

Interval-Arithmetic-Oriented Interval Computing Technique for Global Optimization

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Firstly, a brief survey of the existing works on comparing and ranking any two interval numbers on the real line is presented, and then pointing out the drawbacks of these definitions, a new approach is proposed in the context of decision maker's (optimistic and pessimistic) point of view. Secondly, an interval technique is proposed to solve unconstrained multimodal optimization problems with continuous variables. In this proposed method, the search region is divided into two equal subregions successively and in each subregion, the lower and upper bounds of the objective function are computed with the help of interval arithmetic. Then, by comparing these two interval objective values and considering the subregion containing the better objective value, the global optimal value of the objective function or close to it is obtained. Finally, the proposed method is applied to solve several number of test problems of global optimization with lower as well as higher dimension and is compared with the existing methods with respect to the number of function evaluations.

1 Introduction

In solving the decision-making problems, a set of intervals may appear in the selection of the best alternative, which arises a question related to the comparison of two arbitrary interval numbers. In this area, Moore [12] defined two transitive order relations

of intervals. However, these order relations cannot find the ranking between two partially or fully overlapping intervals. After Moore [12], Ishibuchi and Tanaka [9] suggested two order relations " \leq_{LR} " and " \leq_{CW} ." Recently, Levin [10] defined a remoteness function to compare two arbitrary intervals. However, this process for comparison is very much complicated to find out the best alternative. Then, Sevastjanov and Róg [16] proposed the same thing in probabilistic approach.

Decision-making is an important task to select the best alternative of some conflicting situations. It depends upon the uncertainty. There are two types of decision-making, namely, optimistic and pessimistic decision-making. For optimistic decision-making, the decision-maker selects the best alternative ignoring the uncertainty whereas for pessimistic case, the decision-maker selects the best alternative with less uncertainty.

The solution methodology is very much important factor for finding global optima of a multimodal multidimensional nonconvex non-linear continuous optimization problem with fixed coefficients. There are several deterministic and stochastic methods proposed for obtaining the global optima. These methods are available in Floudas et al. [6] and Hansen and Walster [7]. Ichida [8] developed an interval computing method to find out the global optima of the problems with fixed coefficients. In this interval computing method, the search domain is divided into two subregions and the lower and upper bounds of the objective function are estimated in each subregion. By rejection principle, one can reject one of the subregions. Continuing this process, one can find the global optima. There are different rejection principles for rejection of one subregion among two. In this connection, one may refer to the works of Csendes [5], Markót et al. [11], Csallner et al. [4], and Casado et al. [2].

In this paper, we have studied the existing works on comparing and ranking of any two interval numbers. After pointing out the weakness of these definitions, a new approach is suggested in the context of decision maker's (optimistic and pessimistic) point of view. Secondly, an interval computing technique is proposed to solve nonlinear bound constrained (also known as the box constrained) optimization problems. In the interval computing technique, the original domain of variables is divided into two equal subregions successively and the lower and upper bounds of the objective function are computed in each subregion with the help of interval arithmetic. Now, by comparing these two interval objective values by the proposed order relations, and then considering the subregion containing the better objective value, one can always find out the global optimal value of the objective function or close to it in the form of an interval with negligible width. Finally, this method is tested on several test functions used in literature and is compared with the existing methods with respect to the number of function evaluations.

2 Finite interval arithmetic

An interval number A is a closed interval defined by $A = [a_L, a_R] = \{x : a_L \leq x \leq a_R, x \in \mathbb{R}\}$, where a_L and a_R are the left and right bounds, respectively, and \mathbb{R} is the set of all real numbers. An interval A can also be expressed in terms of center and radius as

$$A = \langle a_C, a_W \rangle = \{x : a_C - a_W \leq x \leq a_C + a_W, x \in \mathbb{R}\}, \quad (2.1)$$

where a_C and a_W are, respectively, the center and radius of the interval A , that is, $a_C = (a_L + a_R)/2$ and $a_W = (a_R - a_L)/2$. Actually, each real number can be regarded as an interval, such as, for all $x \in \mathbb{R}$, x can be written as an interval $[x, x]$, which has zero length.

