An Artificial Financial Market with Herding in Heterogeneous Expectations: Parameter Analysis and Calibration*

Preliminary version

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Abstract

Heterogeneous agent-based financial market models address the behavioural origins of stylized facts in asset returns. However, since these models are often highly parameterized, it is important not only to reproduce the stylized facts, but also to demonstrate which behavioural features affect which stylized fact. For this reason, employing our model of an artificial financial market populated with fundamentalists and mean-variance investors, we examine econometrically the relation between the model parameters and statistical characteristics of the resulting outputs in our agent-based model. Using regression models we calibrate the parameters by minimizing appropriately chosen distances between the empirical characteristics of real-life data and their counterparts quantified by the estimated regression relations. This turns out to be a simple and effective way to calibrate the model parameters. After calibration we make a comparison of the calibrated parameter values and the values used in our model.

Keywords: artificial financial market, heterogeneous expectations, herding behaviour, stylized facts, parameter analysis, calibration.

JEL classification: G 12, C 63.

1 Introduction

Heterogeneous agent-based financial market models (see surveys by Hommes (2006), LeBaron (2006), Lux (2009)) are successful in generating financial time series exhibiting stylized facts, which are common across markets and time periods (Pagan (1996), Cont (2001)). The agent-based modelling approach emphasizes the interaction of possibly boundedly rational agents instead of the optimal choices made by representative agents. The central feature of these models is heterogeneity among agents. This heterogeneity drives the internal dynamics of artificial financial markets. One can distinguish two streams of literature on agent-based modeling of financial markets. One class

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of models employs stylized so-called few-type models that are analytically tractable. This stream includes works of Brock and Hommes (1997, 1998), Chiarella and He (2002), Chiarella et al. (2006a,b), He and Li (2007, 2008), Lux (1998), Lux and Marchesi (2000). They generally consider only a few types of agents in the market, typically fundamentalists, who believe in the existence of a fundamental value, and trend chasers, who try to exploit the most recent trends in prices. These agents adapt their beliefs over time and can sometimes switch type. The endogenous switching can be based on portfolio performance measures or the prevailing majority opinion about the market. As a result, these models of financial markets are able to generate stylized facts and allow for studying behavioural factors as potential drivers of market macro dynamics.

Another stream of literature uses models (see, for example, LeBaron et al. (1999), Levy et al. (2000), LeBaron and Yamamoto (2007), Adriaens (2008), LeBaron (2010)) which are more complex in terms of agents heterogeneity and interactions, compared to the other stream. The financial markets are modeled in such a way that economic agents differ in many aspects, such as, for example, the agents' information sets, ways of processing the available information, or adaptive behaviour. Agents can choose a preferred strategy and can also learn, i.e., have an ability to update and develop their behaviour for investments by switching to a better performing forecasting rule.

In this paper we investigate an artificial financial market with agents of two types distinguished depending on the beliefs about the predictability of asset prices. We consider fundamentalists, who believe that asset prices are predictable and adjust the demand according to market fundamentals, and mean-variance investors, who do not believe that portfolio choice can be based on the predictability of asset prices but instead diversify the portfolio (as mean-variance optimizers, see Markowitz (1959)).

The agents have heterogeneous expectations about the market, which are characterized by a set of different forecasting rules and a different number of observations of data used to make the forecasts. In addition, we implement endogenous switching among different forecasting rules rather than switching among different strategies such as fundamentalists and chartists. We model two types of rule switching behaviour. First, agents choose the forecasting rule after evaluating the forecasting performance of each rule. Agents can change their expectations about asset prices and returns, if they observe that their predictions were performing poorly. Each agent evaluates this measure individually and can learn to forecast better by observing which rule gives more accurate forecasts. Second, in some trading periods agents also observe the majority opinion about the forecasting rules in the market, which is measured in terms of the opinion index (Lux, 1995, 1998, Lux and Marchesi, 2000). Agents also consider the latter in addition to individually evaluated forecasting performance of each rule. This can give rise to herding behaviour.
We include herding as one of the behavioural factors driving the dynamics of financial market. Herding in financial markets is defined as the behaviour of investors and fund managers imitating each other and engaging in risk taking without taking into account adequate risk-reward trade-offs (Bikhchandani and Sharma, 2001). There is strong evidence (see, for example, Menkhoff et al. (2006), Walter and Weber (2006), Demirera et al. (2010)) that investors are more eager to seek for advice and support in situations with great uncertainty, lack of information, or search and selection costs. A number of studies employing heterogeneous agent-based models have included herding behaviour originating among artificial agents (see, for example, Lux and Marchesi (2000), Alfarano et al. (2010), Yamamoto (2011), Wagner (2003)). Our paper also fits in this group of studies.

