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The Statistical Mechanics of the Three-Dimensional Euclidean Black Hole

S. Carlip*
Department of Physics
University of California
Davis, CA 95616
USA

Abstract

In its formulation as a Chern-Simons theory, three-dimensional general relativity induces a Wess-Zumino-Witten action on spatial boundaries. Treating the horizon of the three-dimensional Euclidean black hole as a boundary, I count the states of the resulting WZW model, and show that when analytically continued back to Lorentzian signature, they yield the correct Bekenstein-Hawking entropy. The relevant states can be understood as "would-be gauge" degrees of freedom that become dynamical at the horizon.

* *email: carlip@dirac.ucdavis.edu*

The underlying microscopic source of black hole entropy is not yet understood, but it is natural to conjecture that it originates in quantum gravitational degrees of freedom at the black hole horizon. If this is true, however, then the (2+1)-dimensional black hole of Bañados, Teitelboim, and Zanelli [1] presents a paradox. General relativity in three spacetime dimensions can be rewritten as a Chern-Simons theory [2, 3], and as such, its degrees of freedom are fairly well understood. In particular, the small number of topological degrees of freedom of a Chern-Simons theory are not sufficient to account for the large entropy of a macroscopic BTZ black hole.

In reference [4], and independently in [5], a possible solution to this paradox was suggested. A Chern-Simons theory on a manifold with boundary induces a Wess-Zumino-Witten theory on the boundary [6,7], and this WZW model can have many more degrees of freedom than the original Chern-Simons theory. These new degrees of freedom are "would-be pure gauge" excitations that become physical at the boundary [8, 9]. Let us suppose that the event horizon of a black hole can be treated as a boundary. (I will return to this assumption later.) Then the induced WZW model at the horizon offers a natural source of microscopic degrees of freedom.

A preliminary counting argument in reference [4] indicated that these degrees of freedom can correctly account for the entropy of the BTZ black hole. That analysis was based on plausible but unproven assumptions about the quantization of the SU(1, 1) WZW model. In this paper, I consider the analytic continuation to the better-understood SL(2, **C**) WZW model obtained from Euclidean gravity in three dimensions, and demonstrate that the entropy of the three-dimensional black hole can be derived as the logarithm of the number of microscopic states at the horizon. The results are quite robust: the semiclassical contribution to the entropy is determined by Virasoro zero-modes, and is independent of the details of the rest of the Hilbert space.

1. "Would-be Gauge" Degrees of Freedom

Before proceeding with the computation, it is useful to recall the source of boundary degrees of freedom in Chern-Simons theory [10]. Consider a Chern-Simons theory on a manifold with boundary, described by the action

$$I_{\mathrm{CS}} = \frac{k}{4\pi}\int_M \mathrm{Tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) + \frac{k}{4\pi}\int_{\partial M}\mathrm{Tr}\, A_z A_{\bar z}. \tag{1.1}$$

The boundary term in (1.1) is the one appropriate for fixing the field A_z at ∂M. If M is closed, this term disappears, and $e^{iI_{\mathrm{CS}}}$ is gauge invariant. If M has a boundary, however, this invariance is broken. Indeed, under the decomposition

$$A = g^{-1}dg + g^{-1}\tilde{A}g, \tag{1.2}$$

the action becomes [8, 9]

$$I_{\mathrm{CS}}[A] = I_{\mathrm{CS}}[\tilde{A}] + kI^{+}_{\mathrm{WZW}}[g, \tilde{A}_z], \tag{1.3}$$

where $I^+_{\rm WZW}[g,\tilde{A}_z]$ is the action of a chiral WZW model on the boundary ∂M,

$$I^+_{\rm WZW}[g,\tilde{A}_z] = \frac{1}{4\pi}\int_{\partial M}\mathrm{Tr}\left(g^{-1}\partial_z g\, g^{-1}\partial_{\bar z} g - 2g^{-1}\partial_{\bar z} g\tilde{A}_z\right) + \frac{1}{12\pi}\int_M \mathrm{Tr}\left(g^{-1}dg\right)^3 . \quad (1.4)$$

The "pure gauge" degrees of freedom g are thus promoted to true dynamical degrees of freedom at the boundary.

