\mu|_X - |\nu|_X$ for all $n \in S$. Since X is extensible, $\Delta \neq 0$ and $|\det(X_n)| \to \infty$ as $n \to \infty$. This forces $|\lambda|_X + |\mu|_X - |\nu|_X = 0$ $\square$.

Example 3.14 We refer back to Example (2.7) and keep the same notation here. From the definition, it is easy to see that for $\lambda = \mu = (\mathbf{0}, \epsilon_1)$, we have

$|\lambda|_E = |\mu|_E = 1$. Similarly when ν is one of $(\epsilon_1, \mathbf{0}), (\mathbf{0}, 2\epsilon_1)$ or $(\mathbf{0}, \epsilon_2)$, $|\nu|_E = 2 = |\lambda|_E + |\mu|_E$ while for $\nu = (\mathbf{0}, \mathbf{0})$, $|\nu|_E = 0$. Proposition (3.13) now implies that for $\nu = (\mathbf{0}, \mathbf{0})$, $c^{\nu}_{\lambda\mu}(n) = 0$ eventually, as was stated before.

4 Littelmann paths and the proof of the main theorem

4.1 The notion of *depth*

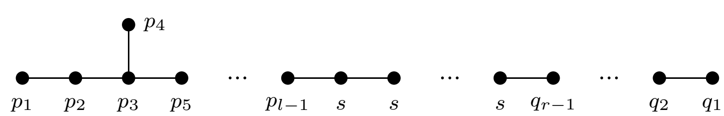
Let X be an extensible marked Dynkin diagram. In light of Proposition (3.13), we now consider $\lambda, \mu, \nu \in \mathcal{H}_2^+$ such that $|\lambda|_X + |\mu|_X = |\nu|_X$. Let $\gamma = \lambda + \mu - \nu \in \mathcal{H}_2$. Let $\gamma = (x, y)$ with $x, y \in \mathcal{H}_1$ and $l := \max(\ell(x), d)$, $r := \ell(y)$. Thus $\ell(\gamma, X) = l + r$. Since $|\gamma|_X = 0$, we know that $\gamma^{(n)} \in Q(X_n)$ for all $n \geq l + r$ for which $\det(X_n) \neq 0$ i.e, $\gamma^{(n)}$ is an integral linear combination of $\alpha_i^{(n)}$ $i = 1, \cdots, n$. The next proposition tells us how the coefficients of this linear combination change as n increases. This proposition allows us to define the useful notion of depth. At the end of this subsection, we shall also restate our main theorem giving an explicit value for N.

With notation as above, we have

Proposition 4.1 *There exist integers p_i $(1 \leq i \leq l-1)$, q_j $(1 \leq j \leq r-1)$ and s such that for $n \geq l + r$*

$$\gamma^{(n)} = \sum_{i=1}^{l-1} p_i \alpha_i^{(n)} + \sum_{i=l}^{n-r+1} s \alpha_i^{(n)} + \sum_{i=n-r+2}^{n} q_{n-i+1} \alpha_i^{(n)} \tag{4.1}$$

Remark 4.2 For the case $X_n = E_n$, the figure shows these coefficients labeling the corresponding nodes.



Thus, as n increases, the expression of $\gamma^{(n)}$ as a linear combination of the simple roots of X_n continues to have the same $l-1$ coefficients on the left and the same $r-1$ coefficients on the right, while the string of s 's in the middle grows longer.
Proof: We first prove the Proposition for some special choices of γ. For $i \geq 1$, consider the following elements of $\mathcal{H}_1$: $\sigma_i = (0, 0, \cdots, -\Delta, 0, 0, \cdots)$ where the $-\Delta$ occurs in the i^{th} position, and $\tau_i = (-a_i, 0, 0, \cdots)$. Let $\gamma_i = (\sigma_i, \tau_i) \in \mathcal{H}_2$. Clearly $|\gamma_i|_X = 0$ for all i by Equation (3.6).

Fix $i \geq 1$ and $n \geq \ell(\gamma_i, X)$ such that $\det(X_n) \neq 0$. We have $\gamma_i^{(n)} = (-\Delta)\omega_i^{(n)} - a_i \overline{\omega}_1^{(n)} \in Q(X_n)$. Let $\gamma_i^{(n)} = \sum_{k=1}^{n} c_k \alpha_k^{(n)}$. By Lemma (3.7) and the fact that $a_n = \det(X_{n-1})$, we get

$$c_n = (-\Delta)\frac{a_i}{\det(X_n)} - a_i\left(\frac{\det(X_{n-1})}{\det(X_n)}\right) = -a_i \tag{4.2}$$

For $(\max(d,i)+1) \leq j \leq n-1$, $\gamma_i^{(n)}(\check{\alpha}_j^{(n)}) = 0$. But $\gamma_i^{(n)}(\check{\alpha}_j^{(n)}) = 2c_j - c_{j-1} - c_{j+1}$. So

$$2c_j - c_{j-1} - c_{j+1} = 0, \text{ if } \max(d,i)+1 \leq j \leq n-1 \tag{4.3}$$

Further

$$\gamma_i^{(n)}(\check{\alpha}_n^{(n)}) = -a_i = 2c_n - c_{n-1} \tag{4.4}$$

Equations (4.2)-(4.4) imply that

$$c_j = -a_i \text{ for } \max(d,i) \leq j \leq n \tag{4.5}$$

We return to our general $\gamma = (x,y)$. Fix $m \geq \ell(\gamma)$ such that $\det(X_m) \neq 0$. Since $|\gamma|_X = 0$, we have

$$\begin{aligned}\gamma^{(m)} &= (-1/\Delta)(-\Delta\gamma^{(m)} - |\gamma|_X\overline{\omega}_1^{(m)}) \\ &= (-1/\Delta)\left(\textstyle\sum_{i=1}^{l} x_i(-\Delta\,\omega_i^{(m)} - a_i\overline{\omega}_1^{(m)}) - \Delta\sum_{i=1}^{r} y_i(\overline{\omega}_i^{(m)} - i\,\overline{\omega}_1^{(m)})\right) \\ &= (-1/\Delta)\textstyle\sum_{i=1}^{l} x_i\gamma_i^{(m)} + \sum_{i=1}^{r} y_i(\overline{\omega}_i^{(m)} - i\,\overline{\omega}_1^{(m)})\end{aligned} \tag{4.6}$$

Now if $\sum_{i=1}^{l} x_i\gamma_i^{(m)} = \sum_{j=1}^{n} c_j\alpha_j^{(m)}$, then Equation (4.5) implies that $c_j = -\sum_{i=1}^{l} x_i a_i$ for $l \leq j \leq m$. Further, Equation (3.5) implies that $\sum_{i=1}^{r} y_i(\overline{\omega}_i^{(m)} - i\overline{\omega}_1^{(m)})$ is a linear combination of $\alpha_j^{(m)}$ for $m-r+2 \leq j \leq m$. These two observations together with Equation (4.6) mean that if $\gamma^{(m)} = \sum_{j=1}^{n} d_j\alpha_j^{(m)}$, then $d_j = (1/\Delta)(\sum_{i=1}^{l} x_i a_i)$ for $l \leq j \leq m-r+1$. We note that this implies $(1/\Delta)(\sum_{i=1}^{l} x_i a_i) \in \mathbb{Z}$.

Define $p_i = d_i$ for $1 \leq i \leq l-1$, $q_j = d_{m-j+1}$ for $1 \leq j \leq r-1$ and $s = (1/\Delta)(\sum_{i=1}^{l} x_i a_i)$. For $n \geq \ell(\gamma, X) = l+r$ define

$$\mu_n = \sum_{i=1}^{l-1} p_i\alpha_i^{(n)} + \sum_{i=l}^{n-r+1} s\alpha_i^{(n)} + \sum_{i=n-r+2}^{n} q_{n-i+1}\alpha_i^{(n)} \in Q(X_n)$$

By definition, $\mu_m = \gamma^{(m)}$. Now for $1 \leq i \leq l$, $\mu_n(\check{\alpha}_i^{(n)})$ only depends on the values p_i, p_j for j running over all neighbors of the node i in X_n and possibly on s (if $i = l$ or $l-1$). Thus $\mu_n(\check{\alpha}_i^{(n)})$ is independent of n. Similarly, $\mu_n(\check{\alpha}_{n-j+1}^{(n)})$ is independent of n for $1 \leq j \leq r$. Further $\mu_n(\check{\alpha}_i^{(n)}) = 0$ for $l+1 \leq i \leq n-r$. These facts combined with $\mu_m = \gamma^{(m)}$ gives us that $\mu_n = \gamma^{(n)}$ for all $n \geq l+r$. □

Definition 4.3 If γ is any element of $\mathcal{H}_2$ such that $|\gamma|_X = 0$, it is clear that Proposition (4.1) still holds. We shall call the number s that occurs in Proposition (4.1) the *depth* of γ. We write

$$\operatorname{dep}(\gamma) := s = (1/\Delta)\sum_{i=1}^{l} x_i a_i = \sum_{j=1}^{r} j\, y_j$$

The last equality follows from $|\gamma|_X = 0$.

Lemma 4.4 *Let $\lambda, \mu, \nu \in \mathcal{H}_2^+$ be such that $|\lambda|_X + |\mu|_X = |\nu|_X$. Suppose $c_{\lambda\mu}^{\nu}(n) > 0$ for some $n \geq \ell(\lambda + \mu - \nu, X)$, then $\operatorname{dep}(\lambda + \mu - \nu) \geq 0$.*

Proof: We have $\lambda^{(n)} + \mu^{(n)} - \nu^{(n)} \in Q^+(X_n)$. So if $\lambda^{(n)} + \mu^{(n)} - \nu^{(n)} = \sum_{i=1}^{n} d_i \alpha_i^{(n)}$, then all the $d_i \geq 0$. By Proposition (4.1), we now conclude that $\operatorname{dep}(\lambda + \mu - \nu) \geq 0$ □

We restate our main Theorem (2.6) for the case $|\lambda|_X + |\mu|_X = |\nu|_X$ giving an explicit value for N.