With our model of an artificial financial market we combine aspects of the two streams of literature on heterogeneous agent-based models. We mainly follow more computationally oriented models but also move closer to the simpler more analytically tractable so-called few-type models by incorporating some features such as defining types of agents and letting them observe the opinion index in the market. Our market with rule switching mechanisms is able to generate stylized facts in financial time series. The rule switching based on learning and herding gives rise to complex behaviour among both the fundamentalists and the mean-variance investors.

However, these models include a large number of parameters describing the agents’ behaviour in the market and the market design. In applications it is desirable not only to be able to reproduce the outputs sharing distributional properties of actual data but also to demonstrate how and which model parameters affect the results. The literature concentrating on this matter started only quite recently. The difficulties of investigating these models from an econometric point of view have been discussed in Chen et al. (2012). These difficulties include analytically intractable objective functions which are hard to apply directly using methods such as maximum likelihood. In such cases so-called indirect estimation is performed, one example of which is method of simulated moments. Winker et al. (2007) estimate a subset of two parameters in Kirman (1993) and a subset of three parameters in Lux and Marchesi (2000) using the method of simulated moments, by minimizing the distance between the moments, such as the the sum of ARCH- and GARCH-parameters, of actual and simulated data. Still, the method of simulated moments approach typically leads to very complicated optimization problems, especially, when a larger number of parameters is considered. Other works on estimating agent-based models of financial markets include Lux (2012), Amilon (2008), Franke (2009), Alfarano et al. (2008), mostly concentrating on relatively simple models or taking into account only two or three parameters.

With too many parameters these models might even become under-identified, meaning that
data, typically used to estimate, is not sufficient to distinguish between different parameter values. Even if it turns out that the model is identified, traditional econometric methods are not feasible due to very complicated objective functions. Another way to approach the problem of finding reliable parameter values is to calibrate parameters by confronting the properties of real-life data and those of model-generated financial series. Calibration examples can be found in Li et al. (2010, 2006), Bianchi et al. (2007). Our approach is aimed at dealing with more complex model by analyzing a larger number of model parameters.

Li et al. (2010) propose systematic procedures to compare two microscopic simulation models and to make a comparison with real-life data. They calibrate the model in He and Li (2007, 2008) by minimizing the average distance between the autocorrelations in asset returns of actual data and the corresponding autocorrelations in asset returns of the model output. Afterwards they make a comparison of the outputs in the calibrated model and real-life data by inspecting whether the model is able to reproduce the autocorrelations and volatility patterns of actual asset returns.

We approach the calibration of the model parameters in a somewhat different way. Differently from previous studies, we estimate the relations between the model parameters and selected statistical properties of the simulated asset returns. For this we use linear regression analysis and simulation results of the multiple runs of our market. Each separate run is done with the input of different model parameter values. We analyze a subset of seven parameters in our model. After each run the relevant statistical properties, which are typically used to check the reproduction of stylized facts in model output, are measured. We use univariate kernel regressions to have a first visual inspection of the relations between the model parameters and these statistics. The calibration of our model parameters relies on the results of the multivariate linear regressions of the statistical characteristics of real-life data. We calibrate the parameter values by minimizing the distance between the statistical properties of real-life data and the counterparts quantified by the linear regression relation of the model parameters and statistical properties of simulated data in our model. We use kernel estimates of density functions to make a comparison between the calibrated values and our values of the model parameters. This analysis helps to get a better understanding of how model parameters affect the reproduction of the stylized facts and allows to draw conclusions about the validity of the parameter values in our model.

The remainder of the paper is organized as follows. Section 2 describes a model of the financial market with heterogeneous agents. In Section 3 we present the simulation results. Section 4 explains calibration procedure and discusses the results. Section 5 concludes.

1Our analysis is done by formalizing preliminary attempts in Tajuddin (2011).
2 The model

In this section we model a financial market in which agents determine their demand for assets, issued by the firms. The equilibrium in every period is set by a market maker who computes a temporary Walrasian equilibrium (see Grandmont, 1988) and a regulator who controls the orderly functioning of the financial market. We begin by describing the assets traded in the market, then we introduce the heterogeneous agents, proceeding with the definition of the market equilibrium and the rule switching mechanisms.