Here, we will give the concise definitions of addition, subtraction, multiplication and division of interval numbers. Let $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$ be two intervals, then

$$A + B = [a_L + b_L, a_R + b_R], \quad A + B = \langle a_C + b_C, a_W + b_W \rangle. \quad (2.2)$$

The multiplication of an interval by a real number λ is defined as

$$\lambda A = \lambda [a_L, a_R] = \begin{cases} [\lambda a_L, \lambda a_R] & \text{if } \lambda \geq 0, \\ [\lambda a_R, \lambda a_L] & \text{if } \lambda < 0, \end{cases} \quad (2.3)$$

$$\lambda A = \lambda \langle a_C, a_W \rangle = \langle \lambda a_C, |\lambda| a_W \rangle.$$

The difference of two intervals A and B is

$$A - B = [a_L - b_R, a_R - b_L]. \quad (2.4)$$

The product of two different intervals is

$$A \times B = [\min(a_L b_L, a_L b_R, a_R b_L, a_R b_R), \max(a_L b_L, a_L b_R, a_R b_L, a_R b_R)]. \quad (2.5)$$

Hence the division of the interval B by the interval A can be defined as

$$\frac{B}{A} = B \times \frac{1}{A} = [b_L, b_R] \times \left[\frac{1}{a_R}, \frac{1}{a_L} \right], \quad \text{provided } 0 \notin [a_L, a_R]. \quad (2.6)$$

Power of an interval. According to Hansen and Walster [7], the definition of power of an interval is as follows:

$$A^n = \begin{cases} [1, 1] & \text{if } n = 0, \\ [a_L^n, a_R^n] & \text{if } a_L \geq 0 \text{ or if } n \text{ is odd,} \\ [a_R^n, a_L^n] & \text{if } a_R \leq 0, n \text{ is even,} \\ [0, \max(a_L^n, a_R^n)] & \text{if } a_L \leq 0 \leq a_R, n > 0 \text{ is even.} \end{cases} \quad (2.7)$$

3 Functions of intervals

Now, the definitions of exponential function, logarithmic function and bounded trigonometric functions will be given below.

3.1 Exponential function

Exponential function being monotonic over any interval, the exponential of an interval $A = [a_L, a_R]$ is defined as

$$\begin{aligned} \exp(A) &= \exp([a_L, a_R]) = [\exp(a_L), \exp(a_R)], \\ \exp(-A) &= \exp([-a_R, -a_L]) = [\exp(-a_R), \exp(-a_L)]. \end{aligned} \quad (3.1)$$

3.2 Logarithm function

The logarithm of an interval $A = [a_L, a_R]$ is defined as

$$\log(A) = \log([a_L, a_R]) = [\log(a_L), \log(a_R)], \quad \text{provided } a_L > 0. \quad (3.2)$$

3.3 Sine and cosine functions

The trigonometric functions $\sin(A)$ and $\cos(A)$ can be evaluated over any given interval A by evaluating the values of the functions at the end points and checking whether the interval contains a point or points where $\sin(A)$ and $\cos(A)$ can have extreme values.

$\sin(A)$ of the interval $A = [a_L, a_R]$ can be defined as

$$\sin([a_L, a_R]) = \begin{cases} [\sin(a_L), \sin(a_R)] & \text{if } a_L = a_R, \\ [\sin(a_L), 1] & \text{if } a_R \geq (m+2)\frac{\pi}{2}, \sin(m+2)\frac{\pi}{2} = 1, \\ & \sin(a_L) < \sin(a_R), m \text{ is odd} \\ & \text{or if } a_R \geq (m+1)\frac{\pi}{2}, \sin(m+1)\frac{\pi}{2} = 1, \\ & \sin(a_L) < \sin(a_R), m \text{ is even,} \\ [\sin(a_R), 1] & \text{if } a_R \geq (m+2)\frac{\pi}{2}, \sin(m+2)\frac{\pi}{2} = 1, \\ & \sin(a_L) > \sin(a_R), m \text{ is odd} \\ & \text{or if } a_R \geq (m+1)\frac{\pi}{2}, \sin(m+1)\frac{\pi}{2} = 1, \\ & \sin(a_L) > \sin(a_R), m \text{ is even,} \\ [-1, \sin(a_R)] & \text{if } a_R \geq (m+2)\frac{\pi}{2}, \sin(m+2)\frac{\pi}{2} = -1, \\ & \sin(a_L) < \sin(a_R), m \text{ is odd} \\ & \text{or if } a_R \geq (m+1)\frac{\pi}{2}, \sin(m+1)\frac{\pi}{2} = -1, \\ & \sin(a_L) < \sin(a_R), \\ & m \text{ is even,} \\ [-1, \sin(a_L)] & \text{if } a_R \geq (m+2)\frac{\pi}{2}, \sin(m+2)\frac{\pi}{2} = -1, \\ & \sin(a_L) > \sin(a_R), \\ & m \text{ is odd} \\ & \text{or if } a_R \geq (m+1)\frac{\pi}{2}, \sin(m+1)\frac{\pi}{2} = -1, \\ & \sin(a_L) > \sin(a_R), m \text{ is even,} \\ [-1, 1] & \text{if } a_R \geq (m+4)\frac{\pi}{2}, m \text{ is odd} \\ & \text{or if } a_R \geq (m+3)\frac{\pi}{2}, m \text{ is even,} \\ [\sin(a_L), \sin(a_R)] & \text{if } a_R \leq (m+2)\frac{\pi}{2}, \\ & \sin(a_L) < \sin(a_R), m \text{ is odd} \\ & \text{or if } a_R \leq (m+1)\frac{\pi}{2}, \\ & \sin(a_L) < \sin(a_R), m \text{ is even,} \\ [\sin(a_R), \sin(a_L)] & \text{if } a_R \leq (m+2)\frac{\pi}{2}, \\ & \sin(a_L) > \sin(a_R), m \text{ is odd} \\ & \text{or if } a_R \leq (m+1)\frac{\pi}{2}, \\ & \sin(a_L) > \sin(a_R), m \text{ is even,} \end{cases} \quad (3.3)$$