Theorem 4.5 *Let X be an extensible marked Dynkin diagram and $\lambda, \mu, \nu \in \mathcal{H}_2^+$ such that $|\lambda|_X + |\mu|_X = |\nu|_X$. Let $\gamma = \lambda + \mu - \nu \in \mathcal{H}_2$ and $N = \ell(\gamma, X) + 2\operatorname{dep}(\gamma)$. Then $c_{\lambda\mu}^{\nu}(m) = c_{\lambda\mu}^{\nu}(n)$ for all $n, m \geq N$. We denote this constant value by $c_{\lambda\mu}^{\nu}(\infty)$ as before.*

We shall prove this theorem in the next few subsections. For the rest of this section, $\lambda, \mu, \nu, \gamma, N$ will be as in the statement of this Theorem. By (4.1) we know that

$$\gamma^{(n)} = \sum_{i=1}^{l-1} p_i \alpha_i^{(n)} + \sum_{i=l}^{n-r+1} s\alpha_i^{(n)} + \sum_{i=n-r+2}^{n} q_{n-i+1}\alpha_i^{(n)}$$

where $s = \operatorname{dep}(\gamma)$. Here l, r, p_i, q_j are all as in Proposition (4.1).

4.2 The path model

As a first step in proving Theorem (4.5) we will need an explicit expression for $c_{\lambda\mu}^{\nu}(n)$ given by Littelmann's path model [L2]. We recall the relevant notions here.

Let $\Pi^{(n)}$ denote the set of all piecewise linear paths $\pi : [0,1] \to \mathfrak{h}^*(X_n)$ such that $\pi(0) = 0$. We identify paths that are reparametrizations of each other. For each simple root $\alpha_i^{(n)}$ $(1 \leq i \leq n)$, we define a *lowering operator* $f_i^{(n)}$ and a *raising operator* $e_i^{(n)}$ on $\mathbb{Z}\Pi$, the free $\mathbb{Z}$ module with basis Π. Given $\pi \in \Pi^{(n)}$, let $\pi_i(t) = \pi(t)(\check{\alpha}_i^{(n)})$ for $0 \leq t \leq 1$. We consider the function $a : [0,1] \to [0,1]$ defined by $a(t) = \min\{1, \pi_i(s) - m_i | t \leq s \leq 1\}$, where $m_i = \min\{\pi_i(t) | 0 \leq t \leq 1\}$. Note that a is an increasing function. If $a(1) < 1$, $f_i^{(n)}\pi := 0$. Otherwise, $f_i^{(n)}\pi$ is the path defined by

$$f_i^{(n)}\pi(t) := \pi(t) - a(t)\alpha_i^{(n)} \tag{4.7}$$

So if $f_i^{(n)}\pi \neq 0$, then

$$f_i^{(n)}\pi(1) = \pi(1) - \alpha_i^{(n)} \tag{4.8}$$

Thus $f_i^{(n)}$ lowers the endpoint of the path π by $\alpha_i^{(n)}$.

Similarly we consider the increasing function $b : [0,1] \to [0,1]$ with $b(t) = \max\{0, 1 - (\pi_i(s) - m_i) | 0 \leq s \leq t\}$. If $b(0) > 0$, we set $e_i^{(n)}\pi = 0$ and otherwise

$$e_i^{(n)}\pi(t) := \pi(t) + b(t)\alpha_i^{(n)} \tag{4.9}$$

If $e_i^{(n)}\pi \neq 0$, then $e_i^{(n)}\pi(1) = \pi(1) + \alpha_i^{(n)}$. For a more "geometric" description of the action of the lowering and raising operators, see Littelmann [L1, L2, L3].

Remark 4.6 We consider the following situation which will occur often. If $\pi_i(t)$ is itself an increasing function with $\pi_i(1) = 1$, then from the definition, we get $a(t) = \pi_i(t)$.

To obtain the value of $c_{\lambda\mu}^{\nu}(n)$, we first consider the straight line path $\pi_{\lambda^{(n)}} \in \Pi^{(n)}$ defined by $\pi_{\lambda^{(n)}}(t) = t\lambda^{(n)}$ for $t \in [0,1]$. The set of all paths that can be obtained by repeated action of the lowering operators on $\pi_{\lambda^{(n)}}$ is called the set of Lakshmibai-Seshadri (L-S) paths of shape $\lambda^{(n)}$. Let

$$\mathcal{P}(\lambda, \mu, \nu, n) := \{ \text{ L-S paths of shape } \lambda^{(n)} \text{ whose endpoint is } \nu^{(n)} - \mu^{(n)} \}$$

If $\pi = f_{i_k}^{(n)} \cdots f_{i_2}^{(n)} f_{i_1}^{(n)}(\pi_{\lambda^{(n)}})$ is an element of $\mathcal{P}(\lambda, \mu, \nu, n)$, then clearly Equation (4.8) implies that $\sum_{j=1}^{k} \alpha_{i_j}^{(n)} = \lambda^{(n)} + \mu^{(n)} - \nu^{(n)}$. A path $\pi \in \mathcal{P}(\lambda, \mu, \nu, n)$ is said to be $\mu^{(n)}$ **dominant** if the translated path $\mu^{(n)} + \pi(t)$ lies completely in the dominant Weyl chamber of $\mathfrak{h}^*(X_n)$. Let

$$\mathcal{P}^+(\lambda, \mu, \nu, n) := \{\pi \in \mathcal{P}(\lambda, \mu, \nu, n) \,:\, \pi \text{ is } \mu^{(n)} \text{ dominant}\}$$

Littelmann's tensor product decomposition formula [L2] now states that the number of elements in $\mathcal{P}^+(\lambda, \mu, \nu, n)$ is the value of $c_{\lambda\mu}^{\nu}(n)$.

Theorem 4.7 (Littelmann) $c_{\lambda\mu}^{\nu}(n) = \#\mathcal{P}^+(\lambda, \mu, \nu, n)$

This theorem will be the main tool in our proof of Theorem (4.5).

4.3

In light of Theorem (4.7), one needs to analyze the set $\mathcal{P}^+(\lambda, \mu, \nu, n)$ better. In this subsection, we introduce certain special lowering operators. It will turn out that paths in $\mathcal{P}^+(\lambda, \mu, \nu, n)$ can be obtained by repeated application of just these special lowering operators on $\pi_{\lambda^{(n)}}$. This fact will imply our main theorem (4.5).

We first consider a larger set of paths. Let

$$V^{(n)} := \{i : l + s < i < (n - r + 1) - s\}$$

and

$$\overline{V}^{(n)} := \{i : l+s \le i \le (n-r+1)-s\}$$

Let $\Sigma^{(n)} \subset \Pi^{(n)}$ be

$$\Sigma^{(n)} := \{\eta \in \Pi^{(n)} \,|\, \eta(t)(\check{\alpha}_i^{(n)}) = 0 \,\forall t \in [0,1];\, \forall i \in V^{(n)}\}$$

i.e, $\Sigma^{(n)}$ is the set of paths that are "supported" on $l+s$ nodes on the left and $r+s$ nodes on the right. Now, $\mathbb{Z}\Sigma^{(n)}$ will no longer be closed under the action of all the lowering operators. We will show below that there are still many lowering operators and certain compositions of them that preserve $\mathbb{Z}\Sigma^{(n)}$. Let

$$g^{(n)} := f_{n-r+1-s}^{(n)} \cdots f_{l+s+1}^{(n)} f_{l+s}^{(n)} \qquad \text{and} \qquad h^{(n)} := f_{l+s}^{(n)} f_{l+s+1}^{(n)} \cdots f_{n-r+1-s}^{(n)}$$

Clearly $g^{(n)}, h^{(n)} \in \operatorname{End} \mathbb{Z}\Pi^{(n)}$.

Lemma 4.8 *Suppose $\eta \in \Sigma^{(n)}$. Then*

1. *Let $j \notin \overline{V}^{(n)}$. If $f_j^{(n)}\eta \neq 0$, then $f_j^{(n)}\eta \in \Sigma^{(n)}$.*
2. *If $g^{(n)}\eta \neq 0$, then $g^{(n)}\eta \in \Sigma^{(n)}$.*
3. *If $h^{(n)}\eta \neq 0$, then $h^{(n)}\eta \in \Sigma^{(n)}$.*

Proof: (1) By Equation (4.7), $f_j^{(n)}\eta(t) = \eta(t) - a(t)\alpha_j^{(n)}$. If $i \in V^{(n)}$, then the nodes i and j are not connected by a line in the Dynkin diagram X_n. Hence $\alpha_j^{(n)}(\check{\alpha}_i^{(n)}) = 0$. This together with $\eta \in \Sigma^{(n)}$ implies that $f_j^{(n)}\eta \in \Sigma^{(n)}$.
(2) Suppose $(f_{l+s}^{(n)}\eta)(t) = \eta(t) - a(t)\alpha_{l+s}^{(n)}$, then

$$\begin{aligned}(f_{l+s}^{(n)}\eta)(t)(\check{\alpha}_{l+s+1}^{(n)}) &= \eta(t)(\check{\alpha}_{l+s+1}^{(n)}) - a(t)\alpha_{l+s}^{(n)}(\check{\alpha}_{l+s+1}^{(n)}) \\ &= a(t)\end{aligned}$$

Since $g^{(n)}\eta \neq 0, f_{l+s}^{(n)}\eta \neq 0$. Hence $a(t)$ is an increasing function with $a(1) = 1$. By Remark (4.6) we have $(f_{l+s+1}^{(n)} f_{l+s}^{(n)}\eta)(t) = \eta(t) - a(t)\alpha_{l+s}^{(n)} - a(t)\alpha_{l+s+1}^{(n)}$. Continuing this process, we have

$$(g^{(n)}\eta)(t) = (f_{n-r+1-s}^{(n)} \cdots f_{l+s+1}^{(n)} f_{l+s}^{(n)}\eta)(t) = \eta(t) - a(t)\sum_{j=l+s}^{n-r+1-s} \alpha_j^{(n)} \tag{4.10}$$