2.1 Assets

There are $J + 1$ assets, numbered 0 to $J$, and characterized by dividend payoffs $D_{jt}$ and prices $S_{jt}$ in period $t$. The numéraire asset has price $S_{0t} = 1$ and dividend payoffs $D_{0t} = 0$ in each period $t$. In modeling the dividend process, we follow, among others, LeBaron (2001), Levy et al. (2000), and He and Li (2007, 2008), and assume it is exogenously given. In particular, we follow LeBaron (2002a) in the dividend rule specification. These dividend processes are assumed to be independent across firms. We use the following dividend payment rule:

$$\log(D_{jt+1}) = c_{Dj} + \log(D_{jt}) + \varepsilon_{D,j,t+1},$$

with $j = 1, \ldots, J$, $c_{Dj}$ the growth rate of log dividends of asset $j$, and $\varepsilon_{D,j,t+1}$ the noise term.

The net return of asset $j$ over period $t$ is calculated by

$$R_{jt} = \frac{S_{jt} + D_{jt} - S_{j,t-1}}{S_{j,t-1}},$$

We assume that dividend payoffs in period $t$, $D_{jt}$, will only be observed after trading of assets in period $t$ has been completed.

2.2 Agents

The financial market is populated with $I$ agents who hold and trade the available assets. Agents use a model to forecast prices, dividends, or returns and base their investment decisions on those forecasts. To estimate their model, they may use the available data which they have observed and memorized in the past. The number of periods of data in the memory of agent $i$ is denoted by $M_i$. We denote $h_{ijt}$ as agent’s $i$ holdings (measured in numbers of assets) of asset $j$ in time period $t$, and $u_{ijt}$ as the value of agent’s $i$ portfolio holdings of asset $j$ in terms of the numéraire. The
wealth of agent \( i \) at the beginning of time period \( t \) is given by \( W_{it} = S_{i,t-1} \).

We model the financial market with two types of agents, a fundamentalist type and a mean-variance (MV) investor. Next, we briefly present their preferences, rules to forecast relevant statistics, and determine portfolio holdings.

**The Fundamentalists.** We assume the fundamentalists believe that market prices are predictable and depend on market fundamentals, such as dividend earnings. They discount expected dividends over future periods of each asset \( j \) independently using the discount rate \( r_{Dj}^* \) in the following way:

\[
F_{ijt} = \mathbb{E}_{it} \left( \sum_{\tau=1}^{\infty} \frac{D_{j,t+\tau}}{(1 + r_{Dj}^*)^\tau} \right),
\]

where \( \mathbb{E}_{it} (\cdot) \) denotes the expectations operator according to the model that agent \( i \) uses for asset \( j \) and using all information available up to and including time \( t \). Under appropriate regularity conditions we can simplify to:

\[
F_{ijt} = \frac{\mathbb{E}_{it}(D_{j,t+1})}{r_{Dj}}.
\]

We set \( r_{Dj} = \frac{r_{Dj}^* - g_j}{1 + g_j} \), with \( g_j \) the expected growth rate of the dividend of asset \( j \) and \( r_{Dj}^* \) the expected return on the asset, in line with the dividend model of Gordon (1962). With \( r_{Dj} = \frac{1 - \beta}{\beta} \), and \( \beta \) the fraction of wealth invested in stocks at every time period, this fundamental price is equivalent with the homogeneous agent equilibrium price in LeBaron (2002a). In our model the fundamentalists trade according to the difference between the perceived fundamental price, \( F_{ijt} \), and the past period price, \( S_{j,t-1} \) (cf., for instance, Chiarella et al. (2006a) and He and Li (2007)). We assume that fundamentalists increase their demand when they believe the asset is underpriced and reduce their demand in the opposite case.

For computing a portfolio adjustment rule we want a functional form that is flexible enough for calibration of the financial market and exhibits the following characteristics. The portfolio adjustment rule for the fundamentalist has to take into account portfolio holdings in the previous period, \( u_{i,j,t-1} \), the direction in which prices were moving compared to the fundamental price, \( F_{ijt} - S_{j,t-1} \), also how much prices changed relative to perceived fundamental price, \( F_{ijt} \).