where m is the integral part of $a_L/(\pi/2)$. The function $\cos([a_L, a_R])$ can be defined similarly.

4 Order relations of interval numbers

In this section, we will discuss the order relations for both maximization problems and minimization problems. Let A and B be two closed intervals. These two intervals may be one of the following three types.

Type 1. two intervals are disjoint.

Type 2. intervals are partially overlapping.

Type 3. one of the intervals contains the other one.

4.1 Order relations for minimization problems

For minimization problems, Ishibuchi and Tanaka [9] have defined the order relations of two closed intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$ in the following way.

Definition 4.1. (i)

$$A \leq_{LR} B \iff a_L \leq b_L \wedge a_R \leq b_R, \quad (4.1)$$

$$A <_{LR} B \iff A \leq_{LR} B \wedge A \neq B,$$

(ii)

$$A \leq_{CW} B \iff a_C \leq b_C \wedge a_W \leq b_W, \quad (4.2)$$

$$A <_{CW} B \iff A \leq_{CW} B \wedge A \neq B.$$

The order relations \leq_{LR} and \leq_{CW} are partial order, as they are reflexive, transitive and antisymmetric, and a decision maker will prefer either A or B in case of minimization problem.

Chanas and Kuchta [3] proposed the generalization of the above relations introducing the concept of t_0, t_1 -cut of an interval $A = [a_L, a_R]$ as

$$A/_{[t_0, t_1]} = [a_L + t_0(a_R - a_L), a_L + t_1(a_R - a_L)], \quad \text{where } 0 \leq t_0 < t_1 \leq 1. \quad (4.3)$$

Recently, Sengupta and Pal [15] defined the ranking of two closed intervals in two different ways. In the first approach, they ordered the intervals introducing the acceptability function given below.

Definition 4.2. The value judgement index or the acceptability function $A_{\prec} : I \times I \rightarrow [0, \infty)$ is given by $A_{\prec}(A, B) = (m(B) - m(A)) / (w(B) + w(A))$, where $w(B) + w(A) \neq 0$, I is the set of all closed intervals on the real line \mathbb{R} and $m(X)$ and $w(X)$ are, respectively, the center and radius of the interval X . A_{\prec} may be interpreted as the grade of acceptability of the “first interval to be inferior to the second interval” in terms of value. If $A_{\prec}(A, B) > 0$, then for a maximizing problem in which A and B are two alternative interval profits and the problem is to choose the maximum profit, the interval B is preferred to A and for a minimization problem in which A and B are two interval costs, A is preferred to B , in terms of value.

According to Sengupta and Pal [15], the acceptability index is only a value-based ranking index and it can be applied partially to select the best alternative from the pessimistic point of view of the decision maker. So, only the optimistic decision maker can use it completely.

In the second approach, Sengupta and Pal [15] introduced the fuzzy preference for the ranking of a pair of intervals on the real line with respect to a pessimistic decision maker’s point of view. They defined a nonlinear membership function, which lies in the interval $[0, 1]$. When the value of this membership function lies within the interval $[0.333, 0.666]$, this definition fails to find out the order relations.

4.2 Order relations for maximization problems

Let the uncertain profits from two alternatives be represented by two closed intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$, respectively. It is also assumed that the profit of each alternative lies in the corresponding interval. Ishibuchi and Tanaka [9] defined the order relations for maximization problem as given below.