But $\eta \in \Sigma^{(n)}$ and $(\sum_{j=l+s}^{n-r+1-s} \alpha_j^{(n)})(\check{\alpha}_i^{(n)}) = 0$ for all $i \in V^{(n)}$. Hence, $g^{(n)}\eta \in \Sigma^{(n)}$ too. The proof of (3) is analogous. □

The definition of $\Sigma^{(n)}$ makes it clear that $\Sigma^{(n)}$ and $\Sigma^{(m)}$ are in some sense the same, since the paths in both sets are basically just supported on $l+r+2s$ nodes. To make this more precise, we define maps $\phi_{nm} : \Sigma^{(n)} \to \Sigma^{(m)}$ for all $n, m \ge l+r+2s$ as follows: Take $\eta \in \Sigma^{(n)}$. Since $\eta(t)(\check{\alpha}_i^{(n)}) = 0$ for all $i \in V^{(n)}$, we can write

$$\eta(t) = \sum_{i=1}^{l+s} d_i(t)\omega_i^{(n)} + \sum_{j=1}^{r+s} \tilde{d}_j(t)\overline{\omega}_j^{(n)}$$

We define

$$\phi_{nm}(\eta)(t) := \sum_{i=1}^{l+s} d_i(t)\omega_i^{(m)} + \sum_{j=1}^{r+s} \tilde{d}_j(t)\overline{\omega}_j^{(m)}$$

Clearly ϕ_{nm} and ϕ_{mn} are inverses of each other and set up bijections between the sets $\Sigma^{(n)}$ and $\Sigma^{(m)}$.

The following lemma ensures that these bijections also respect the actions of the special lowering operators introduced above. We let $\overline{f}_i^{(n)}$ denote the lowering operator $f_{n-i+1}^{(n)}$.

Lemma 4.9 *Let $m, n \geq l + r + 2s$ and $\eta \in \Sigma^{(n)}$.*

1. *If $1 \leq i < l + s$ then $\phi_{nm}(f_i^{(n)}\eta) = f_i^{(m)}\phi_{nm}(\eta)$.*
2. *If $1 \leq j < r + s$ then $\phi_{nm}(\overline{f}_j^{(n)}\eta) = \overline{f}_j^{(m)}\phi_{nm}(\eta)$.*
3. *$\phi_{nm}(g^{(n)}\eta) = g^{(m)}\phi_{nm}(\eta)$.*
4. *$\phi_{nm}(h^{(n)}\eta) = h^{(m)}\phi_{nm}(\eta)$.*

All these equalities also hold if some of the paths involved become 0. We define $\phi_{nm}(0) = 0$

Proof: (1) and (2) follow from the definitions of the lowering operators and ϕ_{nm}. For (3), suppose $\eta(t) = \sum_{i=1}^{l+s} d_i(t)\omega_i^{(n)} + \sum_{j=1}^{r+s} \tilde{d}_j(t)\overline{\omega}_j^{(n)}$, then Equation (4.10) implies that

$$(g^{(n)}\eta)(t) = \eta(t) - a(t) \sum_{j=l+s}^{n-r+1-s} \alpha_j^{(n)}$$

But $\sum_{j=l+s}^{n-r+1-s} \alpha_j^{(n)} = \omega_{l+s}^{(n)} + \overline{\omega}_{r+s}^{(n)}$. Hence

$$(g^{(n)}\eta)(t) = \eta(t) - a(t)(\omega_{l+s}^{(n)} + \overline{\omega}_{r+s}^{(n)})$$

It is easy to see that if we replace n by m and η by $\phi_{nm}(\eta)$ throughout, then the above argument still holds, showing that $(g^{(m)}\phi_{nm}(\eta))(t) = \phi_{nm}(\eta) - a(t)(\omega_{l+s}^{(m)} + \overline{\omega}_{r+s}^{(m)})$. The proof of (4) is similar. $\square$

Since these special lowering operators seem to be natural in our setting, we next consider the subset of L-S paths of shape $\lambda^{(n)}$ which are obtained by repeated actions of only these special lowering operators. More precisely, define

$\mathcal{P}^0(\lambda, \mu, \nu, n) :=$ set of L-S paths π of shape $\lambda^{(n)}$, with $\pi(1) = \nu^{(n)} - \mu^{(n)}$ such that π can be obtained by the action of the operators $\{f_i^{(n)} : 1 \leq i < l + s\} \cup \{\overline{f}_j^{(n)} : 1 \leq j < r + s\} \cup \{g^{(n)}, h^{(n)}\}$ on $\pi_{\lambda^{(n)}}$.

We then have the following:

Lemma 4.10 *1.* $\mathcal{P}^0(\lambda,\mu,\nu,n) \subset \Sigma^{(n)}$.

2. $\phi_{nm}(\mathcal{P}^0(\lambda,\mu,\nu,n)) \subset \mathcal{P}^0(\lambda,\mu,\nu,m)$.

Proof: (1) Since the path $\pi_{\lambda^{(n)}} \in \Sigma^{(n)}$, acting on it by the special lowering operators still gives us a path in $\Sigma^{(n)}$ (by Lemma (4.8)).
(2) If the path π is obtained by the action of the operators $f_i^{(n)}, \overline{f}_j^{(n)}, g^{(n)}$ and $h^{(n)}$ on $\pi_{\lambda^{(n)}}$, Lemma (4.9) implies that $\phi_{nm}(\pi)$ is obtained by the action of the corresponding operators $f_i^{(m)}, \overline{f}_j^{(m)}, g^{(m)}$ and $h^{(m)}$ on $\phi_{nm}(\pi_{\lambda^{(n)}})$. But $\phi_{nm}(\pi_{\lambda^{(n)}}) = \pi_{\lambda^{(m)}}$ since the support of λ is a subset of the first l and last r nodes. Further, since the endpoint of π is $\nu^{(n)} - \mu^{(n)}$, the endpoint of $\phi_{nm}(\pi)$ is clearly $\nu^{(m)} - \mu^{(m)}$. Thus $\phi_{nm}(\pi) \in \mathcal{P}^0(\lambda,\mu,\nu,m)$. □

Clearly $\mathcal{P}^0(\lambda,\mu,\nu,n)$ and $\mathcal{P}^+(\lambda,\mu,\nu,n)$ are both subsets of $\mathcal{P}(\lambda,\mu,\nu,n)$. The next important proposition relates these subsets.

Proposition 4.11 $\mathcal{P}^+(\lambda,\mu,\nu,n) \subset \mathcal{P}^0(\lambda,\mu,\nu,n)$

Before we embark upon the proof of Proposition (4.11), we state a corollary which implies our main Theorem (4.5).

Corollary 4.12 $\phi_{nm}(\mathcal{P}^+(\lambda,\mu,\nu,n)) \subset \mathcal{P}^+(\lambda,\mu,\nu,m)$

Proof: Let $\pi \in \mathcal{P}^+(\lambda,\mu,\nu,n)$. Proposition (4.11) implies that $\pi \in \mathcal{P}^0(\lambda,\mu,\nu,n)$. By Lemma (4.10), $\phi_{nm}(\pi) \in \mathcal{P}^0(\lambda,\mu,\nu,m)$; in particular $\phi_{nm}(\pi)$ is an L-S path. We need to show that $\phi_{nm}(\pi)$ is $\mu^{(m)}$ dominant. Since $\pi \in \mathcal{P}^0(\lambda,\mu,\nu,n \subset \Sigma^{(n)}$, write

$$\pi(t) = \sum_{i=1}^{l+s} d_i(t)\,\omega_i^{(n)} + \sum_{j=1}^{r+s} \tilde{d}_j(t)\,\overline{\omega}_j^{(n)}$$

Let $\mu = (x,y) \in \mathcal{H}_2^+$. Since π is $\mu^{(n)}$ dominant, we have $(\mu^{(n)}+\pi(t))(\check{\alpha}_i^{(n)}) \geq 0\,\forall t$ for all $1 \leq i \leq n$. This is equivalent to the following conditions

1. $x_i + d_i(t) \geq 0\,\forall t$; $1 \leq i \leq l+s$
2. $y_j + \tilde{d}_j(t) \geq 0\,\forall t$; $1 \leq j \leq r+s$

It is clear that these very same conditions imply the fact that $\phi_{nm}(\pi)$ is $\mu^{(m)}$ dominant. □

Proof of Theorem (4.5)**:** Corollary (4.12) together with the fact that ϕ_{nm} and ϕ_{mn} are inverse maps imply that the sets $\mathcal{P}^+(\lambda,\mu,\nu,n)$ and $\mathcal{P}^+(\lambda,\mu,\nu,m)$ are in bijection with each other, for $m,n \geq l+r+2s$. We now appeal to Theorem (4.7) to deduce Theorem (4.5): $c_{\lambda\mu}^{\nu}(n) = c_{\lambda\mu}^{\nu}(m)$ provided $n,m \geq l+r+2s$. □

4.4 Proof of Proposition (4.11)

To prove proposition (4.11), we shall start with a path $\pi \in \mathcal{P}^+(\lambda, \mu, \nu, n)$ and construct a string of raising operators which maps π to $\pi_{\lambda^{(n)}}$. These raising operators will be the analogues of the special lowering operators introduced before.

We first state some properties of raising operators that we will need. We refer to Littelmann's paper [L2] for the proofs.