We choose a specific functional form which satisfies the criteria above. We postulate that the fundamentalists adapt their portfolio at each time period \( t \) in the following way:

\[
\begin{align*}
    u_{i,j,t} & = u_{i,j,t-1} + \alpha_i \text{sgn}(F_{ijt} - S_{j,t-1}) \left( \frac{|F_{ijt} - S_{j,t-1}|}{F_{ijt}} \right)^{\theta_i} F_{ijt}, \\
\end{align*}
\]

where \( \alpha_i > 0 \) is the adjustment speed, \( \theta_i > 0 \) the responsiveness to differences between the funda-
mental price and the price of the previous period, and \( \text{sgn}(\cdot) \) the sign function. The adjustment speed, \( \alpha_i \), is a combined measure of risk tolerance, the agent’s ‘sensitivity’ to perceived mispricing, and confidence (trust) in market fundamentals (Chiarella et al. (2006a), He and Li (2007)). The agent’s responsiveness to differences in prices, \( \theta_i \), is used to implement a nonlinear way of response for \( \theta_i \neq 1 \). If \( \theta_i \) increases and \( |F_{ijt} - S_{j,t-1}| / F_{ijt} < 1 \), the portfolio adjustment will decrease. If the asset is perceived to be underpriced \( (F_{ijt} > S_{j,t-1}) \), \( \text{sgn}(\cdot) \) will be positive and the agent increases her demand for the asset. The size of the adjustment depends positively on the difference between the perceived fundamental price \( F_{ijt} \) and the previous period price, \( S_{j,t-1} \), through the relative price difference. What is more, the adjustment is proportional to \( F_{ijt} \).

**Mean-Variance Investors.** The second type of traders in the market are the mean-variance (MV) investors. These agents do not believe that asset prices are predictable and, thus, diversify their portfolio. We assume that MV investors buy the perceived one period ahead mean-variance efficient portfolio (Markowitz, 1959), found by solving the following optimal investment problem:

\[
\max_u \quad \mathbb{E}_t(W_{i,t+1} - \gamma_{it} \text{Var}_t(W_{i,t+1})) \\
\text{s.t.} \quad W_{i,t+1} = \mathbf{t}'_{t+1} u_{i,t} \\
\text{and} \quad \mathbf{t}' u_{i,t} = W_{i,t},
\]

where \( \text{Var}_t(\cdot) \) denotes the variance operator of agent \( i \) using the available information up to and including time \( t \); \( W_{i,t+1} \) is invested wealth. \( \gamma_{it} \) is the time varying risk aversion parameter of agent \( i \) in time period \( t \). The agent wants an optimal combination of expected return and risk at the end of the planning horizon. This results in the demand for assets in terms of the numéraire:

\[
\tilde{u}_{it} = \frac{1}{2 \gamma_{it}} (\text{Cov}_{it}(R_{t+1}))^{-1} \mathbb{E}_t(R_{t+1}), \quad (2)
\]

\[
u_{it} = W_{it} - \mathbf{t}'_{t} \tilde{u}_{it}.
\]

where \( \tilde{u}_{it} \) is a \((J \times 1)\) vector of portfolio holdings of risky assets by investor \( i \) in time period \( t \), and \( \mathbf{t} \) is a vector of ones.

The MV investors observe that the total wealth in the market is growing due to dividend payments. Furthermore, due to the trend in dividends, the predictions of the fundamental price by the fundamentalists will contain an exponential trend. This, ignoring other influences for the moment, in general leads to an increase of fundamentalists’ investments in the risky assets. We assume that the MV investors adjust their risk aversion in order to keep a similar market share.
of risky assets as from the start of the simulation, reducing it each period to
\[ \gamma_t = \frac{\gamma_{t-1}}{1 + c_D}, \]
where \( c_D \) is the growth rate of dividends. By becoming more risk averse the MV agents stay in the market and can keep up with the fundamentalists who might otherwise gradually take over the market.

2.3 Market Equilibrium

We compute a temporary Walrasian equilibrium at every time period. At time \( t \), the total supply of risky assets is given by \( \sum_{i=1}^{I} \tilde{h}_{i,j,t-1} \), i.e., the sum of portfolio holdings of risky asset \( j \) by all agents \( I \) in the market at the beginning of time period \( t \). The total demand for risky asset \( j \) in time \( t \) in terms of the numéraire is given by \( \sum_{i=1}^{I} \tilde{u}_{i,j,t} \), where \( \tilde{u}_{i,j,t} \) depends on agent’s expectation about the current prices and returns (see equations (1) and (2)) and does not depend on current prices \( S_{jt} \). Hence, the equilibrium price in risky asset market \( j \) is set when market maker equalizes asset demand and supply:

\[ S_{jt} = \frac{\sum_{i=1}^{I} \tilde{u}_{i,j,t}}{\sum_{i=1}^{I} \tilde{h}_{i,j,t-1}}. \]

We model the market maker such that he proposes this temporary Walrasian equilibrium price to a regulator. The regulator observes the equilibrium price and can take actions if the market maker suggests prices that would make the market too volatile or even crash. After prices \( S_{jt} \) have been proposed by the market maker, the regulator controls the price difference by limiting returns within the interval of \( \lambda \) standard deviations, \( \lambda \sigma^\tau_{jt} \), with \( \sigma^\tau_{jt} \) computed in time period \( t \) over returns of the \( \tau \) most recent periods of trading.

