Efficient total domination in digraphs

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Abstract

We generalize the concept of efficient total domination from graphs to digraphs. An efficiently total dominating set \( X \) of a digraph \( D \) is a vertex subset such that every vertex of \( D \) has exactly one predecessor in \( X \). We study graphs that permit an orientation having such a set and give complexity results and characterizations concerning this question. Furthermore, we study the computational complexity of the (weighted) efficient total domination problem for several digraph classes. In particular we deal with most of the common generalizations of tournaments, like locally semicomplete and arc-locally semicomplete digraphs.

Keywords: total domination, efficient total domination, digraphs, domination in digraphs

1. Introduction

A directed graph (or digraph) is a pair \( D = (V, A) \) where \( V \) is a finite set and \( A \subseteq V \times V \) is an irreflexive binary relation. The elements of \( V \) are called the vertices and the elements of \( A \) are the arcs of \( D \). Since digraphs with symmetric arc set \( A \) can be considered as undirected graphs, digraphs are a natural generalization of them. There is a lot of mathematical theory on digraphs. A good introduction into the field is given by Bang-Jensen and Gutin in their book on digraphs [1].

A dominating set of a digraph \( D \) is a vertex subset \( X \) such that any vertex outside of \( X \) has a predecessor in \( X \). Dominating sets in digraphs are discussed in the book by Haynes, Hedetniemi and Slater [3]. A more recent paper, gathering and detailing some results on domination in tournaments, is the paper by Hedetniemi, Hedetniemi, McRae and Reid [4]. However, there is not much theory on domination in digraphs yet and this field is much less studied than domination in undirected graphs. One of the possible reasons may be the following: Even for tournaments, which may be considered as one of the most famous digraph classes, it is not clear if there is an algorithm which efficiently computes the minimal size of a dominating set, the domination number. According to

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our knowledge, the best exact algorithm is, essentially, brute force and runs in sub-exponential time. This fact and some more are surveyed in [4].

Informally speaking, a tournament has quite a lot of structure, but still not enough for the domination problem. There are a lot of open problems in algorithmic domination theory, and these problems are very difficult. Again for some very restricted digraph classes, like De Bruijn and Kautz digraphs, some domination parameters can be explicitly computed (see for example [5, 9, 7, 8, 6]).

Efficient total domination in graphs is a studied topic (see [10, 11]). In this paper, we introduce efficient total domination for digraphs, a natural generalization of efficient total domination in graphs. In fact, if one restricts the attention to digraphs with symmetric arc set, one obtains the efficient total domination problem for graphs. A total dominating set of a digraph \( D \) is a vertex subset \( X \) such that any vertex of \( D \) has a predecessor in \( X \). An efficient total dominating set of a digraph \( D \) is a set \( X \) such that every vertex of \( D \) has exactly one predecessor in \( X \). Any efficient total dominating set is a total dominating set in particular, but the converse is not true.

A reformulation of efficient total domination is the following: Let \( v_1, v_2, \ldots, v_n \) be an ordering of the vertices of \( D \) and let \( A \) be the 01-adjacency matrix of \( D \) with respect to this ordering. That is, \( A_{ij} = 1 \) if there is an arc from \( v_i \) to \( v_j \) and \( A_{ij} = 0 \) otherwise. An efficient total dominating set corresponds to a 01-vector \( x \) for which \( A^t x = 1 \), where \( 1 \) denotes again the vector containing only ones. Hence, an efficient total dominating set corresponds to an exact cover of \( A^t \) and vice versa.

Like for undirected graphs we think that efficient total domination in digraphs is a topic worth studying. According to our knowledge, there is not much theory on efficient domination in digraphs (besides [12]) and efficient total domination in digraphs has not been considered in the literature at all.

1.1. Our contribution

We contribute an in-depth study of the problem, offering an analysis of the relation of efficiently total dominatable digraphs and their underlying graphs. We prove that it is \( NP \)-hard to decide whether a given graph has a (bi-)orientation which admits an efficient total dominating set and characterized the graphs for which any isolate-free induced subgraph has such a biorientation. Moreover, we give a necessary and sufficient condition for an isolate-free graph to have an efficiently total dominatable biorientation in any of its isolate-free induced subgraphs, using results from the theory of structural total domination [15].

We then turn our attention to the computational complexity of the efficient total domination problem on several generalizations of tournaments introduced below. In most of the cases, we can either prove \( NP \)-completeness or give an efficient algorithm to find even a minimum weighted efficient total dominating set. We show that WETD is efficiently solvable on biorientations of \( \{K_{1,p}, qK_2\} \)-free graphs for fixed \( p \) and \( q \), locally out-semicomplete digraphs, quasi-transitive digraphs, \( k \)-partite tournaments, arc-locally out-semicomplete digraphs.
orgraphs and strongly connected arc-locally in-semicomplete orgraphs. We prove \( NP \)-hardness on strongly connected arc-locally semicomplete digraphs, arc-locally in-semicomplete orgraphs, strongly connected \( k \)-partite semicomplete digraphs, orientations of split graphs, strongly connected biorientations of threshold graphs and path-mergeable orientations of planar bipartite graphs of maximum degree 4.

2. Preliminaries

2.1. Digraphs and digraph classes

For standard notations we do not introduce here, the reader is always referred to the introductory chapter of [1].