Proposition 4.13 *Let η be an element of $\Pi^{(n)}$ and $1 \leq i, j \leq n$.*

1. *If the nodes i and j have no edge between them i.e, $\alpha_i^{(n)}(\check{\alpha}_j^{(n)}) = 0 = \alpha_j^{(n)}(\check{\alpha}_i^{(n)})$, then $e_i^{(n)} e_j^{(n)} \eta = e_j^{(n)} e_i^{(n)} \eta$.*
2. *$e_i^{(n)} \eta = 0 \Leftrightarrow \eta(t)(\check{\alpha}_i^{(n)}) \geq 0 \; \forall t \in [0, 1]$.*
3. *If $e_i^{(n)} \eta \neq 0$, then $\min_t (e_i^{(n)} \eta)(t)(\check{\alpha}_i^{(n)}) = \min_t \eta(t)(\check{\alpha}_i^{(n)}) + 1$.*
4. *If $e_i^{(n)} \eta \neq 0$, then $f_i^{(n)} e_i^{(n)} \eta = \eta$*
5. *If η is an L-S path, then η has the* integrality property *i.e, $\min_t \eta(t)(\check{\alpha}_i^{(n)})$ is an integer for all $1 \leq i \leq n$.*
6. *If η is an L-S path of shape $\lambda^{(n)}$ then $\lambda^{(n)} - \eta(1) \in Q^+(X_n)$.*

We shall now prove Proposition (4.11). Let $U_1 = \{l < i < n - r + 1\}$ and $\overline{U}_1 = \{l \leq i \leq n - r + 1\}$. Assume $\pi \in \mathcal{P}^+(\lambda, \mu, \nu, n)$. By definition, this means that $\mu^{(n)}(\check{\alpha}_i^{(n)}) + \pi(t)(\check{\alpha}_i^{(n)}) \geq 0$ for all t and for all $1 \leq i \leq n$. In particular, since $\mu^{(n)}(\check{\alpha}_i^{(n)}) = 0$ for all $i \in U_1$, we have

$$\pi(t)(\check{\alpha}_i^{(n)}) \geq 0 \; \forall t \in [0, 1], \; \forall i \in U_1 \tag{4.11}$$

Secondly, since $\pi(1) = \nu^{(n)} - \mu^{(n)}$, we have

$$\pi(1)(\check{\alpha}_i^{(n)}) = 0 \; \forall i \in U_1 \tag{4.12}$$

Properties (4.11) and (4.12) will be important for us. In fact we will only need these two properties of π and the fact that π is an L-S path to show that $\pi \in \mathcal{P}^0(\lambda, \mu, \nu, n)$.

Since π is an L-S path, there exists a sequence of raising operators which maps π to $\pi_{\lambda^{(n)}}$. Let $e_{i_p}^{(n)} \cdots e_{i_2}^{(n)} e_{i_1}^{(n)} \pi = \pi_{\lambda^{(n)}}$. Clearly $\lambda^{(n)} - \pi(1) = \sum_{k=1}^{p} \alpha_{i_k}^{(n)}$. Pick j minimal such that $i_j \in \overline{U}_1$.

<u>Claim:</u> $i_j \notin U_1$.

Proof: Suppose $i_j \in U_1$. For $1 \leq k \leq j - 1$, $i_k \notin \overline{U}_1$. Hence the nodes i_j and i_k of X_n do not have an edge between them. By (1) of Proposition (4.13), this implies that $e_{i_j}^{(n)}$ commutes with $e_{i_k}^{(n)}$ for all $1 \leq k \leq j-1$. Thus $e_{i_j}^{(n)} \cdots e_{i_2}^{(n)} e_{i_1}^{(n)} \pi = e_{i_{j-1}}^{(n)} \cdots e_{i_2}^{(n)} e_{i_1}^{(n)} e_{i_j}^{(n)} \pi = 0$; since $e_{i_j}^{(n)} \pi = 0$ by Property (4.11) and Proposition (4.13), (2). This contradicts $e^{(n)}_{i_p}\cdots e^{(n)}_{i_2}e^{(n)}_{i_1}\pi = \pi_{\lambda^{(n)}} \neq 0$, proving our claim $\square$

So either $i_j = l$ or $i_j = n-r+1$. **Case 1:** $i_j = l$. Let $\pi' = e^{(n)}_{i_j}\cdots e^{(n)}_{i_2}e^{(n)}_{i_1}\pi$. Then $\pi'(1) = \pi(1) + \sum_{k<j}\alpha^{(n)}_{i_j} + \alpha^{(n)}_l$. By Property (4.12), $\pi'(1)(\check{\alpha}^{(n)}_{l+1}) = \alpha^{(n)}_l(\check{\alpha}^{(n)}_{l+1}) = -1$. Proposition (4.13), (2) implies that $e^{(n)}_{l+1}\pi' \neq 0$. Again, by definition $(e^{(n)}_{l+1}\pi')(1) = \pi'(1) + \alpha^{(n)}_{l+1}$. So $(e^{(n)}_{l+1}\pi')(1)(\check{\alpha}^{(n)}_{l+2}) = -1$. Thus $e^{(n)}_{l+2}e^{(n)}_{l+1}\pi' \neq 0$. We continue this way to conclude that $e^{(n)}_{n-r}\cdots e^{(n)}_{l+2}e^{(n)}_{l+1}\pi' \neq 0$. Note that we cannot go all the way to $e^{(n)}_{n-r+1}$ since $\pi(1)(\check{\alpha}^{(n)}_{n-r+1})$ may not be 0. We set $\pi_1 := e^{(n)}_{n-r}\cdots e^{(n)}_{l+2}e^{(n)}_{l+1}\pi'$. **Case 2:** If $i_j = n-r+1$, the same argument as in Case 1 proves that $e^{(n)}_{l+1}\cdots e^{(n)}_{n-r-1}e^{(n)}_{n-r}\pi' \neq 0$. In this case, we set $\pi_1 := e^{(n)}_{l+1}\cdots e^{(n)}_{n-r-1}e^{(n)}_{n-r}\pi'$.

By Proposition (4.13), (4), we have just shown that π can be obtained by repeated action of operators from the set $\{f^{(n)}_i : i \notin V^{(n)}\} \cup \{g^{(n)}, h^{(n)}\}$ on π_1. We recall that $V^{(n)} = \{l+s < i < n-r+1-s\}$. Note that in either case,

$$\pi_1(1) - \pi(1) \geq \sum_{i=l+1}^{n-r} \alpha^{(n)}_i \tag{4.13}$$

Here we used the 'usual' partial order on $P(X_n)$ defined by $\alpha \geq \beta \Leftrightarrow \alpha - \beta \in Q^+(X_n)$.

Now, π_1 is still an L-S path of shape $\lambda^{(n)}$. We define the sets $U_2 := \{i : l+1 < i < n-r\}$ and $\overline{U}_2 := \{i : l+1 \leq i \leq n-r\}$, obtained by deleting one node from each end of the string of nodes in U_1 and $\overline{U}_1$. One observes that (a) $\pi'(1)(\check{\alpha}^{(n)}_i) = 0$ for all $i \in U_2$ and (b) $(\sum_{i=l+1}^{n-r}\alpha^{(n)}_i)(\check{\alpha}^{(n)}_i) = 0\,\forall i \in U_2$. Since $\pi(1) = \pi'(1) + \sum_{i=l+1}^{n-r}\alpha^{(n)}_i$, these give us

$$\pi_1(1)(\check{\alpha}^{(n)}_i) = 0\,\forall i \in U_2 \tag{4.14}$$

This is similar to Property (4.12) of π. We now claim that the analog of Property (4.11) of π also holds for π_1. More precisely we have:

Lemma 4.14

$$\pi_1(t)(\check{\alpha}^{(n)}_i) \geq 0\,\forall t \in [0,1],\,\forall i \in U_2 \tag{4.15}$$

Proof: Let $i \in U_2$ i.e, $l+2 \leq i \leq (n-r-1)$. We only consider **Case 1:** $\pi_1 = e^{(n)}_{n-r}\cdots e^{(n)}_{l+2}e^{(n)}_{l+1}\pi'$. The other case will follow by a similar argument. Let $\eta = e^{(n)}_{i-1}\cdots e^{(n)}_{l+1}\pi'$. Then by succesively using the definitions of $e^{(n)}_i, e^{(n)}_{i+1}, \cdots, e^{(n)}_{n-r}$ we get

$$\begin{aligned} \pi_1(t) &= (e^{(n)}_{n-r}\cdots e^{(n)}_{i+1}e^{(n)}_i\eta)(t) \\ &= \eta(t) + b_i(t)\alpha^{(n)}_i + b_{i+1}(t)\alpha^{(n)}_{i+1} + \textstyle\sum_{k=i+2}^{n-r} b_k(t)\alpha^{(n)}_k \end{aligned} \tag{4.16}$$

where the $b_j : [0,1] \to [0,1]$ are increasing functions with $b_j(0) = 0$ and $b_j(1) = 1$. Note that $(e_i^{(n)}\eta)(t) = \eta(t) + b_i(t)\alpha_i^{(n)}$. Similarly

$$(e_{i+1}^{(n)}e_i^{(n)}\eta)(t) = (e_i^{(n)}\eta)(t) + b_{i+1}(t)\alpha_{i+1}^{(n)} \tag{4.17}$$

etc. Since $\alpha_k^{(n)}(\check{\alpha}_i^{(n)}) = 0\ \forall k \geq i+2$, we get

$$\pi_1(t)(\check{\alpha}_i^{(n)}) = \eta(t)(\check{\alpha}_i^{(n)}) + 2b_i(t) - b_{i+1}(t) \tag{4.18}$$

We need to show that the left hand side is ≥ 0 for all t. We will first show that it is $\geq -1\ \forall t \in [0,1]$. In fact we claim:

$$\eta(t)(\check{\alpha}_i^{(n)}) + 2b_i(t) \geq 0\ \forall t \in [0,1] \tag{4.19}$$