A similar phenomenon can occur in general relativity. The infinitesimal analog of the decomposition (1.2) may be obtained by performing a transverse splitting of small fluctuations of a background metric $g_{\mu\nu}$,

$$\delta g_{\mu\nu} = h_{\mu\nu} + (K\xi)_{\mu\nu}, \quad (K^\dagger h)_\mu = 0, \qquad \text{with} \quad (K\xi)_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu = \mathcal{L}_\xi g_{\mu\nu}. \quad (1.5)$$

If M is closed, this splitting is unique [11, 12], and provides a standard division into "physical" and "gauge" degrees of freedom. If M has a boundary, however, a unique decomposition requires boundary conditions that make $K^\dagger K$ self-adjoint. The simplest choice is

$$\xi^\mu|_{\partial M} = 0. \quad (1.6)$$

Once again, the "would-be gauge" degrees of freedom $K\xi$ with $\xi^\mu \neq 0$ at ∂M are potential new dynamical degrees of freedom at the boundary. Evidence for this special role comes from the canonical formalism [5]: the generator of the transformation $g_{ij} \rightarrow g_{ij} + (K\xi)_{ij}$ on a spacelike hypersurface Σ is proportional to a constraint, and thus generates a symmetry, only when ξ^i vanishes at $\partial\Sigma$. Note, of course, that vector fields ξ in the kernel of K, the Killing vectors of $g_{\mu\nu}$, do *not* give rise to new degrees of freedom. This will be important to the later analysis.

Unfortunately, the decomposition (1.5) holds only for infinitesimal variations $\delta g_{\mu\nu}$; the finite version is highly nonlocal. The gravitational analog of the WZW action is consequently difficult to find (although see [13]). In three spacetime dimensions, however, we can avoid this difficulty: three-dimensional general relativity can be reformulated as a Chern-Simons theory in which the diffeomorphisms are transmuted into ordinary gauge transformations, and the results from Chern-Simons theory apply directly.

In particular, for Lorentzian gravity with a negative cosmological constant $\Lambda = -1/\ell^2$, we can define an $\mathrm{SU}(1,1)\times\mathrm{SU}(1,1)$ gauge field

$$A^\pm = \left(\tilde\omega^a \pm \frac{1}{\ell}\tilde{e}^a\right)\tilde{T}_a, \quad (1.7)$$

where $\tilde{e}^a = \tilde{e}^a{}_\mu dx^\mu$ is a triad and $\tilde\omega^a = \frac{1}{2}\epsilon^{abc}\tilde\omega_{\mu bc}dx^\mu$ is a spin connection. The standard Einstein action can then be written as

$$I_{\rm grav} = I_{\rm CS}[A^+] - I_{\rm CS}[A^-], \quad (1.8)$$

where $I_{\rm CS}[A]$ is the Chern-Simons action (1.1) with a coupling constant*

$$k = -\frac{\ell}{4G}. \quad (1.9)$$

*I take $\tilde{T}_0 = i\sigma_3/2$, $\tilde{T}_1 = \sigma_1/2$, $\tilde{T}_2 = \sigma_2/2$, with Tr the matrix trace. This normalization differs from that of [4]. In the notation of [15], my k is is, with s pure imaginary. Equation (1.9) can be checked by comparing the extremal action for a closed manifold in the metric and Chern-Simons formalisms.

We can now continue to Euclidean signature by setting $e^3 = i\tilde{e}^0$, $e^1 = \tilde{e}^1$, $e^2 = \tilde{e}^2$. The action then becomes

$$I_{\text{grav}} = I_{\text{CS}}[A] - I_{\text{CS}}[\bar{A}], \tag{1.10}$$

where

$$A = \left(\omega^a + \frac{i}{\ell}e^a\right)T_a, \qquad \bar{A} = \left(\omega^a - \frac{i}{\ell}e^a\right)T_a \tag{1.11}$$

is an $\mathrm{SL}(2,\mathbf{C})$ gauge field (with $T_a = -i\sigma_a/2$). Our goal is to count the boundary states in this theory for the three-dimensional black hole.

2. Quantization

$\mathrm{SL}(2,\mathbf{C})$ is a noncompact group, and the techniques developed for quantizing Chern-Simons theories with compact gauge groups require some modification. However, the action (1.10) looks tantalizingly like the difference between two SU(2) Chern-Simons actions. It is thus tempting to treat A and $\bar{A}$, and the corresponding gauge transformations g and $\bar{g}$, as independent fields, and write

$$Z_{\mathrm{SL}(2,\mathbf{C})}[\tilde{A},\bar{\tilde{A}}] = \left|Z_{\mathrm{SU}(2)}[\tilde{A}]\right|^2, \tag{2.1}$$

where Z denotes the partition function for the WZW action (1.4) on ∂M. Note that since the integrand in the partition function is $\exp\{iI\}$, the complex conjugation in (2.1) automatically leads to the difference in sign between the two terms in the action (1.10).