Definition 4.3. (i)

$$A \leq_{LR} B \iff a_L \leq b_L \wedge a_R \leq b_R, \quad (4.4)$$

$$A <_{LR} B \iff A \leq_{LR} B \wedge A \neq B,$$

(ii)

$$A \leq_{CW} B \iff a_C \leq b_C \wedge a_W \geq b_W, \quad (4.5)$$

$$A <_{CW} B \iff A \leq_{CW} B \wedge A \neq B.$$

However, Chanas and Kuchta [3] and Sengupta and Pal [15] did not give the definition separately for maximization problems.

As the existing definitions of order relations are not sufficient, we will define the order relations of two closed intervals in the context of decision-maker's point of view.

Example 4.4. Let $A = [1, 15] = \langle 8, 7 \rangle$ and $B = [2, 12] = \langle 7, 5 \rangle$. The existing definitions of order relations are not sufficient to order these two intervals that are of Type 3. Without using the existing definitions of order relations, it is clear from the observation that A is superior to B in case of minimization as well as maximization problem from the optimistic decision maker's point of view. However, A is superior to B in case of maximization problem from pessimistic decision maker's point of view and B is superior to A in case of minimization problems from pessimistic decision maker's point of view. However, we cannot conclude about the inferiority of the other interval in case of optimistic decision-making as well as pessimistic decision-making.

Thus, we see that the order relations of a pair of intervals depend on the type of the problem as well as on the decision maker's point of view. So, we will define the order relations between the interval numbers according to the decision maker's point of view for both minimization and maximization problems.

5 Revised definitions for order relations of interval numbers

In this section, the order relations which represent the decision maker's point of view between interval costs/times for minimization problems and interval profits for maximization problems are defined. Let the uncertain costs/times or profits from two alternatives be represented by the intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$.

5.1 Optimistic decision-making

In this case, the decision maker chooses the lowest cost/time for minimization problems and the highest profit for maximization problems ignoring the uncertainty.

Definition 5.1. For minimization problems, let us define the order relation $\leq_{o \min}$ between the intervals $A = [a_L, a_R]$ and $B = [b_L, b_R]$ as

$$\begin{aligned} A \leq_{o \min} B &\iff a_L \leq b_L, \\ A <_{o \min} B &\iff A \leq_{o \min} B \wedge A \neq B. \end{aligned} \tag{5.1}$$

According to this definition, A is superior to B and A is accepted. This order relation is not symmetric.

Definition 5.2. For maximization problems, let us define the order relation $\geq_{o \max}$ between the intervals $A = [a_L, a_R]$ and $B = [b_L, b_R]$ as

$$\begin{aligned} A \geq_{o \max} B &\iff a_R \geq b_R, \\ A >_{o \max} B &\iff A \geq_{o \max} B \wedge A \neq B. \end{aligned} \tag{5.2}$$

According to this definition, the optimistic decision maker accepts A . Here also, the order relation $\geq_{o \max}$ is not symmetric.

5.2 Pessimistic decision-making

In this case, the decision maker expects the minimum costs/times for minimization problems and the maximum profit for maximization problems according to the principles “Less uncertainty is better than more uncertainty” or “More uncertainty is worse than less uncertainty.”

Definition 5.3. For minimization problems, let us define the order relation $<_{p \min}$ between the intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$ for a pessimistic decision maker as

(i)

$$A <_{p \min} B \iff a_C < b_C \quad \text{for Types 1 and 2 intervals,} \tag{5.3}$$

(ii)

$$A <_{p \min} B \iff a_C \leq b_C \wedge a_W < b_W \quad \text{for Type 3 intervals.} \tag{5.4}$$

However, for Type 3 intervals with $a_C < b_C \wedge a_W > b_W$ a pessimistic decision cannot be taken. Here we will take the optimistic decision.

Definition 5.4. For maximization problems, let us define the order relation $>_{p \max}$ between the intervals $A = [a_L, a_R] = \langle a_C, a_W \rangle$ and $B = [b_L, b_R] = \langle b_C, b_W \rangle$ for a pessimistic decision maker as

(i)

$$A >_{p \max} B \iff a_C > b_C \quad \text{for Types 1 and 2 intervals,} \tag{5.5}$$

(ii)

$$A >_{p \max} B \iff a_C \geq b_C \wedge a_W < b_W \quad \text{for Type 3 intervals.} \tag{5.6}$$

However, for Type 3 intervals with $a_C > b_C \wedge a_W > b_W$ pessimistic decision cannot be taken. Here we also take the optimistic decision.