To prove this observe that $\eta(t)(\check{\alpha}_i^{(n)}) + 2b_i(t) = (\eta(t) + b_i(t)\alpha_i^{(n)})(\check{\alpha}_i^{(n)}) = (e_i^{(n)}\eta)(t)(\check{\alpha}_i^{(n)})$. By Proposition (4.13), (3) we have $\min_t(e_i^{(n)}\eta)(t)(\check{\alpha}_i^{(n)}) = \min_t \eta(t)(\check{\alpha}_i^{(n)}) + 1$. Equation (4.19) would thus follow if we show that

$$\min_t \eta(t)(\check{\alpha}_i^{(n)}) = -1 \tag{4.20}$$

But $\eta(t)(\check{\alpha}_i^{(n)}) = \pi'(t)(\check{\alpha}_i^{(n)}) + \sum_{k=l+1}^{i-2} b_k(t)\alpha_k^{(n)}(\check{\alpha}_i^{(n)}) + b_{i-1}(t)(-1)$ where the b_k are increasing functions with $b_k(0) = 0$ and $b_k(1) = 1$. Now, $\pi'(t)(\check{\alpha}_i^{(n)}) \geq 0$ by (4.11), $b_{i-1}(t) \leq 1$ and the intermediate terms in the sum are 0 since $\alpha_k^{(n)}(\check{\alpha}_i^{(n)})$ for $k \leq i-2$. Thus $\eta(t)(\check{\alpha}_i^{(n)}) \geq -1$. In fact, $\eta(1)(\check{\alpha}_i^{(n)}) = -1$ since $\pi'(1)(\check{\alpha}_i^{(n)}) = 0$. This proves Equation (4.20) and hence Equation (4.19). Looking back at Equation (4.18), this means that $\pi_1(t)(\check{\alpha}_i^{(n)}) \geq -1$. Our next step is to show that $\pi_1(t)(\check{\alpha}_i^{(n)})$ never attains the value -1 for any t.

To see this, suppose $\pi_1(t_0)(\check{\alpha}_i^{(n)}) = -1$. Then we must have that

$$\eta(t_0)(\check{\alpha}_i^{(n)}) + 2b_i(t_0) = 0 \tag{4.21a}$$

$$b_{i+1}(t_0) = 1 \tag{4.21b}$$

We look more closely at Equation (4.21b). By Equation (4.9), $b_{i+1}(t)$ is determined by the values of the function $(e_i^{(n)}\eta)(t)(\check{\alpha}_{i+1}^{(n)})$. It is easy to see that if we replace η by $e_i^{(n)}\eta$ and $\check{\alpha}_i^{(n)}$ by $\check{\alpha}_{i+1}^{(n)}$ in Equation (4.20), then it still holds (*the proof is similar*). We record this as

$$\min_t(e_i^{(n)}\eta)(t)(\check{\alpha}_{i+1}^{(n)}) = -1 \tag{4.22}$$

By the definition of $b_{i+1}(t_0)$ (Equation (4.9)) and Equation (4.22), there must exist s, $0 \leq s \leq t_0$ such that $(e_i^{(n)}\eta)(s)(\check{\alpha}_i^{(n)}) = -1$ i.e,

$$\eta(s)(\check{\alpha}_{i+1}^{(n)}) - b_i(s) = -1 \tag{4.23}$$

But observe that the first term on the left hand side is ≥ 0. This is because $\eta(s)(\check{\alpha}_{i+1}^{(n)}) = \pi'(s)(\check{\alpha}_{i+1}^{(n)}) + \sum_{k=l+1}^{i-1} b_k(s)\alpha_k^{(n)}(\check{\alpha}_{i+1}^{(n)}) = \pi'(s)(\check{\alpha}_{i+1}^{(n)}) \geq 0$ by Equation (4.11). So the only way Equation (4.23) can hold is if $\eta(s)(\check{\alpha}_{i+1}^{(n)}) = 0$ and $b_i(s) = 1$. Since b_i is an increasing function and $s \leq t_0$, this means that $b_i(t_0) = 1$ as well. Substituting in Equation (4.21a) we get $\eta(t_0)(\check{\alpha}_i^{(n)}) = -2$. This clearly contradicts Equation (4.20). We have thus shown that

$$\pi_1(t)(\check{\alpha}_i^{(n)}) > -1 \; \forall t \in [0,1]$$

But π_1 being an L-S path, has the integrality property (Proposition (4.13), (5)). Thus $\min_t \pi_1(t)(\check{\alpha}_i^{(n)}) \geq 0$ proving Fact (4.15). □

We have thus shown that the path π_1 is an L-S path, which satisfies Equations (4.14) and (4.15). The situation is now analogous to the path π which satisfies Equations (4.12) and (4.11). So we can repeat all the arguments that came between Equation (4.12) and Equation (4.15) replacing π with π_1 and U_1 with U_2 throughout. We thus obtain an L-S path π_2 of shape $\lambda^{(n)}$ which satisfies

$$\pi_2(1) - \pi_1(1) \geq \sum_{i=l+2}^{n-r-1} \alpha_i^{(n)} \tag{4.24}$$

$$\begin{aligned} \pi_2(t)(\check{\alpha}_i^{(n)}) &= 0 \, \forall i \in U_3 \\ \pi_2(t)(\check{\alpha}_i^{(n)}) &\geq 0 \; \forall t \in [0,1], \, \forall i \in U_3 \end{aligned} \tag{4.25}$$

where $U_3 := \{i : l+2 < i < n-r-1\}$.

It is clear that this process has to stop before the s^{th} stage where $s = \mathrm{dep}(\lambda + \mu - \nu)$. This is because the coefficient of $\alpha_i^{(n)}$ for $(l+s) < i < (n - r + 1 - s)$ in $\lambda^{(n)} - \pi_k(1)$ decreases by at least 1 each time k increases by 1 (by Equations (4.13), (4.24), etc). To start with however, we know that $\lambda^{(n)} - \pi(1) = \gamma^{(n)}$, which is given by Equation (4.1). Thus the coefficient of these $\alpha_i^{(n)}$ is 's' to begin with. By Proposition (4.13), (6) the coefficient of these $\alpha_i^{(n)}$ in $\lambda^{(n)} - \pi_k(1)$ must be ≥ 0, forcing $k \leq s$. In fact one can show that k must equal s, but we will not need this fact.

Let π_k denote the last path in the list. Then clearly $\lambda^{(n)} - \pi_k(1) = \sum_{i=1}^n c_i\alpha_i^{(n)}$ with $c_i = 0$ for $i \in V^{(n)}$. Recall here that $V^{(n)} = \{l+s < i < n-r+1-s\}$. We can thus write $\pi_{\lambda^{(n)}} = e_{i_p}^{(n)} \cdots e_{i_2}^{(n)} e_{i_1}^{(n)} \pi_k$ for some $i_1, i_2, \cdots, i_p \notin V^{(n)}$. In summary , if we define $\pi_0 = \pi$ and $\pi_{k+1} = \pi_{\lambda^{(n)}}$, then for each $0 \leq j \leq k$, we have shown that π_j can be obtained from π_{j+1} by repeated action of elements of $T = \{f_i^{(n)} : i \notin V^{(n)}\} \cup \{g^{(n)}, h^{(n)}\}$. Thus $\pi_0 = \pi$ can be obtained from $\pi_{k+1} = \pi_{\lambda^{(n)}}$ by the action of elements of T. Hence $\pi \in \mathcal{P}^0(\lambda, \mu, \nu, n)$. This concludes the proof of Proposition (4.11). □

4.5 Special cases

We now restrict λ, μ, ν to be certain special types of double-headed weights and say something more about the stable multiplicities for these types.

1. The first type we shall consider was the starting point for this present work. Let λ, μ and ν be *single headed* dominant weights supported on the "tail" portion i.e, $\lambda = (0,x)$, $\mu = (0,y)$, $\nu = (0,z)$ for $x,y,z \in \mathcal{H}_1^+$. If $x = (x_1,x_2,x_3,\cdots)$, $y = (y_1,y_2,y_3,\cdots)$ $z = (z_1,z_2,z_3,\cdots)$, then $|\lambda|_X + |\mu|_X = |\nu|_X$ implies that $\sum_i i\,(x_i+y_i) = \sum_i i\,z_i$. So, when thought of as partitions, the numbers of boxes in the Young diagrams of x and y add up to that of z. It is clear from the proof of Proposition (4.1) (or alternatively from Equation (3.5)) that in this case, there exist integers $q_1,q_2,\cdots,q_{r-1}$ such that

$$\lambda^{(n)} + \mu^{(n)} - \nu^{(n)} = \sum_{i=1}^{r-1} q_i \alpha_{n-i+1}^{(n)} \quad \forall n \geq r \tag{4.26}$$

(the p_i and s in Proposition (4.1) are 0). Here $r = \max(\ell(x),\ell(y),\ell(z))$.

Let us now consider the Dynkin diagrams A_n. Clearly x,y and z define dominant weights of A_n by setting

$$x^{(n)} := \sum_{i=1}^{\ell(x)} x_i \overline{\omega}_i^{(n)}$$

where $\overline{\omega}_i^{(n)}$ denotes the fundamental weight of A_n corresponding to the node $n-i+1$. The definitions of $y^{(n)}$ and $z^{(n)}$ are similar. It is also clear that

$$x^{(n)} + y^{(n)} - z^{(n)} = \sum_{i=1}^{r-1} q_i \tilde{\alpha}_{n-i+1}^{(n)} \quad \forall n \geq r \tag{4.27}$$

where r and q_i are the same integers as above, and the $\tilde{\alpha}_i^{(n)}$ denote the simple roots of A_n (as opposed to X_n).