If \( D \) is a digraph with no specified vertex or arc set, \( V(D) \) denotes its vertices and \( A(D) \) denotes its arcs. Let \( D = (V, A) \) be a digraph. For any vertex \( v \) of \( D \) its out-neighborhood, denoted by \( N^+_D(v) \), is defined as the set of vertices \( u \) with \( (v, u) \in A \). Such vertex \( u \) is then called an out-neighbor of \( v \). The in-neighborhood of \( v \), denoted by \( N^-_D(v) \), is defined as the set of vertices \( u \) with \( v \in N^+(u) \). Such vertex \( u \) is then called an in-neighbor of \( v \). The out-degree \( d^+_D(v) \) of \( D \) is the function with \( d^+_D(v) = |N^+_D(v)| \) for any \( v \in V \). The maximum out-degree \( \Delta^+(D) \) is defined as \( \Delta^+(D) = \max_{v \in V} d^+(v) \). If there is a \( k \) such that \( d^+_D \equiv k \), \( D \) is said to be \( k \)-out-regular or just out-regular. The notions in-degree \( d^-_D \), maximum in-degree \( \Delta^-(D) \) and \( (k-) \)-in-regularity are defined analogously.

If \( D \) is clear from the context, we sometimes omit it from our notation, e.g. we may write \( N^+(v) \) instead of \( N^+_D(v) \).

An efficient total dominating set of \( D \) is a set \( X \subseteq V \) such that for any \( v \in V \) there is exactly one vertex \( x \in X \) with \( (x, v) \in A \). That is, \( |X \cap N^-(v)| = 1 \) for any \( v \in V \). If \( D \) has an efficient total dominating set, \( D \) is called an efficiently total dominatable digraph. Note that not every digraph has an efficient total dominating set, e.g. acyclic digraphs. In fact, the corresponding decision problem is \( NP \)-hard, even if restricted to some very special digraph classes (as shown below). We denote the decision problem associated to the existence of efficient total dominating sets by \( ETD \). If the vertices have a real-valued weight, we can consider minimum weight efficient total dominating sets. The related minimization problem is denoted by \( WETD \). A solution of the \( WETD \) problem is either a minimum weight efficient total dominating set or the information that the input digraph is not efficiently total dominatable.

All of the following digraph properties and digraph classes are discussed in detail in [1].

Let \( D = (V, A) \) be a digraph. The notions of a subdigraph and an induced subdigraph are defined in analogy to the undirected case. In this sense, if \( U \) is a vertex subset, \( D[U] \) denotes the induced subdigraph on \( U \). Two arcs \( (u, v) \) and \( (v, u) \) are called anti-parallel. If \( D \) does not have anti-parallel arcs, it is called an oriented graph (orgraph for short). If \( (u, v) \in A \) or \( (v, u) \in A \), \( u \) and \( v \) are said to be adjacent. Thus adjacency is an irreflexive and symmetric binary relation. The underlying graph of \( D \) is the graph \( G \) with vertex set \( V \) defined by...
this adjacency relation. Hence, $G$ is obtained from $D$ by loosing the direction of the arcs and then identifying parallel edges. $D$ is then called a biorientation of $G$. If furthermore $D$ is an orgraph, $D$ is called an orientation of $G$. $D$ is said to be connected if $G$ is connected. The connected components of a digraph are the inclusionwise maximal subdigraphs that are connected graphs.

A directed path or just path in a digraph is a sequence $(v_1, v_2, \ldots, v_k)$ of mutually distinct vertices such that $(v_i, v_{i+1}) \in A$ for any $1 \leq i \leq k-1$. The length of the path $(v_1, v_2, \ldots, v_k)$ is $k-1$, i.e. the number of vertices decreased by one. A directed cycle or cycle is a path $(v_1, v_2, \ldots, v_k)$ where $(v_k, v_1) \in A$. The length of the cycle $(v_1, v_2, \ldots, v_k)$ is $k$. An induced cycle is a cycle $P = (v_1, v_2, \ldots, v_k)$ where no arc can be removed without destroying the property of being a cycle. A digraph without cycles is called acyclic. $D$ is called strongly connected if for any two vertices $u$ and $v$ there is a directed path from $u$ to $v$ and a directed path from $v$ to $u$. In particular, any strongly connected digraph is also connected. As the digraph consisting of a single arc shows, the opposite does not hold in general. A strongly connected component of a digraph is a maximal strongly connected subdigraph.

If $D$ is the biorientation of a complete graph, it is called semicomplete. If $D$ is furthermore an orgraph, it is called a tournament. $D$ is called locally out-semicomplete (locally in-semicomplete) if $D[N^+(v)] (D[N^-(v)])$ is semicomplete for all $v \in V$. If $D$ is both locally out-semicomplete and locally in-semicomplete, $D$ is simply called locally semicomplete. $D$ is called $k$-partite semicomplete if it is the biorientation of a complete $k$-partite graph. $D$ is called a $k$-partite tournament if it is the orientation of a complete $k$-partite graph. $D$ is called arc-locally out-semicomplete (arc-locally in-semicomplete) if for every arc $(u, v) \in A$ it holds that every out-neighbor (in-neighbor) of $u$ is identical or adjacent to every out-neighbor (in-neighbor) of $v$. If $D$ is both arc-locally out-semicomplete and arc-locally in-semicomplete, $D$ is simply called arc-locally semicomplete.

$D$ is called transitive if for all three distinct vertices $u$, $v$ and $w$ with $(u, v), (v, w) \in A$ it holds that $(u, w) \in A$. $D$ is called quasi-transitive if for all three distinct vertices $u$, $v$ and $w$ with $(u, v), (v, w) \in A$ it holds that $u$ is adjacent to $w$. Of course, any transitive digraph is quasi-transitive. As the directed cycle of length 3 shows, the opposite does not hold in general. A digraph is called path-mergeable if for any two vertices $u$ and $v$ the following holds: For any two directed paths $P$ and $P'$ from $u$ to $v$ that do not have common vertices (except $u$ and $v$), there is a directed path $P''$ from $u$ to $v$ with $V(P'') = V(P) \cup V(P')$.