From Definitions 5.3(ii) and 5.4(ii) and their complementary cases, it is observed that

$$A < B \implies a_C < b_C \quad \text{for Type 3 intervals.} \quad (5.7)$$

Therefore, in case of pessimistic decision-making we may consider the following definition for order relations of two closed intervals for all types of intervals, namely, Types 1, 2, and 3:

$$A <_p B \iff a_C < b_C. \quad (5.8)$$

The equality sign will hold if and only if $a_L = b_L$ and $a_W = b_W$.

6 Solution procedures of optimization problems

Let us consider an unconstrained optimization (maximization or minimization) problem with fixed coefficients as follows:

$$\begin{aligned} Z &= f(x), \quad l \leq x \leq u, \quad x = (x_1, x_2, \dots, x_n), \\ l &= (l_1, l_2, \dots, l_n), \quad u = (u_1, u_2, \dots, u_n), \end{aligned} \quad (6.1)$$

where n represents the number of variables, x_j is the j th variable whose prescribed upper and lower bounds are l_j and u_j , respectively. Hence, the search region of the above problem is as follows:

$$D = \{x \in \mathbb{R}^n : l_j \leq x_j \leq u_j, j = 1, 2, \dots, n\}. \quad (6.2)$$

6.1 Interval methodology

The prescribed domain is defined as

$$D = \{x \in \mathbb{R} : l_j \leq x_j \leq u_j, j = 1, 2, \dots, n\}. \quad (6.3)$$

Then, D can be divided into two subregions R_1 and R_2 with respect to the variable x_k ($k = 1, 2, \dots, n$) defined as follows:

$$\begin{aligned} R_1 &= \left\{ x \in \mathbb{R}^n : l_k \leq x_k \leq \frac{l_k + u_k}{2}, l_j \leq x_j \leq u_j; j = 1, 2, \dots, k-1, k+1, \dots, n \right\}, \\ R_2 &= \left\{ x \in \mathbb{R}^n : \frac{l_k + u_k}{2} < x_k \leq u_k, l_j \leq x_j \leq u_j; j = 1, 2, \dots, k-1, k+1, \dots, n \right\}. \end{aligned} \quad (6.4)$$

Let $F(R_1) = [f_1, \bar{f}_1]$ and $F(R_2) = [f_2, \bar{f}_2]$ be the interval values of $f(x)$ in the subregions R_1 and R_2 , respectively, where f_k, \bar{f}_k ($k = 1, 2$) denote the upper and lower bounds of $f(x)$ in R_k , calculated by applying interval arithmetic. Then comparing $F(R_1)$ and $F(R_2)$, the subregion either R_1 or R_2 that contains the better objective value, is accepted. This process for each subregion is repeated till the domain of each variable reduces to an interval with negligible width. Finally, the global optimal value or close to the optimal value of the given objective function has been obtained. For the entire process, Algorithm 6.1 is developed for minimization/maximization problems.

Algorithm 6.1

Step 1. Initialize the number of variables n .

Step 2. Initialize the lower and upper bounds l_j and u_j ($j = 1, 2, \dots, n$) of the variables.

Step 3. Compute the widths $\varepsilon_j = u_j - l_j; j = 1, 2, \dots, n$.

Step 4. If $\varepsilon_j < \alpha$, a preassigned very small positive number, then go to Step (7); otherwise go to the next step.

Step 5. (i) Divide the accepted subregion or region X into two other smaller distinct subregions R_1 and R_2 such that $R_1 \cup R_2 = X$.

(ii) Applying interval arithmetic, compute the interval value $F(R_1) = [f_1, \bar{f}_1]$ and $F(R_2) = [f_2, \bar{f}_2]$ of the objective function in the subregions R_1 and R_2 , respectively.

(iii) Select the subregion R_1 or R_2 as the new search region which contains the better objective function value by comparing $F(R_1)$ and $F(R_2)$ with the help of pessimistic order relations between two intervals defined in Definitions 5.3 and 5.4 for minimization and maximization problems, respectively.

Step 6. Go to Step 3.

Step 7. Print the values of the variables and the objective function in the form of intervals.

Step 8. Stop.