By Littelmann's tensor product decomposition formula, $c_{\lambda\mu}^{\nu}(n)$ is the number of $\mu^{(n)}$ dominant L-S paths $\eta = f_{i_k}^{(n)} f_{i_{k-1}}^{(n)} \cdots f_{i_1}^{(n)}(\pi_{\lambda^{(n)}})$ which satisfy $\sum_{j=1}^{k} \alpha_{i_j}^{(n)} = \sum_{i=1}^{r-1} q_i \alpha_{n-i+1}^{(n)}$. Similarly, if $\tilde{c}_{xy}^{z}(n) :=$ multiplicity of $L(z^{(n)})$ in $L(x^{(n)}) \otimes L(y^{(n)})$ as representations of $\mathfrak{g}(A_n)$, then $\tilde{c}_{xy}^{z}(n)$ is the number of L-S paths $\tilde{\eta}$ (in $\mathfrak{h}^*(A_n)$) of the form $\tilde{f}_{i_k}^{(n)} \tilde{f}_{i_{k-1}}^{(n)} \cdots \tilde{f}_{i_1}^{(n)}(\pi_{x^{(n)}})$ which are $y^{(n)}$ dominant and satisfy $\sum_{j=1}^{k} \tilde{\alpha}_{i_j}^{(n)} = \sum_{i=1}^{r-1} q_i \tilde{\alpha}_{n-i+1}^{(n)}$. The $\tilde{f}_i^{(n)}$ denote lowering operators of A_n.

Observe that the lowering operators involved in both cases correspond to the rightmost $r-1$ nodes. The above expressions for tensor product multiplicities for X_n and A_n clearly imply that $c_{\lambda\mu}^{\nu}(n) = \tilde{c}_{xy}^{z}(n)$. Taking n large enough, Theorem (4.5) implies that $c_{\lambda\mu}^{\nu}(n) = c_{\lambda\mu}^{\nu}(\infty)$. By the classical theory for A_n, we know that $\tilde{c}_{xy}^{z}(n)$ equals the Littlewood-Richardson coefficient $LR_{x,y}^{z}$ corresponding to x,y,z (considered as partitions). Thus $c_{\lambda\mu}^{\nu}(\infty) = LR_{x,y}^{z}$.

In summary, as long as all three dominant weights of X_n under consideration are supported on the "tail", their behavior is exactly like dominant weights of A_n.

2. Let us consider a different situation. Let λ, μ be single headed weights, supported on the tail as above, but now let ν be a single headed weight supported on the "head" i.e, near the X portion of the Dynkin diagram. So $\lambda = (\mathbf{0}, x)$, $\mu = (\mathbf{0}, y)$, $\nu = (z, \mathbf{0})$ for $x, y, z \in \mathcal{H}_1^+$.

   If $X_n = A_n$, then $c^{\nu}_{\lambda\mu}(\infty) = LR^{\nu}_{\lambda,\mu} = 0$, since the Littlewood-Richardson rule implies that any ν for which $LR^{\nu}_{\lambda,\mu} \neq 0$ has to be supported on the rightmost k nodes, where $k = \ell(x) + \ell(y)$. For large n, $\nu = (z, \mathbf{0})$ fails to meet this criterion. One can also obtain this fact from our point of view. Notice that $|\nu|_A = |(z, \mathbf{0})|_A = \sum_{i=1}^{\ell(z)} i\, z_i > 0$, while $|\lambda|_A + |\mu|_A = |(\mathbf{0}, x + y)|_A = -\sum_i i\,(x_i + y_i) < 0$. Hence $|\nu|_A \neq |\lambda|_A + |\mu|_A$. Proposition (3.13) implies $c^{\nu}_{\lambda\mu}(\infty) = 0$.

   However for $X_n = E_n$, $c^{\nu}_{\lambda\mu}(\infty)$ could be positive, as we saw in Example (2.7) for $x = y = z = \epsilon_1$. Observe that the contradiction obtained above for A_n in terms of the number of boxes function disappears for E_n. From the definition, it follows that $|\lambda|_E + |\mu|_E > 0$ while $|\nu|_E = \sum_i a_i z_i = 2z_1 + 4z_2 + \cdots + (10 - k)z_k$ which is positive for many choices of z_i.

   In summary, for A_n, if λ and μ are single headed and supported on the tail portion, any ν for which $c^{\nu}_{\lambda\mu}(\infty) > 0$ must also be single headed and supported on the tail portion. However for E_n, this is not the case. In fact, there can even exist a ν, supported on the "head" portion for which $c^{\nu}_{\lambda\mu}(\infty) > 0$. In a sense, information that is localized at one end (the tail) of the Dynkin diagram of E_n propagates to the other end.

5 The Stable Representation Ring

Having established that the multiplicities $c^{\nu}_{\lambda\mu}(n)$ stabilize, we shall now use the stable values $c^{\nu}_{\lambda\mu}(\infty)$ as structure constants to define a multiplication operation $*$ on a space Λ^X. We shall call Λ^X the *stable representation ring of type* X.

In type A, the associativity of $*$ will follow directly from the associativity of the tensor product. But for general type X, using $c^{\nu}_{\lambda\mu}(\infty)$ as structure constants means that we only keep the stable terms in the tensor product decomposition and discard the "transient" ones. Associativity of $*$ is no longer obvious. The goal of this section is to show that associativity still holds and that Λ^X becomes a genuine $\mathbb{C}$ - algebra.

We assume X is an extensible marked Dynkin diagram with d nodes. We shall consider the tensor product of three or more irreducible integrable highest weight representations and study its decomposition. First we will need a technical lemma concerning the large n behavior of the set of dominant weights

$P^+(X_n)$. We prove this so called Interval Stabilization lemma in Section (5.1) and then use it in Section (5.2) to look at stable multiplicities in k-fold tensor products . The stable representation ring will be defined in Section (5.3).

5.1 Interval Stabilization

First, some notation that will be needed to state our lemma: Let $\lambda_1 = (x, y)$, $\lambda_2 = (z, w) \in \mathcal{H}_2$ be such that $|\lambda_1|_X = |\lambda_2|_X$. Let $l = \max(d, \ell(x), \ell(z))$ and $r = \max(\ell(y), \ell(w))$. Proposition (4.1) implies that there exist integers p_i $(1 \leq i \leq l-1)$, q_j $(1 \leq j \leq r-1)$ and s such that for $n \geq l + r$

$$\lambda_1^{(n)} - \lambda_2^{(n)} = \sum_{i=1}^{l-1} p_i \alpha_i^{(n)} + \sum_{i=l}^{n-r+1} s\alpha_i^{(n)} + \sum_{i=n-r+2}^{n} q_{(n-i+1)} \alpha_i^{(n)} \tag{5.1}$$

We define a partial order $\geq$ on $\mathcal{H}_2$ by requiring that $\lambda_1 \geq \lambda_2$ iff $|\lambda_1|_X = |\lambda_2|_X$ and the p_i, q_j, s which occur in Equation (5.1) are all non-negative.

It is easy to check that $\geq$ is a partial order on $\mathcal{H}_2$ and that $\lambda_1 \geq \lambda_2$ implies that $\lambda_1 + \mu \geq \lambda_2 + \mu$. We also have these equivalent conditions which follow from the arguments of Section 3:

$$\begin{aligned} \lambda_1 \geq \lambda_2 \quad & \Leftrightarrow \lambda_1^{(n)} - \lambda_2^{(n)} \in Q^+(X_n) \;\forall \text{ large } n \\ & \Leftrightarrow \lambda_1^{(n)} - \lambda_2^{(n)} \in Q^+(X_n) \;\text{ for infinitely many values of } n \\ & \Leftrightarrow |\lambda_1|_X = |\lambda_2|_X \text{ and } \lambda_1^{(n)} - \lambda_2^{(n)} \in Q^+(X_n) \;\text{ for some value of } n \geq l + r \end{aligned}$$

Recall that the usual partial order $\geq$ on $\mathfrak{h}^*(X_n)$ is defined by $\beta \geq \beta'$ iff $\beta - \beta' \in Q^+(X_n)$ $(\beta, \beta' \in \mathfrak{h}^*(X_n))$. Hence for $\lambda_1, \lambda_2 \in \mathcal{H}_2$, $\lambda_1 \geq \lambda_2$ iff $\lambda_1^{(n)} \geq \lambda_2^{(n)}$ in $\mathfrak{h}^*(X_n)$ for all large n. We now state our main lemma.

Lemma 5.1 *(Interval Stabilization) Let $\lambda_1, \lambda_2 \in \mathcal{H}_2^+$ with $\lambda_1 \geq \lambda_2$. Let $I(\lambda_1, \lambda_2) := \{\gamma \in \mathcal{H}_2^+ : \lambda_1 \geq \gamma \geq \lambda_2\}$ and $I^{(n)}(\lambda_1, \lambda_2) := \{\beta \in P^+(X_n) : \lambda_1^{(n)} \geq \beta \geq \lambda_2^{(n)}\}$ for n larger than the lengths of λ_1 and λ_2. Then*

1. *$I(\lambda_1, \lambda_2)$ is a finite set*
2. *There exists N such that for all $n \geq N$, $I^{(n)}(\lambda_1, \lambda_2) = \{\gamma^{(n)} : \gamma \in I(\lambda_1, \lambda_2)\}$*

Proof: Let $\lambda_1 = (x, y)$, $\lambda_2 = (z, w)$. Let $l = \max(d, \ell(x), \ell(z))$ and $r = \max(\ell(y), \ell(w))$. We had by Equation (5.1)

$$\lambda_1^{(n)} - \lambda_2^{(n)} = \sum_{i=1}^{l-1} p_i \alpha_i^{(n)} + \sum_{i=l}^{n-r+1} s\alpha_i^{(n)} + \sum_{i=n-r+2}^{n} q_{(n-i+1)} \alpha_i^{(n)}$$

Since $\lambda_1 \geq \lambda_2$, the p_i, q_j and s are all non-negative. Set $N = l + r + 2s$. Fix $n \geq N$. Let $\overline{U}^{(n)} := \{l \leq i \leq n - r + 1\}$, $U^{(n)} := \{l < i < n - r + 1\}$, $\overline{V}^{(n)} := \{l + s \leq i \leq n - r + 1 - s\}$ and $V^{(n)} := \{l + s < i < n - r + 1 - s\}$.