2.2. Graphs and hypergraphs

Let $G$ be a graph. A total dominating set $X$ is a vertex subset such that any vertex of $G$ is adjacent to a member of $X$. Hence, $G[X]$ has minimum degree at least 1. A pendant vertex of $G$ is a vertex with exactly one neighbor. The corona of $G$, denoted by $Cr(G)$, is obtained from $G$ by simultaneously attaching a pendant vertex to any vertex of $G$. A graph is planar if it can be drawn into the plane without crossing edges. A threshold graph is a graph that can be constructed from the empty graph by repeatedly adding either an
isolated vertex or a dominating vertex. A graph is a **split graph** if its vertices admit a partition into a clique and a stable set. Detailed information on these graph classes are given in the survey by Brandstädt, Le and Spinrad in [13].

A **hypergraph** \( H = (V,E) \) is an ordered pair where \( E \) is a nonempty finite family of nonempty finite sets and \( V = \bigcup E \). The elements of \( V \) are called **vertices** and the elements of \( E \) **hyperedges**. The **bipartite incidence graph** of a hypergraph \( H = (V,E) \) is the bipartite graph \((V \cup E, \{\{v,e\} : v \in e \in E\})\). A **cover** of \( H \) is a set \( C \subseteq E \) such that \( \bigcup C = V \). A **matching** of a hypergraph \( H = (V,E) \) is a set \( M \subseteq E \) such that \( m \cap n = \emptyset \) for all \( m \neq n \in M \). A matching \( M \) which is also a cover is a **perfect matching**. Not all hypergraphs have a perfect matching; in fact it is \( NP \)-complete to decide if a given hypergraph has a perfect matching (see **exact cover** in Gary and Johnson [14]).

If we prove \( NP \)-completeness of ETD for certain digraph classes, we always give a polynomial reduction from the perfect matching problem for hypergraphs. In our figures, we always choose the same hypergraph to be the instance of the perfect matching problem. It is defined by

\[
V = \{v_1, v_2, v_3\}, \\
E = \{e_1 = \{v_1, v_2\}, e_2 = \{v_1\}, e_3 = \{v_2, v_3\}\}.
\]

The bipartite incidence graph of this hypergraph is displayed in Figure 1.

![Figure 1: The bipartite incidence graph of the hypergraph used in our figures.](image)

### 3. Underlying graphs

#### 3.1. Efficiently total dominatable orientations

Given a graph, it is a natural question to ask if it can be oriented or bioriented in a way that the resulting digraph is efficiently total dominatable. Not every graph has an efficiently total dominatable orientation or biorientation, e.g. the graph displayed in Figure 2.

Let \( D \) be an efficiently total dominatable digraph with efficient total dominating set \( X \). The efficient total domination condition says that every vertex of \( D \) has exactly one in-neighbor among the set \( X \). Hence, \( D[X] \) is 1-in-regular (a so-called **contrafunctional** digraph). The connected components of contrafunctional digraphs have the following structure: Any connected component has
Figure 2: A graph not having an efficiently total dominatable biorientation.

exactly one directed cycle and this cycle is induced. If a single arc of this cycle is removed, the resulting digraph is the orientation of a tree which has exactly one vertex of in-degree 0. An example of a contrafunctional digraph is displayed in Figure 3.

Figure 3: A connected contrafunctional digraph.

This leads us to our first lemma.

**Lemma 1.** Let $G$ be a graph.

1. $G$ has an efficiently total dominatable biorientation if and only if it has a total dominating set $X$ such that the connected components of $G[X]$ have at most one cycle each.

2. $G$ has an efficiently total dominatable orientation if and only if it has a total dominating set $X$ such that the connected components of $G[X]$ have exactly one cycle each.

**Proof.** Let $G$ be a graph.

Assume $G$ has an efficiently total dominatable biorientation $D$. Let $X$ be an efficient total dominating set of $D$. Since $D[X]$ is contrafunctional, $G[X]$ is the disjoint union of graphs having at most one cycle and at least two vertices each. Furthermore, $X$ is a total dominating set of $G$. In the case of $D$ being an orientation of $G$, $G[X]$ is the disjoint union of graphs having exactly one cycle and at least two vertices each.

To complete the proof, we have to show the following: If a graph $G$ has a total dominating set $X$ such that any connected component of $G[X]$ has at most one cycle, it has an efficiently total dominatable biorientation. Furthermore,
Figure 4: A graph with a total dominating subgraph with exactly one cycle, and an efficiently total dominatable orientation.

if a graph $G$ has a total dominating set $X$ such that any connected component of $G[X]$ has exactly one cycle, it also has an efficiently total dominatable orientation.

Given such a total dominating set one constructs an efficiently total dominatable (bi-)orientation of $G$ as follows: The edges between the vertices of $V(G) \setminus X$ we direct in an arbitrary way. The edges between the vertices contained in the total dominating set $X$ can be (bi-)oriented such that the resulting (bi-)orientation of $G[X]$ is contrafunctional. For each vertex $v \in V(G) \setminus X$ there is at least one edge joining $v$ to a member of $X$, since $X$ is a total dominating set. We direct exactly one of these edges from $X$ to $v$ and the other ones from $v$ to $X$. Now, $|N^-(v) \cap X| = 1$ for each $v \in V(G)$.

A graph having a total dominating subgraph with exactly one cycle, and an efficiently total dominatable orientation of it is displayed in Figure 3.1.