7 Numerical examples

Numerical experiments have been carried out to test the performance of the proposed approach described in this paper. A number of well-known test functions have been selected from the literature. Problems 1–3 and 6–16 are taken from Casado et al. [2], problems 4 and 5 from Ratz [13], problems 17 and 22 from Salhi and Queen [14]. These test functions have different features like convex/nonconvex, continuous, unimodal/multimodal and low/high dimension. Solving these test problems, the global optimal solution or an approximation of it have been obtained by applying Algorithm 6.1. The test problems with their results have been given in the appendix with corresponding number of function evaluation and evaluation time in Table 7.1 with error tolerance $\varepsilon = 10^{-8}$. Table 7.1 also shows the comparison in number of function evaluations (NFE) among the methods: TIAM (Traditional Interval analysis global minimization Algorithm with Monotonicity test), IAG (Interval analysis global minimization Algorithm using Gradient information), GA (SQ) (Genetic Algorithm by Salhi and Queen [14]), and the proposed method IAIOICT (Interval Arithmetic Oriented Interval Computing Technique) in this paper.

The approach for computing the best-found value in each subregion of a given search region of the test problem has been coded in C++ programming language and implemented on a Pentium 4 PC, 3.0 GHz with 1 GB RAM in LINUX environment.

8 Concluding remarks

In this paper, we have presented a new method dependent on interval computing technique to solve the unconstrained optimization problems. The basis of the proposed method is the comparison of intervals according to the decision maker's point of view. The definitions of bounded trigonometric functions (like, sine and cosine) of an interval are given. From the numerical experiments, it has been observed that the proposed method possesses the merits of global exploration, fast convergence, and it can find the optimal or close-to-optimal solutions. However, for some cases, the interval computing technique, which has been applied for finding the best value in each subregion, gives the

Table 7.1 Comparison of TIAM, IAG, GA (SQ), and the proposed method (IAOICT) with CPU time.

Problem no.	No. of variables	Search domain	No. of function evaluations				CPU time in IAOICT (in s)
			TIAM	IAG	GA(SQ)	IAOICT	
1	1	$-10 \leq x \leq 10$	112	88	—	62	0.05
2	1	$2.7 \leq x \leq 7.5$	132	81	—	58	0.05
3	1	$-10 \leq x \leq 10$	167	114	—	62	0.05
4	1	$-10 \leq x \leq 10$	78	62	—	62	0.05
5	1	$-10 \leq x \leq 10$	66	56	—	62	0.05
6	1	$-10 \leq x \leq 10$	107	88	—	62	0.05
7	1	$-10 \leq x \leq 10$	199	116	—	62	0.05
8	1	$-10 \leq x \leq 9$	459	333	—	62	0.05
		$-9 \leq x \leq 10$				62	0.05
		$0 \leq x \leq 9$				62	0.05
9	1	$-20 \leq x \leq 20$	207	114	—	64	0.05
10	1	$0 \leq x \leq 10$	139	113	—	60	0.05
11	1	$-5 \leq x \leq 5$	191	77	—	60	0.05
12	1	$0 \leq x \leq 10$	824	288	—	60	0.05
13	1	$0 \leq x \leq 20$	104	75	—	62	0.00
14	1	$-10 \leq x \leq 10$	218	146	—	62	0.05
15	1	$-4 \leq x \leq 0$	807	498	—	64	0.11
		$0 \leq x \leq 4$				64	0.15
16	1	$-10 \leq x \leq 20$	6041	1364	—	70	0.05
		$0 \leq x \leq 20$				70	0.05
17	2	$-2 \leq x_j \leq 2,$ $j = 1, 2$	—	—	200	116	0.16
18	2	$-100 \leq x_j \leq 100,$ $j = 1, 2$	—	—	—	140	0.22
19	2	$-100 \leq x_j \leq 100,$ $j = 1, 2$	—	—	—	140	0.21
20	3	$-100 \leq x_j \leq 100,$ $j = 1, 2, 3$	—	—	—	180	0.0025

Table 7.1 Continued.

Problem no.	No. of variables	Search domain	No. of function evaluations				CPU time in IAOICT (in s)
			TIAM	IAG	GA(SQ)	IAOICT	
21	5	$-100 \leq x_j \leq 100$ $j = 1, 2, \dots, 5$	—	—	—	300	0.005
	10	$-100 \leq x_j \leq 100$ $j = 1, 2, \dots, 10$	—	—	—	600	0.01
	50	$-100 \leq x_j \leq 100$ $j = 1, 2, \dots, 50$	—	—	—	3000	0.09
22	10	$-2 \leq x_j \leq 2$, $j = 1, 2, \dots, 10$	—	—	168220	560	3.02

better result for smaller subregion. For future research, one may apply the same methodology for constrained optimization problems modifying the rejection principle for subregion of a search domain. It can also be used in interval-valued optimization problems.