2.4 Rule Switching and Herding

Agents in this financial market are able to choose from a set of rules for forecasting the relevant quantities, such as the expected return or price. We endow each type of agent with two forecasting rules. The rules that agents choose may be different in each period of time. Below we describe how rules are evaluated. We, also, present the rule switching mechanism and describe the forecasting rules available to each type of agents.

**Rule performance.** In every time period agents evaluate the performance of the available rules before choosing one which they use to form the forecasts. The performance is measured in terms
of the Exponentially Weighted Mean Squared Prediction Error (EWMSPE), following LeBaron (2002b). Since the fundamentalist and MV investor forecast different quantities, we define the prediction error for MV investors and the fundamentalists separately:

\[
\text{MV: } v^l_{i,j,t} = (1 - \delta_i) v^l_{i,j,t-1} + \delta_i (R_{jt} - \mathcal{E}^l_{i,t-1}(R_{jt}))^2 \tag{4a}
\]

\[
\text{Fund: } v^l_{i,j,t} = (1 - \delta_i) v^l_{i,j,t-1} + \delta_i (S_{jt} - F_{ij,t})^2 \tag{4b}
\]

where \(\delta_i\) is a weighting parameter, also denoting different degrees of agent’s myopia when evaluating forecasting accuracy. \(\mathcal{E}^l_{i,t-1}(R_{jt})\) denotes the return on asset \(j\) as predicted by agent \(i\) using forecasting rule \(l\) in time \(t-1\). Every period, agent \(i\) computes the performance of each prediction rule \(l \in L\).

A relatively high rule performance error (EWMSPE) indicates that a rule forecasts poorly, possibly leading to suboptimal investment decisions. After each trading period before forming a demand for risky assets for the next trading period agents can learn about the prediction accuracy of each forecasting rule. Next, we will introduce the forecasting rules available for each type of agents in this artificial financial market.

**Rule switching and herding.** Every trading period each agent decides to use one of the two forecasting rules by minimizing the prediction error encountered when using the specific rule (equations (4a) and (4b)). This way, rule switching can be considered as a rational change of expectations about the market. In addition, agents might herd in their choices over the forecasting rules, i.e., they switch between two forecasting rules based on two factors, namely, rule performance measure and the observations of the majority opinion in the market. The majority opinion is measured in terms of the relative fraction of total number of agents employing one or the other rule in the previous trading period.

In implementing this mechanism, we follow closely Lux and Marchesi (2000). They model the endogenous switching of agents investing in the market according to the prevailing mood among chartists and fundamentalists. Following Lux and Marchesi (2000) we define the **opinion index** among a type of investors (for the fundamentalists and for the MV investors separately) with the forecasting rules \(l_1, l_2 \in \mathcal{L}_i\) (available to agent \(i\)) in the following way:

\[
x_{jt} = \frac{n^l_{1jt} - n^l_{2jt}}{n^l_{1jt} + n^l_{2jt}} \tag{5}
\]

where \(n^l_{1jt}\) (\(n^l_{2jt}\)) refers to the number of agents applying rule \(l_1\) (\(l_2\)) to predict asset returns, scaled
by $n_{jt}^{l_1 + l_2}$, the total number of agents applying one or another rule, for every asset $j$ in time period $t$.

We assume that an agent switches to another forecasting rule with a probability which is driven by two factors, the forecasting performance measure of the rule and the fractions of agents using a specific rule in the market. We construct transition probabilities for every agent $i$ in the following way:

$$
\pi_{ij}^{l_1} = \eta_{ijt} \exp \left( +U_{ij}^{l_1} \right) \tag{6a}
$$

$$
\pi_{ij}^{l_2} = \eta_{ijt} \exp \left( -U_{ij}^{l_1} \right) \tag{6b}
$$

where

$$
U_{ij}^{l_1} = \psi_{i_1} x_{j,t-1} + \psi_{i_2} \left( v_{i_2}^{l_1} - v_{i_1}^{l_1} \right) \tag{7}
$$

denotes the utility function of agent $i$ for asset $j$ when using forecasting rule $l_1$. $v_{i_1}^{l_1}$ ($v_{i_2}^{l_1}$) are defined either by equation (4a) in MV investor’s case or by equation (4b) in the fundamentalist’s case. The parameters $\psi_{i_1} > 0$ and $\psi_{i_2} > 0$ represent the strength of each factor determining transition probabilities; $\eta_{ijt}$ is a normalization parameter such that $\pi_{ij}^{l_1} + \pi_{ij}^{l_2} = 1$.