The question arises, whether the conditions of Lemma 1 can be recognized efficiently. However, we have the following negative result:

**Theorem 1.** The following decision problems are $NP$-hard: Given a graph $G$, does $G$ admit an efficiently total dominatable orientation? Does $G$ admit an efficiently total dominatable biorientation?

**Proof.** Let $H = (V, E)$ be a hypergraph. To prove $NP$-hardness, we define a graph $G$ by

- $V(G) = \{a, b, c, a', b', c'\} \cup V \cup E$,
- $A(G) = \{\{a, a'\}, \{b, b'\}, \{c, c'\}, \{a, b\}, \{a, c\}, \{b, c\}\} \cup \{\{a, e\} : e \in E\}
  \cup \{\{e, f\} : e, f \in E, e \cap f \neq \emptyset\} \cup \{\{e, v\} : v \in e \in E\}$,

where $\{a, b, c, a', b', c'\}$ is assumed to be disjoint to $V \cup E$. The constructed graph $G$ is displayed schematically in Figure 5.

It is easy to see that for every total dominating set $X$ of $G$, $G[X]$ is connected and $\{a, b, c\} \subseteq X$. Hence, $G[X]$ has at least one cycle. Thus by Lemma 1, $G$ has an efficiently total dominatable orientation iff it has an efficiently total dominatable biorientation. We claim that there is a total dominating set $Y$ of $G$ such that $G[Y]$ has exactly one cycle iff $H$ has a perfect matching. First we assume that there is a total dominating set $X$ of $G$ such that any connected
component of $G[X]$ has exactly one cycle. Since $G[X]$ is connected, $G[X]$ has exactly one cycle. Since $\{a, b, c\} \subseteq X$ induces a cycle, $Y = X \cap E$ is a stable set. That is, $Y$ is a matching of $H$. Since $X$ is a total dominating set and $V$ is stable, $Y$ is also a cover of $H$. Hence, $Y$ is a perfect matching of $H$. On the other hand, if $M \subseteq E$ is a perfect matching of $H$, $M$ is a stable set in $G$ and hence $X = \{a, b, c\} \cup M$ is a connected total dominating set such that $G[X]$ has exactly one cycle.

Lemma 1 and the fact that the perfect matching decision problem is $NP$-hard complete the proof.

In contrast, any graph admits an orientation which has an efficient dominating set, as observed by Bange, Barkauskas, Host and Clark [12]. (An efficient dominating set of a digraph is a set of mutually non-adjacent vertices such that any vertex outside the set has exactly one predecessor in the set.)

As the proof of Theorem 1 shows, the problem remains $NP$-hard if one asks for efficiently total dominatable (bi-)orientations with connected efficient total dominating sets. On the other hand, the theory of the structure of total dominating subgraphs (developed in [15]) allows the following characterization. As Theorem 2 of [15] shows, the following holds for a graph $G$: Any isolate-free induced subgraph of $G$ has a total dominating set $X$ such that the connected components of $G[X]$ have at most one cycle each if and only if $G$ does not contain the corona of a graph with two cycles as induced subgraph. Together with Lemma 1 this gives the following:

**Theorem 2.** Let $G$ be an isolate-free graph. $G$ and any of its isolate-free induced subgraphs have an efficiently total dominatable bi-orientation if and only if $G$ does not contain the corona of a graph with two cycles as induced subgraph.

**3.2 Underlying graphs**

A sharp non-trivial bound on the size of an efficient total dominating set is given by the stability number of the underlying graph. This number, denoted by $\alpha$, equals the size of a maximum stable set of the graph.

**Theorem 3.** For each efficiently total dominatable digraph $D$ with underlying graph $G$ any efficient total dominating set has size at most $3\alpha(G)$. This bound is sharp for efficiently total dominatable tournaments.
Proof. Let $D$ be an efficiently total dominatable digraph with underlying graph $G$ and let $X$ be an efficient total dominating set of $D$. Since each connected component of $G[X]$ contains at most one cycle, it is 3-partite. Hence, $3\alpha(G) \geq 3\alpha(G[X]) \geq |X|$. The bound is sharp, since for each $n$, each efficient total dominating set of an efficiently total dominatable tournament has size $3 = 3\alpha(K_n)$. □

The following results are obtained by a straightforward structural analysis leading to digraph classes on which WETD can be solved by a complete enumeration. For fixed $p$ and $q$ a $\{K_{1,p},qK_2\}$-free graph is a graph that does not contain the complete bipartite graph $K_{1,q}$ or $q$ disjoint copies of $K_2$ as induced subgraph.

**Lemma 2.** For fixed $p$ and $q$, the maximal size of an efficient total dominating set of an efficiently total dominatable biorientation of a $\{K_{1,p},qK_2\}$-free graph is bounded by a constant.

**Proof.** Let $G$ be a $\{K_{1,p},qK_2\}$-free graph and let $D$ be an efficiently total dominatable biorientation of $G$. Let $X$ be an efficient total dominating set of $D$. As described above, $D[X]$ is a contrafunctional digraph. Hence, $G[X]$ does not have isolated vertices and is the disjoint union of graphs having at most one cycle and at least two vertices each. Since $G$ is $K_{1,p}$-free, the maximum degree of $G[X]$ is $p$. Since $G$ is $qK_2$-free, each connected component of $G[X]$ contains at most $p(p-1)^{q-2}$ vertices. By $qK_2$-freeness again, $G[X]$ contains at most $q$ connected components. All in all $|X| \leq q(p-1)^{q-2}$. □

This gives the following.

**Theorem 4.** For fixed $p$ and $q$, WETD is efficiently solvable on the class of biorientations of $\{K_{1,p},qK_2\}$-free graphs.