Witten has shown that this procedure is essentially correct [15]. If one chooses a real polarization, which in our case amounts to fixing A_z and $\bar{A}_{\bar{z}}$ at ∂M, then the dependence of a wave function on A and $\bar{A}$ is determined entirely by its dependence on ω. In particular, it is sufficient to evaluate the partition function at $e^a = 0$ and then "analytically continue." But for $e^a = 0$, the two terms in (1.10) are ordinary SU(2) Chern-Simons actions, and wave functions are basically products of two conjugate SU(2) wave functions. Hayashi has worked out the resulting $\mathrm{SL}(2,\mathbf{C})$ wave functions for the solid torus in great detail [16], and has shown explicitly that a basis can be constructed from products of holomorphic affine SU(2) Weyl-Kac characters (from the first SU(2)) and their complex conjugates (from the second).†

Note, however, that in the standard quantization of an SU(2) WZW model, and in Hayashi's computations, the coupling constant k must be a positive integer—that is, by (1.9), we must analytically continue to negative G. This sign change is identical to that described by Henningson et al. [17], who show that the partition function for an $\mathrm{SU}(1,1)$ WZW model is formally identical to an SU(2) WZW partition function analytically continued to $k < -2$. To obtain a final answer for the entropy in the Lorentzian theory,

†In references [15, 16] the focus was on Chern-Simons states, but a simple reinterpretation extends the results to the partition function. Viewed as a functional of boundary data, a Chern-Simons state on ∂M may be defined as a path integral over M, weighted by appropriate Wilson lines; the partition function on a manifold with boundary is thus formally equivalent to a particular state.

we will therefore start with the Euclidean partition function (2.1) with positive integral k, and continue to negative k at the end of the computation.

The advantage of the Euclidean approach is that the path integral for an SU(2) WZW theory is well understood. In particular, if ∂M is a two-torus with modulus $\tau = \tau_1 + i\tau_2$, the partition function $Z_{\mathrm{SU}(2)}[\tilde{A}]$ can be described as follows [6, 14, 18]. We first perform a gauge transformation to set the gauge field $\tilde{A}_z$ on ∂M to a constant value

$$a = -\frac{\pi i}{\tau_2} u\, T_3 \tag{2.2}$$

in the Cartan algebra. Then for k a positive integer,

$$Z_{\mathrm{SU}(2)}[\tilde{A}] = \sum_{n=0}^{k} \psi_{nk}(0)\psi_{nk}(a) \tag{2.3}$$

with

$$\psi_{nk}(a) = \exp\left\{\frac{\pi k}{4\tau_2}\, \bar{u}^2\right\} \bar{\chi}_{nk}(\bar{\tau}, \bar{u}), \tag{2.4}$$

where χ_{nk} are the Weyl-Kac characters for affine SU(2). Later we will need the asymptotic behavior of the characters for large τ_2:

$$\chi_{nk}(\tau, u) \sim \exp\left\{\frac{\pi i}{2}\left[\frac{(n+1)^2}{k+2} - \frac{1}{2}\right]\tau\right\} \frac{\sin \pi(n+1)u}{\sin \pi u}. \tag{2.5}$$

Our interest is not the partition function per se, but the number of states. For the partition function on a torus with modulus τ, standard WZW theory [19] tells us that

$$Z_{\mathrm{SL}(2,\mathbf{C})}(\tau)[\tilde{A}, \bar{\tilde{A}}] = \mathrm{Tr}\left\{e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0}\right\} = \sum \rho(N, \bar{N}) q_1^{N-\bar{N}} q_2^{N+\bar{N}}, \tag{2.6}$$

where $q_1 = e^{2\pi i\tau_1}$, $q_2 = e^{-2\pi\tau_2}$, and $\rho(N, \bar{N})$ is the number of states for which the Virasoro generators L_0 and $\bar{L}_0$ have eigenvalues N and $\bar{N}$. This number can be extracted from (2.6) by a standard contour integral:

$$\rho(N, \bar{N}) = -\frac{1}{4\pi^2}\int \frac{dq_1}{q_1^{N-\bar{N}+1}} \int \frac{dq_2}{q_2^{N+\bar{N}+1}} Z_{\mathrm{SL}(2,\mathbf{C})}(\tau)[\tilde{A}, \bar{\tilde{A}}], \tag{2.7}$$

where the integrals are along circles surrounding the origin in the complex q_1 and q_2 planes.