Appendix

List of test functions

Single variable functions

(1)

$$f(x) = 2x^2 - 0.03e^{-(200(x-0.0675))^2}, \quad -10 \leq x \leq 10; \quad (\text{A.1})$$

one global minimum: $x^* \in [0.067388, 0.067388]$, $f^* \in [-0.020903, -0.020903]$.

(2)

$$f(x) = \sin(x) + \sin \frac{10x}{3} + \ln(x) - 0.84x, \quad 2.7 \leq x \leq 7.5; \quad (\text{A.2})$$

one global minimum: $x^* \in [5.199778, 5.199778]$, $f^* \in [-4.601308, -4.601308]$.

(3)

$$f(x) = \sin^2 \left(1 + \frac{x-1}{4} \right) + \left(\frac{x-1}{4} \right)^2, \quad -10 \leq x \leq 10; \quad (\text{A.3})$$

one global minimum: $x^* \in [-0.787880, -0.787880]$, $f^* \in [0.475689, 0.475689]$.

(4)

$$f(x) = (x-1)^2 (\sin^2(1+x)) + 1, \quad -10 \leq x \leq 10; \quad (\text{A.4})$$

one global minimum: $x^* \in [1.000000, 1.000000]$, $f^* \in [1.000000, 1.000000]$.

(5)

$$f(x) = -\frac{1}{(x-2)^2 + 3}, \quad -10 \leq x \leq 10; \quad (\text{A.5})$$

one global minimum: $x^* \in [2.000000, 2.000000]$, $f^* \in [-0.333333, -0.333333]$.

(6)

$$f(x) = (x-x^2)^2 + (x-1)^2, \quad -10 \leq x \leq 10; \quad (\text{A.6})$$

one global minimum: $x^* \in [1.000000, 1.000000]$, $f^* \in [0.000000, 0.000000]$.

(7)

$$f(x) = \exp(x^2), \quad -10 \leq x \leq 10; \quad (\text{A.7})$$

one global minimum: $x^* \in [0.000000, 0.000000]$, $f^* \in [1.000000, 1.000000]$.

(8)

$$f(x) = -\sum_{k=1}^5 k \cdot \sin((k+1)x + k), \quad -10 \leq x \leq 10; \quad (\text{A.8})$$

three global minima: $x^* \in [-6.774576, -6.774576]$ in $[-10, 0]$, $x^* \in [-0.491391, -0.491391]$ in $[-9, 10]$ and $x^* \in [5.791794, 5.791794]$ in $[0, 9]$, $f^* \in [-12.031249, -12.031249]$.

(9)

$$f(x) = \frac{x^2}{20} - \cos(x) + 2, \quad -20 \leq x \leq 20; \quad (\text{A.9})$$

one global minimum: $x^* \in [0.000000, 0.000000]$, $f^* \in [1.000000, 1.000000]$.

(10)

$$f(x) = -\sum_{i=1}^{10} \frac{1}{(k_i(x - a_i))^2 + c_i}, \quad 0 \leq x \leq 10; \quad (\text{A.10})$$

where

$$\begin{aligned} a &= (3.040, 1.098, 0.674, 3.537, 6.173, 8.679, 4.503, 3.328, 6.937, 0.700), \\ k &= (2.983, 2.378, 2.439, 1.168, 2.406, 1.236, 2.868, 1.378, 2.348, 2.268), \\ c &= (0.192, 0.140, 0.127, 0.132, 0.125, 0.189, 0.187, 0.171, 0.188, 0.176); \end{aligned} \quad (\text{A.11})$$

(11) one global minimum: $x^* \in [0.685844, 0.685844]$, $f^* \in [-14.572917, -14.572917]$.

$$f(x) = x^2 - \cos(18x), \quad -5 \leq x \leq 5; \quad (\text{A.12})$$

(12) one global minimum: $x^* \in [0.000000, 0.000000]$, $f^* \in [-1.000000, -1.000000]$.

$$f(x) = x^4 - 12x^3 + 47x^2 - 60x - 20e^{-x}, \quad 0 \leq x \leq 10; \quad (\text{A.13})$$

(13) one global minimum: $x^* \in [0.713667, 0.713667]$, $f^* \in [-32.781261, -32.781261]$.

$$f(x) = e^{-3x} - \sin^3 x, \quad 0 \leq x \leq 20; \quad (\text{A.14})$$

(14) one global minimum: $x^* \in [14.137167, 14.137167]$, $f^* \in [-1.000000, -1.000000]$.