We now prove $NP$-completeness of ETD on (bi-)orientations of certain graph classes.

**Theorem 5.** ETD is $NP$-complete on the following digraph classes:

1. orientations of split graphs,
2. path-mergeable orientations of planar bipartite graphs of maximum degree 4,
3. strongly connected biorientations of threshold graphs,
4. strongly connected biorientations of complete $k$-partite graphs for all fixed $k \geq 2$.

**Proof.** Let $H = (V,E)$ be a hypergraph on the vertices $V = \{v_1, v_2, \ldots, v_n\}$.

To see Claim 1, we define an orgraph $D$ by

$$V(D) = V \cup E \cup \{a, b, c, d\},$$

$$A(D) = \{(a, b), (b, c), (c, a), (a, d)\} \cup \{(a, e) : e \in E\}$$

$$\cup \{(c, v) : v \in E\} \cup \{(v_i, v_j) : 1 \leq j < i \leq n\}$$

$$\cup \{(v, a), (v, c), (v, d) : v \in V\}.$$
The underlying graph of $D$ is a split graph: $D|\{b, d\} \cup E|$ is arc-less and $D|\{a, c\} \cup V|$ is semicomplete. $D$ is displayed schematically in Figure 6.

If $X \subseteq E$ is a perfect matching of $H$, then $X \cup \{a, b, c\}$ is an efficient total dominating set of $D$. On the other hand, let $X$ be an efficient total dominating set of $D$. $N^-(b) = \{a\}$ gives $a \in X$. Since $N^-(d) = \{a\} \cup V$, $X \cap V = \emptyset$. Hence, $N^-(v) \cap E \subseteq X$ for all $v \in V$ and so $X \cap E$ is a perfect matching of $H$.

Therefore, $H$ has a perfect matching if and only if $D$ is efficiently total dominatable and this completes the proof of Claim 1.

As is shown in [16], the decision problem of the existence of a perfect matching is $NP$-complete if restricted to the class of hypergraphs for which the bipartite incidence graph is planar and has maximum degree 3. Thus we can assume that the bipartite incidence graph of $H$ is planar and has maximum degree 3.

We define a path-mergeable orgraph $D$ by

$$V(D) = V \cup E \cup \{a_e, b_e, c_e, d_e : e \in E\},$$

$$A(D) = \{(a_e, b_e), (b_e, c_e), (c_e, d_e), (d_e, a_e) : e \in E\} \cup \{(a_e, e) : e \in E\}$$

and observe that the underlying graph of $D$ is a planar bipartite graph of maximum degree four.

If $X \subseteq E$ is a perfect matching of $H$, then $X \cup \{a_e, b_e, c_e, d_e : e \in E\}$ is an efficient total dominating set of $D$. On the other hand, let $X$ be an efficient total dominating set of $D$. Since $N^-(v) = \{e : v \in V\}$ for all $v \in V$, it follows that $X \cap E$ is a perfect matching of $H$.

Therefore, $H$ has a perfect matching if and only if $D$ is efficiently total dominatable and this completes the proof of Claim 2.

To see Claim 3, we define a digraph $D$ by

$$V(D) = V \cup E \cup \{a, b\},$$

$$A(D) = \{(a, b), (b, a)\} \cup \{(a, e) : e \in E\}$$

$$\cup \{(e, v) : v \in e \in E\} \cup \{(v, e) : v \notin e \in E\}$$

$$\cup \{(v_i, v_j) : 1 \leq j < i \leq n\} \cup \{(v, a), (v, b) : v \in V\}$$

We observe that the underlying graph of $D$ is a threshold graph. It is constructed by iteratively adding $\{b\} \cup E$ as isolated vertices and then $\{a\} \cup V$ as dominating vertices. We can assume that every vertex of $V$ is contained in at least one hyperedge and there is no empty hyperedge. Thus, $D$ is strongly connected. $D$ is displayed schematically in Figure 6.

If $X \subseteq E$ is a perfect matching of $H$, then $X \cup \{a, b\}$ is an efficient total dominating set of $D$. On the other hand, let $X$ be an efficient total dominating set of $D$. $N^-(b) = \{a\} \cup V$ gives $|X \cap (\{a\} \cup V)| = 1$. Let $x \in X \cap (\{a\} \cup V)$ Clearly any vertex of $E$ is dominated by $x$. Since any vertex of $H$ is contained in a hyperedge, $x \notin V$. Hence, $a \in X$ and $X \cap V = \emptyset$. Thus, $N^-(v) \cap E \subseteq X$ for all $v \in V$ and so $X \cap E$ is a perfect matching of $H$.

Therefore, $H$ has a perfect matching if and only if $D$ is efficiently total dominatable and this completes the proof of Claim 3.
To see Claim 4, let $k \geq 2$ be arbitrary. We define a digraph $D$ by

$$V(D) = V \cup E \cup \{a, b\} \cup \{u_i : 1 \leq i \leq k - 2\},$$

$$A(D) = \{(a, b), (b, a)\} \cup \{(a, e) : e \in E\} \cup \{(v, b) : v \in V\}$$

$$\cup \{(e, v) : v \in E\} \cup \{(v, e) : v \notin e \in E\}$$

$$\cup \{(a, u_i) : 1 \leq i \leq k - 2\} \cup \{(v, u_i) : v \in V, 1 \leq i \leq k - 2\}$$

$$\cup \{(u_i, b) : 1 \leq i \leq k - 2\} \cup \{(u_i, e) : e \in E, 1 \leq i \leq k - 2\}$$

$$\cup \{(u_i, u_j) : 1 \leq j < i \leq n\}$$

and observe that the underlying graph of $D$ is a complete $k$-partite graph. Thereby, the $k$ partitions are $\{b\} \cup E$, $\{a\} \cup V$ and $\{u_1\}, \ldots, \{u_{k-2}\}$. Furthermore, $D$ is easily seen to be strongly connected.