In addition, we assume that agents are influenced by the opinion index with probability equal to $\omega$. In periods when they cannot take into account the majority opinion in the market, they choose the forecasting rule by minimizing the prediction error.

**Forecasting rules.** We endow each agent with two possible forecasting rules. The fundamentalists can choose from the *Last dividend rule* and the *Dividend trend prediction rule*. When using the former rule, the fundamentalist predicts $E_{i,t-1}(D_{jt})$, the expected dividend payment in time $t$, of asset $j$, using the last dividend known, $D_{j,t-1}$. This forecasting rule is one of the simplest methods of predicting the non-stationary dividend process. With the latter rule the fundamentalist uses the following model for asset $j$ to predict the expected dividend payment, $E_{i,t-1}(D_{jt})$:

$$
E_{i,t-1}(D_{jt}) = \exp(\hat{c}_{Dj} + \log(D_{j,t-1}))
$$

where $\hat{c}_{Dj}$ denotes the estimate of $c_{Dj}$. Fundamentalists estimate $c_{Dj}$ using the sample average of the first order differences in log dividends from:

$$
\log(D_{jt}) = c_{Dj} + \log(D_{j,\tau-1}) + \varepsilon_{D,j,\tau}, \quad \tau = t - M_i + 1, \ldots, t - 1,
$$
with $\mathbb{E}(\varepsilon_{D,j,\tau}) = 0$. This is a more advanced method of predicting the non-stationary dividend process.

MV investors predict future returns, $E_{it}(R_{t+1})$, with the Average rule or the Median rule. With the former forecasting rule an MV investor forecasts the next period’s return using the average return over the time periods in an agent’s memory. When using the latter rule, an MV investor predicts $E_{it}(R_{t+1})$ using the median of the return observations in the memory. This might be a more robust forecast rule for the mean return than the sample average rule in case there are outliers in the returns.

3 Simulation results

In this section, we, first, present parameter values for running the experiments with the artificial financial market. Then, we discuss the simulation results.

3.1 Parameter setting

We set $I = 200$ agents in the financial market, half of them characterized as fundamentalists, the other half as MV investors. They trade $J = 3$ risky assets in addition to the risk-free asset. Each agent has a different memory length which is randomly drawn from a uniform distribution with support 100 to 800 periods. After the equilibrium price is determined, a regulator checks it by making sure that realized return lies within $\lambda = 4$ standard deviations computed over the returns of the $\tau = 2000$ most recent periods.

The dividend processes is modelled on a weekly basis. Following LeBaron (2002a) we use $c_D = 0.0252$ for the growth rate of log dividends $c_{D,j}$ for all assets. We assume that $\varepsilon_{D,j,t+1}$, the noise term of the log dividend process, is iid $N\left(0, 0.0252^2\right)$ distributed. We further set the discount factor of fundamentalists, $r_{D,j}$, equal to $0.0252$, the same as the growth rate of log dividends $c_D$, for all assets. The starting value of the dividend processes of all risky assets is also set equal to $c_D$.

We set parameters for agents in the following way. In equation (1) we choose the parameters $\alpha_i = 3.9$ and $\theta_i = 1.1$ for all $i$. The market is, thus, able to generate higher kurtosis with sufficiently low autocorrelations in asset returns. Furthermore, we set $\delta_i = 1/75$ (following LeBaron (1999)) for the rule performance measures in (4a) and (4b). For the herding mechanism we set the parameters $\psi_{i1} = 0.7$ and $\psi_{i2} = 10^5$ in the MV investor’s case, while $\psi_{i1} = 0.7$ and $\psi_{i2} = 10^3$ in the fundamentalist’s case.\footnote{The parameters $\psi_{i2}$ are chosen such that the values of the prediction error are more comparable with the values of the opinion index in transition probabilities' setting in (6a) and (6b).} These parameters are chosen such that the market is able to generate
higher volatility, volatility clustering, and at the same time low autocorrelations in asset returns. The probability to observe the opinion index is set to $\omega = 0.8$.

We run each simulation for 5000 periods, selecting the last 1000 (which is about 20 years) to be presented here. The burn-in period of the 4000 periods is used to let the economy start up and wash out the effects of the starting information which was not generated by the economy.\(^3\)

### 3.2 Simulations

#### 3.2.1 The characteristics of simulated asset returns

In Table 1 we record the sample average and sample covariance matrix of returns in percentages. In addition, we measure kurtosis whose value is higher than 3 for all assets indicating excess kurtosis compared to the normal distribution.