Pick $\beta \in I^{(n)}(\lambda_1, \lambda_2)$ i.e, $\beta \in P^+(X_n)$ and $\lambda_1^{(n)} \geq \beta \geq \lambda_2^{(n)}$. Hence

$$0 \leq \beta - \lambda_2^{(n)} \leq \lambda_1^{(n)} - \lambda_2^{(n)} \tag{5.2}$$

If $\beta - \lambda_2^{(n)} = \sum_{i=1}^{n} b_i \alpha_i^{(n)}$, Equations (5.2) and (5.1) imply that

$$(i)\ \ 0 \leq b_i \leq s\ \forall i \in \overline{U}^{(n)}$$

Since $\beta \in P^+(X_n)$ and $\lambda_2^{(n)}(\check{\alpha}_i^{(n)}) = 0\ \forall i \in U^{(n)}$, we have $(\beta - \lambda_2^{(n)})(\check{\alpha}_i^{(n)}) \geq 0\ \forall i \in U^{(n)}$. This gives us the following additional condition on the $b_i's$

$$(ii)\ \ 2b_i - b_{i-1} - b_{i+1} \geq 0\ \forall i \in U^{(n)}$$

Claim: b_i is a constant on $\overline{V}^{(n)}$ i.e, $b_i = b_j\ \forall i, j \in \overline{V}^{(n)}$. *Proof:* Suppose not, then there exists i such that $i, i+1 \in \overline{V}^{(n)}$, but $b_i \neq b_{i+1}$. **Case 1:** Suppose $b_i > b_{i+1}$. Condition (ii) implies that $b_{i+2} \leq 2b_{i+1} - b_i < b_{i+1}$. Similarly we conclude $b_{i+3} < b_{i+2}$ etc. So we have a strictly descending sequence $b_i > b_{i+1} > b_{i+2} > \cdots > b_{n-r+1}$. The number of terms in this sequence is $(n - r - i + 2) \geq s + 2$ (since $i + 1 \in \overline{V}^{(n)}$ means that $i + 1 \leq n - r + 1 - s$) and by (i) we know that each term in the sequence lies between 0 and s . This is a clear contradiction. **Case 2:** Suppose $b_i < b_{i+1}$. We proceed as above to conclude that $b_l < b_{l+1} < \cdots < b_i < b_{i+1}$. The number of terms in this ascending sequence is $(i - l + 2) \geq s + 2$ (since $i \in \overline{V}^{(n)}$ implies $i \geq l + s$). Again a contradiction. □

We denote the constant value by k. Hence $k = b_i\ \forall i \in \overline{V}^{(n)}$.

Consequences: (1) $(\beta - \lambda_2^{(n)})(\check{\alpha}_i^{(n)}) = 0\ \forall i \in V^{(n)}$. This is clear since the left hand side is just $2b_i - b_{i-1} - b_{i+1}$ and $i, i-1, i+1 \in \overline{V}^{(n)}$.

Since $\lambda_2^{(n)}(\check{\alpha}_i^{(n)}) = 0\ \forall i \in V^{(n)}$, this also means that $\beta(\check{\alpha}_i^{(n)}) = 0\ \forall i \in V^{(n)}$. This implies that if $\gamma = (t, u) \in I(\lambda_1, \lambda_2)$, then $\max(d, \ell(t)) \leq l + s$ and $\ell(u) \leq r + s$, since $\gamma^{(m)} \in I^{(m)}(\lambda_1, \lambda_2)$ for all large m. Since $n \geq l + r + 2s$, we get a well defined, injective map $\phi_n : I(\lambda_1, \lambda_2) \to I^{(n)}(\lambda_1, \lambda_2)$ defined by $\phi_n(\gamma) := \gamma^{(n)}$. Since $I^{(n)}(\lambda_1, \lambda_2) = \{\beta \in P^+(X_n) : \lambda_1^{(n)} \geq \beta \geq \lambda_2^{(n)}\}$ is a finite set, $I(\lambda_1, \lambda_2)$ must be finite too. This proves statement (1) of the Lemma.

(2) Since $\beta(\check{\alpha}_i^{(n)}) = 0\ \forall i \in V^{(n)}$, we can write

$$\beta = \sum_{i=1}^{l+s} c_i \omega_i^{(n)} + \sum_{j=1}^{r+s} \tilde{c}_j \overline{\omega}_j^{(n)}$$

Define $c := (c_1, c_2, \cdots, c_{l+s}, 0, 0, \cdots)$, $\tilde{c} := (\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_{r+s}, 0, 0, \cdots)$ and set $\gamma := (c, \tilde{c}) \in \mathcal{H}_2^+$. Then $\beta = \gamma^{(n)}$. If we show that $|\gamma|_X = |\lambda_2|_X = |\lambda_1|_X$, then γ would be an element of $I(\lambda_1, \lambda_2)$ since $\gamma^{(n)} \in I^{(n)}(\lambda_1, \lambda_2)$. This would prove statement (2) of the Lemma as well.

We know that

$$\gamma^{(n)} - \lambda_2^{(n)} = \sum_{i=1}^{l+s-1} b_i \alpha_i^{(n)} + k \sum_{i=l+s}^{n-r-s+1} \alpha_i^{(n)} + \sum_{j=1}^{r+s-1} \tilde{b}_j \alpha_{n-j+1}^{(n)} \tag{5.3}$$

where for $1 \leq j \leq r+s-1$, $\tilde{b}_j := b_{n-j+1}$. Since γ is supported on the first $l+s$ and last $r+s$ nodes, it is clear from Equation (5.3) above that for all $m \geq n$, $\gamma^{(m)} - \lambda_2^{(m)}$ is given by

$$\gamma^{(m)} - \lambda_2^{(m)} = \sum_{i=1}^{l+s-1} b_i \alpha_i^{(m)} + k \sum_{i=l+s}^{m-r-s+1} \alpha_i^{(m)} + \sum_{j=1}^{r+s-1} \tilde{b}_j \alpha_{m-j+1}^{(m)}$$

obtained by "elongating" the string of k's in the middle. Hence $\gamma^{(m)} \equiv \lambda_2^{(m)}$ (mod $Q(X_m)$) for all $m \geq n$. By the arguments of Section 3, this implies that $|\gamma|_X = |\lambda_2|_X$. This finishes the proof of Lemma (5.1) $\square$

5.2 k-fold tensor products

To extend our main theorem (2.6), we now turn to tensor products of three or more irreducible representations. We ask if multiplicities in k-fold tensor products also stabilize. We shall first show that this remains true. Secondly, it is not obvious that one can understand stable multiplicities in k-fold tensor products by understanding stable multiplicities in successive binary tensor products. Happily it turns out that this can also be done.

Definition 5.2 Let $\lambda_1, \ldots, \lambda_k$ and $\nu \in \mathcal{H}_2^+$ be double-headed weights. Define $c^{\nu}_{\lambda_1\lambda_2\ldots\lambda_k}(n)$ to be the multiplicity of the representation $L(\nu^{(n)})$ in the k-fold tensor product $L(\lambda_1^{(n)}) \otimes \cdots \otimes L(\lambda_k^{(n)})$. If this is independent of n when n is large, let $c^{\nu}_{\lambda_1\lambda_2\ldots\lambda_k}(\infty)$ denote its stable value.

This generalizes the preceding use of $c^{\nu}_{\lambda\mu}(n)$.

Theorem 5.3 *If $|\lambda_1|_X + \cdots + |\lambda_k|_X = |\nu|_X$, then $c^{\nu}_{\lambda_1\lambda_2\ldots\lambda_k}(n)$ is indeed independent of n for n sufficiently large. Moreover, the stable value is related to the stable multiplicities in successive binary tensor products in the usual way:*

$$c^{\nu}_{\lambda_1\lambda_2\ldots\lambda_k}(\infty) = \sum_{\mu_1,\ldots,\mu_{k-2} \in \mathcal{H}_2^+} c^{\mu_1}_{\lambda_1\lambda_2}(\infty) c^{\mu_2}_{\mu_1\lambda_3}(\infty) \cdots c^{\mu_{k-2}}_{\mu_{k-3}\lambda_{k-1}}(\infty) c^{\nu}_{\mu_{k-2}\lambda_k}(\infty) \tag{5.4}$$

If n is finite, then equation (5.4) is clearly true, if we replace the ∞'s by n and let the sum range over all $\mu_i \in P^+(X_n)$. This holds since

$$L(\lambda_1^{(n)}) \otimes \cdots \otimes L(\lambda_k^{(n)}) = (\ldots((L(\lambda_1^{(n)}) \otimes L(\lambda_2^{(n)})\,) \otimes L(\lambda_3^{(n)})\,) \otimes \ldots \otimes L(\lambda_k^{(n)})\,)$$

We will use the Interval stabilization lemma (5.1) to show that when n is large enough, then the ranges of μ_i we must sum over also stabilize. This will prove both parts of Theorem (5.3).
Proof of Theorem (5.3)**:** The essence of the proof is the case $k = 3$. The general case follows by making modifications in the obvious places.