If $X \subseteq E$ is a perfect matching of $H$, then $X \cup \{a, b\}$ is an efficient total dominating set of $D$. On the other hand, let $X$ be an efficient total dominating set of $D$. $N^{-}(a) = \{b\}$ gives $b \in X$. Since $N^{-}(b) = \{a\} \cup V \cup \{u_i : 1 \leq i \leq k - 2\}$, $X \cap \{(a) \cup V \cup \{u_i : 1 \leq i \leq k - 2\}\}$ contains exactly one vertex. Let $x$ be that vertex. Since $D[X]$ is a contrafunctional digraph, it is not acyclic. If $x \neq a$, any cycle of $D[X]$ necessarily contains at least three vertices, in contradiction to $|X \cap N^{-}(b)| = 1$. Hence, $x = a$. Thus, $N^{-}(v) \cap E \subseteq X$ for all $v \in V$ and so $X \cap E$ is a perfect matching of $H$.

Therefore, $H$ has a perfect matching if and only if $D$ is efficiently total dominatable and this completes the proof of Claim 4.

4. Generalized tournaments

This section deals with the algorithmic complexity of ETD and WETD on digraph classes generalizing tournaments. All of these digraphs are rich in structure and thus some allow simple combinatorial algorithms even for WETD.
Some of our proofs make use of the following observation:

**Lemma 3.** When \( m \) is the number of arcs, \( \Delta^- \) the maximum in-degree and \( \Delta^+ \) is the maximum out-degree of the graph considered, the following holds:

1. A minimum weighted efficient total dominating set that induces a cycle of length 2 or 3 can be found in \( \mathcal{O}(m\Delta^+ \max\{\Delta^- , \Delta^+\}) \) time.
2. A minimum weighted efficient total dominating set that induces a cycle of length 2, 3 or 4 can be found in \( \mathcal{O}(m\Delta^- \Delta^+ + \max\{\Delta^- , \Delta^+\}) \) time.

**Proof.** All cycles of length 2 can clearly be found in \( \mathcal{O}(m\Delta^+) \) time. Now, the efficient total domination property can be checked in \( 2\Delta^+ \) steps for each such cycle.

The cycles of length three can be found in the following way. For each arc \( a = (u, v) \), it can be checked if there is a vertex \( w \in N^-(u) \cap N^+(v) \) in \( \mathcal{O}(\max\{\Delta^- , \Delta^+\}) \) time. Again, the efficient total domination property can be checked in \( 3\Delta^+ \) steps for each such cycle.

The cycles of length four can be found in the following way. For each arc \( a = (u, v) \), and each two \( t \in N^-(u) \) and \( w \in N^+(v) \), it is checked in \( \Delta^+ \) steps whether \( (w, t) \in A(D) \). Furthermore, the efficient total domination property of a set of size 4 is checked in \( 4\Delta^+ \) steps. This completes the proof.

#### 4.1. Quasi-transitive digraphs and \( k \)-partite tournaments

In this section we study WETD for quasi-transitive digraphs and \( k \)-partite tournaments. It turns out that the efficient total dominating sets in these digraphs have a very special structure. This yields that efficient total dominating sets can be efficiently enumerated.

**Lemma 4.** If \( D \) is a connected efficiently total dominatable quasi-transitive digraph, each efficient total dominating set of \( D \) induces a cycle of length 2 or 3.

**Proof.** We observe that the only connected contrafunctional quasi-transitive digraphs are cycles of length 2 or 3.

Let \( D = (V, A) \) be a connected quasi-transitive digraph and \( X \) be an efficient total dominating set of \( D \). Thus, \( D[X] \) is the disjoint union of cycles of length 2 or 3. Assume for contradiction that \( D[X] \) is not a single cycle. Furthermore, assume there are two cycles in \( D[X] \), say \( C_1 \) and \( C_2 \), and a vertex \( v \) with \( N^-(v) \cap V(C_1) \neq \emptyset \) and \( N^+(v) \cap V(C_2) \neq \emptyset \). Since \( D \) is quasi-transitive, there is a vertex in \( C_1 \) which is adjacent to some vertex in \( C_2 \), a contradiction. Assume there are two cycles in \( D[X] \), say \( C_1 \) and \( C_2 \), and two vertices, say \( u \) and \( v \), with the following property: \( u \) is dominated by some \( x \in V(C_1) \), \( v \) is dominated by some \( y \in V(C_2) \), and \( (x, y) \in A \). By quasi-transitivity, \( (v, x) \in A \) and hence \( x \) is adjacent to \( y \). This is a contradiction to the efficient total domination property of \( X \).  

Furthermore, connected efficiently total dominatable quasi-transitive digraphs are strongly connected. Lemma 3.1 and Lemma 4 give
Theorem 6. WETD can be solved in $O(m\Delta^+ \max\{\Delta^-,\Delta^+\})$ time on quasi-transitive digraphs, where $m$ is the number of arcs, $\Delta^-$ the maximum in-degree and $\Delta^+$ is the maximum out-degree of the graph considered.

Another easy observation is the following

Lemma 5. If $D$ is an efficiently total dominatable $k$-partite tournament, each efficient total dominating set of $D$ induces a cycle of length 3 or 4.