3. The Euclidean Black Hole

We are now ready to count the states of the three-dimensional Euclidean black hole. Note first that not all states on the black hole horizon are physical. As we saw above, the diffeomorphisms generated by Killing vectors—vectors in the kernel of K—remain genuine gauge symmetries even at a boundary. For the BTZ black hole, Killing vectors generate time translations and rotations, and the corresponding requirement on states is that

$$L_0|\text{phys}\!>\,=\bar{L}_0|\text{phys}\!>\,=0, \tag{3.1}$$

since the Virasoro operators L_0 and $\bar{L}_0$ generate the rigid displacements. Equation (3.1) can be viewed as a remnant of the Wheeler-DeWitt equation. The number of states at the horizon is thus given by $\rho(0,0)$.

We next need the boundary fields $\tilde{A}$ and $\bar{\tilde{A}}$. For this purpose, the black hole metric is most conveniently expressed in an upper half-space form [20],

$$ds^2 = \frac{\ell^2}{R^2\sin^2\chi}\left[dR^2 + R^2 d\chi^2 + R^2\cos^2\chi d\theta^2\right], \tag{3.2}$$

with the identifications

$$(\ln R, \theta, \chi) \sim (\ln R, \theta+\Theta, \chi) \sim (\ln R + \frac{2\pi r_+}{\ell}, \theta + \frac{2\pi|r_-|}{\ell}, \chi). \tag{3.3}$$

Here $r_\pm$ are the Euclidean continuations of the radii of the outer and inner horizons, and $2\pi-\Theta$ is the deficit angle of the conical singularity at the (Euclidean) horizon; the on-shell condition is $\Theta = 2\pi$. The relationship of the coordinates (R,θ,χ) to standard Schwarzschild coordinates is described in [20]. For our purposes, we need only know that χ is related to the usual radial coordinate, and that a surface $\chi = \text{const.}$ is a torus (the two circumferences are a circle around the horizon and a circle in periodic time); the horizon is the degenerate surface $\chi = \pi/2$.

The connection A^a corresponding to the metric (3.2) is easily found to be

$$A^1 = -\csc\chi(d\theta - i\frac{dR}{R}), \qquad A^2 = i\csc\chi d\chi, \qquad A^3 = i\cot\chi(d\theta - i\frac{dR}{R}). \tag{3.4}$$

When restricted to a "stretched horizon" $\chi=\chi_0$, A is conjugate to $(d\theta - i\frac{dR}{R})T_3$, independent of χ_0. To use this boundary data in the partition function, we must express this connection as $a(dx+\tau dy)$, where x and y are coordinates on the torus with period one. Using the identifications (3.3), or equivalently rewriting a in terms of the holonomies of $\tilde{A}$, we obtain

$$a = -\frac{\pi}{\tau_2}\left[\left(\frac{\Theta\tau_2}{2\pi} + \frac{r_+}{\ell}\right) + i\left(\frac{\Theta\tau_1}{2\pi} - \frac{|r_-|}{\ell}\right)\right]T_3. \tag{3.5}$$

For k a positive integer, we can now insert this expression, along with the asymptotic form (2.5) of the Weyl-Kac characters, into (2.4) to compute the partition function $Z_{\mathrm{SL}(2,\mathbf{C})}$. The integral (2.7) may then be evaluated by steepest descent. Consider, for example, the contribution from $n=0$ in (2.3). The corresponding term in the partition function is

$$\begin{aligned} Z_{\mathrm{SL}(2,\mathbf{C})}[a,\bar{a}] &\approx \exp\left\{\frac{\pi k}{4\tau_2}\left(u^2+\bar{u}^2\right) + \frac{\pi k}{k+2}\tau_2\right\} \\ &= \exp\left\{-\frac{\pi k}{2\tau_2}\left[\left(\frac{\Theta\tau_2}{2\pi}+\frac{r_+}{\ell}\right)^2 - \left(\frac{\Theta\tau_1}{2\pi}-\frac{|r_-|}{\ell}\right)^2\right] + \frac{\pi k}{k+2}\tau_2\right\}. \end{aligned} \tag{3.6}$$

The steepest descent approximation of integral (2.7) for $N = \bar{N} = 0$ then gives

$$\rho(0,0) = \exp\left\{-\frac{k\Theta r_+}{\ell} + \left(\frac{2\pi}{\Theta}\right)\frac{\pi r_+}{\ell}\right\} = \exp\left\{\frac{\Theta r_+}{4G} + \left(\frac{2\pi}{\Theta}\right)\frac{\pi r_+}{\ell}\right\} \tag{3.7}$$

up to terms of order $1/k$. Note that the relevant saddle point occurs at $\tau_2 = 2\pi r_+/\Theta\ell$, so the approximation (2.5) is justified as long as the black hole is large, $M/G = 8r_+^2/\ell^2 \gg 1$. A straightforward calculation also shows that the contributions to $\rho(0,0)$ coming from terms in (2.3) with $n \neq 0$ are exponentially suppressed relative to (3.7).