In Table 1 we observe that for example asset 1 has an average excess return of 0.131% per week or 6.812% on a yearly basis, while the annualized volatility is 23.42%. We also notice that the average excess returns of all assets are somewhat similar but the variation of asset 1 returns is nearly twice higher than in case of asset 2 or asset 3. This shows that the three assets are quite different among each other.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Mean (%)</th>
<th>Covariance matrix (%)</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.131</td>
<td>10.549 -0.030 0.428</td>
<td>4.827</td>
</tr>
<tr>
<td>2</td>
<td>0.092</td>
<td>-0.030 5.813 0.206</td>
<td>5.805</td>
</tr>
<tr>
<td>3</td>
<td>0.100</td>
<td>0.428 0.206 4.699</td>
<td>4.076</td>
</tr>
</tbody>
</table>

Table 1: Summary statistics of asset returns.

Figure 2 panel a shows simulated prices and returns for each asset over time. Clearly, this artificial financial market is able to generate volatility clusters in asset returns. Figure 2 panel b shows the autocorrelations of returns, squared returns, and absolute returns for up to 50 lags. The dashed lines indicate the 5% critical values for testing whether the autocorrelations are insignificantly different from zero. We observe that the market generates only a few significant lags of autocorrelations in the asset returns (mostly for the first lags). In cases of asset 1 and 2 we find significant autocorrelations of squared and absolute returns. However, the patterns in all graphs suggest that slow decay of autocorrelations in absolute and squared returns is not present in our results.

\(^3\)We simulate the dividend process and compute prices from the dividends using equation to generate the initial information. The asset returns are computed from the prices and dividends using equation, but we add normally distributed noise with standard deviation 0.01. We calibrate the total supply of assets in the first period of the simulation to align the simulation data with the generated initial data.
Figure 1: Asset prices and returns: prices and returns (in %) of the economy with herding and rule switching; assets in rows, prices in column 1, returns in column 2.

Figure 2: Asset prices and returns (continued): autocorrelations of the economy with herding and rule switching; assets in rows, return in column 1, squared returns in column 2, absolute returns in column 3.

3.2.2 The dynamics of agents’ behaviour in the market

In Figure 3 we report the fractions of agents using one of the two available rules. Panel a reports the fractions of the fundamentalists using the last dividend rule. The dividend trend rule is used by a bigger number of agents throughout the simulations. For example, in the case of asset 3, the dividend trend rule is used by all agents very often. This indicates that the dividend trend rule performs better, except some periods where a fraction of agents (or in rear occasions all agents) switch to the last dividend rule. However, in panel b, we observe quite different patterns in the MV investors’ behaviour where we report the fractions of the MV investors using the mean rule to forecast future asset returns. The fractions of the MV investors vary much more than compared
to the ones of the fundamentalists in panel a. We cannot claim that the mean or the median rule is used more often. For example, in cases of asset 1 and 2, apart from some periods, the mean rule is used by more agents, and, in case of asset 3, the median rule is used more frequently.

Figure 3: The fractions of agents.

In Figures 4 and 5 we provide the simulation results for the fundamentalist and MV investor. The prediction errors and the transition probabilities in panels a and b are plotted for a single agent. In some periods the patterns of the transition probabilities (panel b in Figures 4 and 5) resemble the ones of the opinion index (panels c in Figures 4 and 5). This is caused by the fact that in some periods both prediction rules (available either to the fundamentalists or to the MV investors) give similar values. Thus, the differences $v_{i,j,t}^{l_2} - v_{i,j,t}^{l_1}$ in (6a) are close to zero and the opinion index remains as the main driver of the transition probabilities. Thus, even both rules are performing equally, agents keep switching with a certain probability. This reflects herding behaviour: after observing the opinion index and relying on publicly observed dynamics of the opinion in the market, an agent sometimes chooses the forecasting rule which is performing worse.
Next we examine the simulation results for each type of agent. First, we analyze the fundamentalists’ behaviour. Figure 4 panel a shows the differences in the prediction error between the two rules for a single fundamentalist. Undoubtedly, the dividend trend rule outperforms the last dividend rule except for some simulation periods in case of asset 1. However, in Figure 4 panel c this pattern is not prevailing for all agents in the market. For example, in the case of asset 1, fundamentalists keep using the dividend trend rule almost all periods except the first 100 periods and during the periods 4680 - 4780 when they gradually switch to the last dividend rule, and then gradually switch back to the dividend trend rule. This behaviour occurs due to changes in the prediction error: once the last dividend rule starts performing better, more and more fundamentalists start using this rule, and once it starts to perform poorer, they switch back to the dividend trend rule. In case of rule switching based on learning (minimizing the prediction...
error) only, we would observe a sudden switch of the majority of agents to the better performing rule. But in rule switching with herding effect the switch to a better performing rule is slower due to observations of the opinion index. Similarly, gradual shifts from the dividend trend rule towards the last dividend rule can be found in the case of assets 2 and 3. This indicates that for a fraction of agents the last dividend rule gives more precise predictions.