Proof. The only contrafunctional $k$-partite tournaments are cycles of length 3 or 4. It is clear that a $k$-partite tournament does not have an induced subdigraph that is the disjoint union of two cycles. \qed

Again, efficiently total dominatable $k$-partite tournaments are strongly connected. Lemma 3.2 and Lemma 5 give

Theorem 7. WETD can be solved in $O(m\Delta^-\Delta^{+2})$ time on $k$-partite tournaments, where $m$ is the number of arcs, $\Delta^-$ the maximum in-degree and $\Delta^+$ is the maximum out-degree of the graph considered.

In contrast, Theorem 5 shows that ETD is $NP$-complete on $k$-partite semicomplete digraphs for all fixed $k \geq 2$. In fact, the $k$-partite semicomplete digraphs constructed in the proof of Theorem 5 only have a single anti-parallel arc. Hence, the existence of a single anti-parallel arc leads to the $NP$-hardness of the problem.

For bipartite tournaments one easily obtains the following nice characterization.

Theorem 8. A bipartite tournament $T$ is efficiently total dominatable if and only if there is a cycle $(u_1,v_1,u_2,v_2)$ in $T$ such that $N^+(u_1) = N^-(u_2)$ and $N^+(v_1) = N^-(v_2)$.

4.2. Locally semicomplete digraphs

Locally semicomplete digraphs generalize semicomplete digraphs in a natural way and may be among the most studied generalizations of tournaments [1]. As the following result shows, WETD can be solved efficiently even on the more general class of locally out-semicomplete digraphs. In contrast to the quasi-transitive and $k$-partite semicomplete case addressed in Section 4.1, the efficient total dominating sets of locally out-semicomplete digraphs are not just cycles of bounded length.

Theorem 9. WETD can be solved in $O(m\Delta^+)$ time on the class of locally out-semicomplete digraphs, where $m$ is the number of arcs and $\Delta^+$ is the maximum out-degree of the graph considered.

Proof. WETD can be solved using the following procedure. Let $D = (V,A)$ be a locally out-semicomplete digraph with real-valued vertex weight $c$. First we determine the set $B$ of all arcs $a = (u,v)$ of $D$ with $N^+(u) \cap N^+(v) = \emptyset$. Then we determine the strongly connected components of the digraph $D_B$ defined by
the arcs contained in $B$. Next we check the vertex set of each of these strongly connected components for being an efficient total dominating set of $D$. For all of those efficient total dominating sets, we return as output the one with minimal total weight.

To see correctness of the procedure, let $X$ be an arbitrary efficient total dominating set of $D$. Hence $N^+(u) \cap N^+(v) = \emptyset$ holds for each arc $a = (u, v)$ of $D[X]$ and thus $D[X]$ is an induced subdigraph of $D_B$. Since $D$ is locally out-semicomplete, $D_B$ is 1-out-regular and thus the strongly connected components of $D_B$ are exactly the contrafunctional subdigraphs of $D_B$ (they are exactly the cycles of $D_B$). Therefore $D[X]$ is a strongly connected component of $D_B$ and gets detected during the procedure.

Strongly connected components can be found in linear time by the famous algorithm of Kosaraju (or Tarjans [17] more efficient refinement). Hence, the time of each step of the procedure is bounded by $O(m \Delta^+)$ time.

Furthermore, connected efficiently total dominatable locally out-semicomplete digraphs are strongly connected.

Note that the locally out-semicomplete digraphs properly include (locally) semicomplete digraphs and (local) tournaments. We did not yet discover the complexity of ETD on locally in-semicomplete digraphs. However, Theorem 5.2 shows that for path-mergeable orgraphs ETD remains hard. As stated in [1], this is a common superclass of locally out-semicomplete and locally in-semicomplete orgraphs.

4.3. Arc-locally semicomplete digraphs

A generalization of bipartite semicomplete digraphs are arc-locally semicomplete digraphs. Since any bipartite semicomplete digraph is arc-locally semicomplete in particular, Theorem 5.4 has the following consequence:

**Theorem 10.** ETD is NP-complete on strongly connected arc-locally semicomplete digraphs.

Another negative result is the following.

**Theorem 11.** ETD is NP-complete on arc-locally in-semicomplete orgraphs.

**Proof.** Let $H = (V, E)$ be a hypergraph. We define an arc-locally in-semicomplete orgraph $D$ by

\[
\begin{align*}
V(D) &= V \cup E \cup \{a, b, c\}, \\
A(D) &= \{(a, b), (b, c), (c, a)\} \cup \{(b, e) : e \in E\} \cup \{(e, v) : v \in e \in E\}.
\end{align*}
\]

We observe the following: If $M \subseteq E$ is a perfect matching of $H$, then $\{a, b, c\} \cup M$ is an efficient total dominating set of $D$. On the other hand, if $X$ is an efficient total dominating set of $D$, then $X \cap E$ is a perfect matching of $H$. Hence, $D$ is efficiently total dominatable if and only if $H$ has a perfect matching. This completes the proof. \qed
To obtain positive results, we need further details on the structural properties of arc-locally semicomplete digraphs. A recent paper by Wang and Wang [18] gives a complete description of the strongly connected arc-locally in-semicomplete digraphs. To state this characterization, we need the following notions:

An extended cycle of length $k \geq 2$ is a digraph $C = (V, A)$ where $V$ admits a partition into $k$ non-empty sets $U_1, U_2, \ldots, U_k$ such that $A = \{ (u, v) : u \in U_i, v \in U_{i+1} \text{ for some } 1 \leq i \leq k \}$ where the index is taken modulo $k$. It is clear that any extended cycle has an efficient total dominating set. In fact, the efficient total dominating sets of extended cycles are exactly the sets that contain exactly one element of $U_i$ for each $1 \leq i \leq k$.