$$f(x) = (x + \sin(x))e^{-x^2}, \quad -10 \leq x \leq 10; \quad (\text{A.15})$$

(15) one global minimum: $x^* \in [-0.679579, -0.679579]$, $f^* \in [-0.824239, -0.824239]$.

$$f(x) = x^6 - 15x^4 + 27x^2 + 250, \quad -4 \leq x \leq 4; \quad (\text{A.16})$$

two global minima: $x^* \in [-3.000000, -3.000000]$ in $[-4, 0]$, $x^* \in [3.000000, 3.000000]$ in $[0, 4]$, $f^* \in [7.000000, 7.000000]$.

(16)

$$f(x) = x^4 - 10x^3 + 35x^2 - 50x + 24, \quad -10 \leq x \leq 20; \quad (\text{A.17})$$

two global minima: $x^* \in [1.381966, 1.381966]$ in $[-10, 20]$, $x^* \in [3.618034, 3.618034]$ in $[0, 20]$, $f^* \in [-1.000000, -1.000000]$.

Double variable functions

(17) *Rosenbrock R2 (2 variables) function:*

$$R2(x_1, x_2) = 100(x_2 - x_1)^2 + (1 - x_1)^2, \quad -2 \leq x_j \leq 2, \quad j = 1, 2; \quad (\text{A.18})$$

one global minimum: $x_1^* \in [1.000000, 1.000000]$, $x_2^* \in [1.000000, 1.000000]$, $[R2(x_1, x_2)]^* \in [0.000000, 0.000000]$.

(18) *Eason (ES) (2 variables):*

$$ES(x_1, x_2) = -\cos(x_1) \times \cos(x_2) \times \exp\{-[(x_1 - \pi)^2 + (x_2 - \pi)^2]\}, \\ -100 \leq x_j \leq 100, \quad j = 1, 2; \quad (\text{A.19})$$

one global minimum: $x_1^* \in [3.141593, 3.141593]$, $x_2^* \in [3.141593, 3.141593]$, $[ES(x_1, x_2)]^* \in [-1.000000, -1.000000]$.

(19) *Bohachevsky (BH) (2 variables):*

$$BH(x_1, x_2) = x_1^2 + 2 \times x_2^2 - 0.3 \times \cos(3\pi \times x_1) \times \cos(4\pi \times x_2) + 0.3, \\ -100 \leq x_j \leq 100, \quad j = 1, 2; \quad (\text{A.20})$$

one global minimum: $x_1^* \in [0.000000, 0.000000]$, $x_2^* \in [0.000000, 0.000000]$, $[BH(x_1, x_2)]^* \in [0.000000, 0.000000]$.

Functions of three variables

(20) *De Joung F1 (3 variables):*

$$F1(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2, \quad -100 \leq x_j \leq 100, \quad j = 1, 2, 3; \quad (\text{A.21})$$

one global minimum: $x_1^* \in [0.000000, 0.000000]$, $x_2^* \in [0.000000, 0.000000]$, $x_3^* \in [0.000000, 0.000000]$, $[F1(x_1, x_2, x_3)]^* \in [0.000000, 0.000000]$.

Functions of n variables

(21)

$$f(X) = \sum_{i=1}^n x_i^2, \quad -100 \leq x_i \leq 100, \quad i = 1, 2, \dots, n; \quad (\text{A.22})$$

(i) for $n = 5$,one global minimum: $x_i^* \in [0.000000, 0.000000]$, $i = 1, 2, \dots, 5$, $f^* \in [0.000000, 0.000000]$ (ii) for $n = 10$,one global minimum: $x_i^* \in [0.000000, 0.000000]$, $i = 1, 2, \dots, 10$, $f^* \in [0.000000, 0.000000]$ (iii) for $n = 50$,one global minimum: $x_i^* \in [0.000000, 0.000000]$, $i = 1, 2, \dots, 50$, $f^* \in [0.000000, 0.000000]$.(22) *Rosenbrock* (R_n) *function*

$$R_n(x) = \sum_{j=1}^{n-1} [100(x_j^2 - x_{j+1})^2 + (x_j - 1)^2], \quad -2 \leq x_i \leq 2, \quad i = 1, 2, \dots, n,$$

taking $n = 10$,

(A.23)

one global minimum: $x_i^* \in [1.000000, 1.000000]$, $i = 1, 2, \dots, 10$, $f^* \in [0.000000, 0.000000]$.**Acknowledgments**

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