Consider $c^{\nu}_{\lambda_1\lambda_2\lambda_3}(n)$. Since

$$L(\lambda_1^{(n)}) \otimes L(\lambda_2^{(n)}) \otimes L(\lambda_3^{(n)}) \cong (L(\lambda_1^{(n)}) \otimes L(\lambda_2^{(n)})\,) \otimes L(\lambda_3^{(n)}) \tag{5.5}$$

we have

$$c^{\nu}_{\lambda_1\lambda_2\lambda_3}(n) = \sum_{\beta \in P^+(X_n)} c^{\beta}_{\lambda_1^{(n)},\,\lambda_2^{(n)}} \cdot c^{\nu^{(n)}}_{\beta,\,\lambda_3^{(n)}} \tag{5.6}$$

By a mild abuse of notation, we let $c^{\beta_3}_{\beta_1,\beta_2}$ denote the multiplicity of $L(\beta_3)$ in $L(\beta_1) \otimes L(\beta_2)$ (all representations of X_n), for $\beta_i \in P^+(X_n)$, $i = 1, 2, 3$. Now if

$$c^{\beta}_{\lambda_1^{(n)},\,\lambda_2^{(n)}} > 0 \text{ and } c^{\nu^{(n)}}_{\beta,\,\lambda_3^{(n)}} > 0 \tag{5.7}$$

we get $\lambda_1^{(n)} + \lambda_2^{(n)} \geq \beta$ and $\beta + \lambda_3^{(n)} \geq \nu^{(n)}$. Hence

$$\lambda_1^{(n)} + \lambda_2^{(n)} + \lambda_3^{(n)} \geq \beta + \lambda_3^{(n)} \geq \nu^{(n)}$$

We note that $\sum_{i=1}^3 \lambda_i^{(n)} \geq \nu^{(n)}$ together with $\sum_{i=1}^3 |\lambda_i|_X = |\nu|_X$ implies that $\sum_{i=1}^3 \lambda_i \geq \nu$ in the partial order on $\mathcal{H}_2^+$.

Let $\tilde{\beta} = \beta + \lambda_3^{(n)}$. We can now apply the Interval Stabilization Lemma (5.1). This gives us an integer N' such that for $n \geq N'$, $\tilde{\beta} = \gamma^{(n)}$ for some $\gamma \in I(\lambda_1 + \lambda_2 + \lambda_3, \nu)$. So $\beta = \gamma^{(n)} - \lambda_3^{(n)}$. Let

$$F_{\lambda_3} := \{\gamma - \lambda_3 : \gamma \in I(\lambda_1 + \lambda_2 + \lambda_3, \nu)\} \cap \mathcal{H}_2^+$$

The only possible solutions β to (5.7) are $\beta = \delta^{(n)}$, for $\delta \in F_{\lambda_3}$. Thus

$$c^{\nu}_{\lambda_1\lambda_2\lambda_3}(n) = \sum_{\delta \in F_{\lambda_3}} c^{\delta}_{\lambda_1\lambda_2}(n)\, c^{\nu}_{\delta\lambda_3}(n)$$

Since the number of terms in this sum is finite, we can pick $N \geq N'$ such that for all $n \geq N$ and all $\delta \in F_{\lambda_3}$, $c^{\delta}_{\lambda_1\lambda_2}(n) = c^{\delta}_{\lambda_1\lambda_2}(\infty)$ and $c^{\nu}_{\delta\lambda_3}(n) = c^{\nu}_{\delta\lambda_3}(\infty)$. Hence for all $n, m \geq N$, $c^{\nu}_{\lambda_1\lambda_2\lambda_3}(n) = c^{\nu}_{\lambda_1\lambda_2\lambda_3}(m)$. We've thus shown that the multiplicities of representations in the triple tensor product do stabilize. We've in fact also shown:

$$c^{\nu}_{\lambda_1\lambda_2\lambda_3}(\infty) = \sum_{\delta \in F_{\lambda_3}} c^{\delta}_{\lambda_1\lambda_2}(\infty)\, c^{\nu}_{\delta\lambda_3}(\infty) \;=\; \sum_{\gamma \in \mathcal{H}_2^+} c^{\gamma}_{\lambda_1\lambda_2}(\infty)\, c^{\nu}_{\gamma\lambda_3}(\infty) \tag{5.8}$$

For the last equality, observe by usual arguments that $c^{\gamma}_{\lambda_1\lambda_2}(\infty) > 0$ and $c^{\nu}_{\gamma\lambda_3}(\infty) > 0$ imply that $\lambda_1 + \lambda_2 \geq \gamma$ and $\gamma + \lambda_3 \geq \nu$ in the partial order on $\mathcal{H}_2$. Hence $\gamma + \lambda_3 \in I(\lambda_1 + \lambda_2 + \lambda_3, \nu)$. So $\gamma \in \{\delta - \lambda_3 : \delta \in I(\lambda_1 + \lambda_2 + \lambda_3, \nu)\} \cap \mathcal{H}_2^+ = F_{\lambda_3}$. $\square$

Remark 5.4 In the above proof, instead of (5.5) we could have started from the fact that $L(\lambda_1^{(n)}) \otimes L(\lambda_2^{(n)}) \otimes L(\lambda_3^{(n)}) \cong L(\lambda_1^{(n)}) \otimes (L(\lambda_2^{(n)}) \otimes L(\lambda_3^{(n)}))$. It is clear that we would have obtained the following equation analogous to equation (5.8):

$$c^{\nu}_{\lambda_1\lambda_2\lambda_3}(\infty) = \sum_{\delta \in F_{\lambda_1}} c^{\nu}_{\lambda_1\delta}(\infty)\, c^{\delta}_{\lambda_2\lambda_3}(\infty) = \sum_{\gamma \in \mathcal{H}_2^+} c^{\nu}_{\lambda_1\gamma}(\infty)\, c^{\gamma}_{\lambda_2\lambda_3}(\infty) \tag{5.9}$$

5.3 The stable representation ring Λ^X

Theorem (5.3) is key to our definition of Λ^X. First let $\mathcal{R}$ denote the $\mathbb{C}$ vector space with basis $\{v_\lambda : \lambda \in \mathcal{H}_2^+\}$ and $\widehat{\mathcal{R}}$ be its formal completion i.e, $\widehat{\mathcal{R}}$ is the set

$$\left\{ \sum_{\lambda \in \mathcal{H}_2^+} c_\lambda v_\lambda \,:\, c_\lambda \in \mathbb{C} \right\}$$

of all formal infinite series in the v_λ.

We define a multiplication operation on the basis elements v_λ.

$$v_\lambda * v_\mu := \sum_{\gamma \in \mathcal{H}_2^+} c_{\lambda\mu}^{\gamma}(\infty)\, v_\gamma$$

Equation (5.8) shows that $(v_\lambda * v_\mu) * v_\nu := \sum_{\pi \in \mathcal{H}_2^+} \left(\sum_{\gamma \in \mathcal{H}_2^+} c_{\lambda\mu}^{\gamma}(\infty)\, c_{\gamma\nu}^{\pi}(\infty) \right) v_\pi$ is equal to $\sum_\pi c_{\lambda\mu\nu}^{\pi}(\infty) v_\pi$ and hence well defined. Analogously, equation (5.9) guarantees that $v_\lambda * (v_\mu * v_\nu)$ is also well defined and equal to $\sum_\pi c_{\lambda\mu\nu}^{\pi}(\infty) v_\pi$. Thus:

$$(v_\lambda * v_\mu) * v_\nu = v_\lambda * (v_\mu * v_\nu) \tag{5.10}$$

Looking back on section (5.2), we see that this associativity is essentially a consequence of the associativity of the tensor product:

$$(L(\lambda) \otimes L(\mu)) \otimes L(\nu) \cong L(\lambda) \otimes (L(\mu) \otimes L(\nu))$$

Further, theorem (5.3) on k-fold tensor products shows that the product $v_{\lambda_1} * v_{\lambda_2} * \cdots * v_{\lambda_k}$ of finitely many v_{λ_i}'s is necessarily well defined, since it is equal to $\sum_{\nu \in \mathcal{H}_2^+} c_{\lambda_1 \lambda_2 \ldots \lambda_k}^{\nu}(\infty) v_\nu$. We then make the following definition:

Definition 5.5 Let Λ^X denote the subspace of $\widehat{\mathcal{R}}$ spanned by the set

$$\{v_{\lambda_1} * v_{\lambda_2} * \cdots * v_{\lambda_k} : k \geq 0, \lambda_i \in \mathcal{H}_2^+\}$$

consisting of all finite products of the v_λ's.

Clearly Λ^X is an associative, commutative $\mathbb{C}$ algebra with respect to the operations of adddition and $*$. We call Λ^X the *stable representation ring of type X*. One thinks of Λ^X as encoding information about how tensor products decompose as $n \to \infty$.

When X is of type A, Λ^A can be identified with the polynomial algebra $\mathbb{C}[x_1, y_1, x_2, y_2, \cdots]$ via the map that sends $x_i \mapsto v_{(\epsilon_i, 0)}$ and $y_i \mapsto v_{(0, \epsilon_i)}$. Here ϵ_i denotes the element $(0, 0, \cdots, 1, 0, \cdots) \in \mathcal{H}_1^+$ with the 1 in the i^{th} place. If we introduce $\mathbb{Z}$-gradations on these two algebras, by setting $\deg(x_i) = i = -\deg(y_i)$ and $\deg(v_\lambda) = |\lambda|_A$ for $\lambda \in \mathcal{H}_2^+$, then the above map defines an isomorphism of *graded* algebras.

Equivalently, one can view Λ^A as the tensor product of two copies of the ring of symmetric functions by identifying x_i and y_i with the i^{th} elementary symmetric polynomials in the variables z_i and w_i respectively. Here the gradation would be $\deg(z_i) = 1 = -\deg(w_i)$. In this picture, the subalgebra of Λ^A generated by the elements $\{v_{(\lambda,0)} : \lambda \in \mathcal{H}_1^+\}$ is isomorphic to the algebra of symmetric functions. Our map above sends $v_{(\lambda,0)}$ to the Schur function $s_\lambda(z_1, z_2, \cdots)$.

For general X, a better understanding of the structure of Λ^X might shed more light on the representation theory of the X_n. We conclude by mentioning an important open problem: How far does the ring Λ^X characterize the series X_n ? Can there exist an isomorphism $\Lambda^X \cong \Lambda^Y$ for two different "types" X and Y ?

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