(There is a sign ambiguity here: in the identifications (3.3), we could have taken r_+ and $|r_-|$ to be negative, corresponding to a different fundamental region. The relevant saddle point would then be $\tau_2 = -2\pi r_+/\Theta\ell$, and the first term in (3.7) would no longer appear. For this choice, however, it may be shown that the contribution coming from $n = k$ in (2.3) reproduces (3.7), with Θ replaced by $4\pi - \Theta$.)

The first term in the exponent of (3.7) is the correct semiclassical expression for the entropy of the (2+1)-dimensional black hole. The second term is a one-loop correction. This one-loop expression differs from that of reference [20] by a factor of two, but I believe it is correct; the expression in [20] was based on a computation of determinants in reference [21] (eqn. A16) which, I believe, has an incorrect factor of two in the exponent.

The Euclidean computation has been carried out for k a positive integer, implying that $G<0$. As noted above, however, the analytic continuation to Lorentzian signature requires a change of the sign of k. It might be possible to repeat this computation directly in the Lorentzian theory, using the results of reference [17] for the $\mathrm{SU}(1,1)$ partition function. In fact, though, the semiclassical contribution to the entropy is largely independent of the detailed form of the characters $\chi_{nk}(\tau,u)$. The leading contribution to S comes from the prefactor

$$Z_0(\tau,u) = \exp\left\{\frac{\pi k}{4\tau_2}\left(u^2 + \bar{u}^2\right)\right\} \tag{3.8}$$

in the partition function (3.6). This term may be understood as follows. In a Chern-Simons theory on a manifold with boundary, the WZW current $J(z)$ at the boundary is proportional to the gauge field A_z [22]. By (1.2), this field contains the usual current $g^{-1}\partial g$, but it also has an added zero-mode contribution $\tilde{A}_z$, whose value is determined by the boundary data. The Virasoro generator L_0 has a corresponding zero-mode term proportional to $\mathrm{Tr}\tilde{A}_z\tilde{A}_z$, and the prefactor (3.8) is precisely the contribution of this zero-mode to the partition function. As originally argued in reference [4], it is this zero-mode that determines the dependence of the black hole entropy on the horizon size.

This means that although the entropy counts microscopic states at the horizon, its semiclassical value will not depend sensitively on the quantum theory describing those states, as long as the current zero-modes are fixed. For example, although we do not know the detailed form of the characters corresponding to $\chi_{nk}(\tau,u)$ for k nonintegral, and we do not fully understand the Lorentzian theory, the semiclassical expression for black hole entropy should not be affected: provided the partition function has the form

$Z_0(\tau,u)F(\tau,u)$ with $F \sim 1$ for large τ_2, we will obtain the correct Bekenstein-Hawking entropy.[‡]

Finally, let me return to the question of whether it is sensible to treat a black hole horizon as a boundary. The horizon is not, of course, a physical boundary. It is, however, a place at which one must impose "boundary conditions" in quantum gravity. Statements about black holes in quantum gravity are necessarily statements about conditional probabilities: for instance, "If a black hole with given characteristics is present, then one will observe a certain spectrum of Hawking radiation." To compute such probabilities, one must include the appropriate restrictions on the path integral, by restricting the admissible boundary data at the horizon. Such restrictions are sufficient to generate a WZW action at the horizon, and thus to justify the computations of this paper [10].

4. Conclusion

We have seen that the Chern-Simons formulation of three-dimensional Euclidean gravity permits an explicit description of horizon degrees of freedom, and that these degrees of freedom can provide a microscopic explanation for the entropy of the black hole. The obvious question is whether these results can be generalized to four dimensions. The particular methods described here certainly cannot. The key advantage of the Chern-Simons formalism is that it allows diffeomorphisms to be expressed as local gauge transformations, permitting the decomposition (1.2) and the exact derivation of a boundary action. No such formulation is known in 3+1 dimensions.

Nevertheless, the basic physical mechanism discussed here should generalize to 3+1 dimensions. The canonical formulation of general relativity offers strong evidence for the existence of "would-be gauge" degrees of freedom that can become dynamical at a boundary [5], and some progress has been made towards finding the corresponding boundary action [13, 23]. While much work remains, the approach developed here provides a promising direction for understanding the origin of black hole entropy.

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