A T-digraph is a strongly connected digraph $T = (V, A)$ with the following properties. $V$ admits a partition into the sets $V_1, V_2, V_3$ and $V_4$ such that $T[V_1]$ and $T[V_3]$ have no arcs, $|V_2| = 1$ and $T[V_4]$ is semicomplete. $V_3$ and $V_4$ may not be empty at the same time. If $V_1$ is empty, $V_3$ must be empty, too (a case the authors of [18] forgot). The arcs of $T$ are as follows: For each $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$ and $v_4 \in V_4$ we have

- $(v_1, v_2) \in A$,  
- $(v_2, v_3) \in A$, but $(v_3, v_2) \notin A$,  
- $(v_3, v_1) \in A$, but $(v_1, v_3) \notin A$,  
- $(v_4, v_1) \in A$, but $(v_1, v_4) \notin A$,  
- $(v_4, v_3) \in A$, but $(v_3, v_4) \notin A$.

Furthermore, there may be some more arcs between $V_2$ and $V_1 \cup V_4$ since $T$ is strongly connected.

Using the notion of a T-digraph, Wang and Wang give the following characterization.

**Theorem 12** (Wang, Wang [18]). Let $D$ be a strongly connected arc-locally in-semicomplete digraph. $D$ is either semicomplete, semicomplete bipartite, an extended cycle or a T-digraph. If $D$ has an induced cycle of length at least 5, it is an extended cycle.

For a given digraph the reverse digraph is obtained by changing the direction of each arc. Since the reverse digraph of an arc-locally in-semicomplete digraph is an arc-locally out-semicomplete digraph and the property of being strongly connected is preserved by the reversing operation, we obtain the following from Theorem 12:

**Corollary 1** (Wang, Wang [18]). Let $D$ be a strongly connected arc-locally out-semicomplete digraph. If $D$ has an induced cycle of length at least 5, it is an extended cycle.

Using these characterizations, we obtain the next
Theorem 13. \( \text{WETD can be solved in } O(m\Delta^-\Delta^+^2) \) time on arc-locally out-semicomplete orgraphs, where \( m \) is the number of arcs, \( \Delta^- \) the maximum in-degree and \( \Delta^+ \) is the maximum out-degree of the graph considered.

**Proof.** We observe that any connected contrafunctional arc-locally out-semicomplete orgraph is a cycle. Hence, the subdigraphs induced by efficient total dominating sets in arc-locally out-semicomplete orgraphs are the disjoint union of cycles. A similar argument to the one used in the proof of Theorem 6 shows that in a connected arc-locally out-semicomplete orgraph each efficient total dominating set induces a single cycle. Again, connected efficiently total dominatable arc-locally out-semicomplete orgraphs are strongly connected.

Let \( D \) be a connected arc-locally out-semicomplete digraph. We can solve WETD using the following procedure. First we check if \( D \) is the extension of a cycle. This can be done in linear time easily. If this is the case, a minimum weight efficient total dominating set is obtained by a greedy technique in linear time.

If \( D \) is not the extension of a cycle, there is no induced cycle of length at least 5 in \( D \), by Corollary 1. Hence, we only have to search for a minimum weight efficient total dominating set of \( D \) that induces a cycle of length at most four. By Lemma 3, this can be done in \( O(m\Delta^-\Delta^+^2) \) time. If such a set does not exist, \( D \) is not efficiently total dominatable.

Using a similar algorithm, we obtain the same time complexity for strongly connected arc-locally in-semicomplete digraphs.

Theorem 14. \( \text{WETD can be solved in } O(m\Delta^-\Delta^+^2) \) time on strongly connected arc-locally in-semicomplete orgraphs, where \( m \) is the number of arcs, \( \Delta^- \) the maximum in-degree and \( \Delta^+ \) is the maximum out-degree of the graph considered.

**Proof.** Let \( D \) be a strongly connected arc-locally in-semicomplete digraph. By Theorem 12, \( D \) is either semicomplete, semicomplete bipartite, an extended cycle or a T-digraph. In the case that \( D \) is semicomplete or semicomplete bipartite it is clear that each efficient total dominating set induces a cycle of length at most 4. A few easy case distinctions show that the same holds if \( D \) is a T-digraph.

Hence, we can use the same procedure to solve WETD as for arc-locally out-semicomplete orgraphs.

5. Overview

Summarizing our results on the time complexity of the efficient total domination problem we obtain the following overview. Polynomially solvable instances for WETD are the following:

- biorientations of \( \{K_{1,p}, qK_2\} \)-free graphs for fixed \( p \) and \( q \),
- locally out-semicomplete digraphs (in \( O(m\Delta^+) \) time),
• quasi-transitive digraphs (in $O(m\Delta^+ \max\{\Delta^-, \Delta^+\})$ time),
• $k$-partite tournaments (in $O(m\Delta^+ \Delta^+) \Delta^{-2}$ time),
• arc-locally out-semicomplete orgraphs (in $O(m\Delta^+ \Delta^{-2})$ time),
• strongly connected arc-locally in-semicomplete orgraphs (in $O(m\Delta^+ \Delta^{-2})$ time).

$NP$-completeness holds for ETD in the following cases:
• orientations of split graphs,
• path-mergeable orientations of planar bipartite graphs of maximum degree 4,
• strongly connected biorientations of threshold graphs,
• strongly connected $k$-partite semicomplete digraphs,
• strongly connected arc-locally semicomplete digraphs,
• arc-locally in-semicomplete orgraphs.