In Figure 4 panel b transition probabilities indicate that this specific fundamentalist still encounters a probability to use the last dividend rule even though the dividend trend rule is performing better. The agent takes into account the opinion index and observes that some fundamentalists in the market are using the last dividend rule. This is absorbed in transition probabilities in each trading period and gives rise to herding behaviour. However, in the case of asset 3, when the last dividend rule is performing much worse than the dividend trend rule, the transition probabilities are equal to zero or extremely small, so that the prediction error is high enough and overweighs the influence of the opinion index in determining the transition probabilities.

Further, we analyze the behaviour of a single MV investor in Figure 5. In panel c, the graphs of the opinion index exhibit that MV investors switch between the two rules more heavily compared to the fundamentalists in Figure 4 panel c, causing higher volatility in the market. This is due to the fact that the differences of prediction errors of both rules change frequently, preventing dominance of one rule. The big switches from the mean rule to the median rule (or vice versa) occur after the median rule outperforms the mean rule (or vice versa) and agents are not able to observe the opinion index, so they choose a rule with a better forecasting performance. All agents for whom the mean rule is giving better predictions, switch to it and all agents, for whom the median rule is giving better forecasts, choose the latter. When agents trade in periods in which they cannot observe the opinion index, they only use the forecasting performance measure to evaluate the accuracy of their forecasts. In those periods we observe that the big fractions of MV investors using one rule or the other change frequently. Since there are periods where one rule is outperforming the other, all of them switch to the better performing rule. We do not find high variation in the opinion index for the fundamentalists (Figure 4 panel c) due to the fact that the number of fundamentalists, for whom the last dividend rule is performing better, is similar to the number of fundamentalists, for whom the dividend trend rule is giving better forecasts.

In Figure 5 panel a we plot the differences in the forecasting rule performance for a single MV investor. For example, in case of asset 2, during the last 200 periods both rules are performing more or less equally. Still, the transition probabilities in panel b are higher than 0.5 indicating that after observing the opinion index, this MV investor has higher probability to switch to the mean rule. In case of asset 3, in the periods 4350 - 4800, MV investor observes that the mean
rule is performing poorer than the median rule. However, the transition probabilities in the same trading periods are between 0 and 0.5, indicating that the MV investor sometimes chooses the mean rule which gives less accurate predictions of future asset returns. Therefore, even though there are no significant differences in the forecasting performance of two rules, MV investor still switches between the forecasting rules.

When we compare the patterns in the aggregate market returns in Figure 2 panel c and the behaviour of each type of agents in Figures 4 and 5, we see that volatility clusters are generated due to MV investors’ switching between the forecasting rules. For example, in the cases of assets 2 and 3, the big clusters in asset returns can be associated with the big variations in the opinion index for the MV investors. In some periods the majority of these investors switch the forecasting rule rather frequently. This result is in line with the findings in the earlier studies where volatility clustering in agent-based financial markets was generated by agents’ endogenous switching between investment strategies.
3.2.3 The reproduction of stylized facts

Below we briefly present the statistics which we use to test the presence of the following stylized facts: absence of autocorrelations in returns, heavy tails, and volatility clustering. The computed statistics are provided in Table 2.

Earlier we presented the values of the kurtosis of the simulated asset returns. They were all significantly higher than the value 3, indicating excess kurtosis. The values of the Jarque-Bera statistic confirm that the distribution of asset returns is significantly different from the normal distribution. We use the asymptotic Jarque-Bera critical value for the 5% significance level, which is 5.99.

To test the presence of volatility clustering in simulated asset returns, we use the statistic of Engle’s ARCH test (Engle (1982)), following He and Li (2007) (we use one lag in the ARCH
The values of statistics in Table 2 are all higher than the 5% critical value of 3.84. The test confirms that ARCH effects are present in the asset returns.

<table>
<thead>
<tr>
<th>Asset</th>
<th>Absence of autocorrelations</th>
<th>Heavy tails</th>
<th>Jarque-Bera</th>
<th>Volatility clustering</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.4014*</td>
<td>4.827*</td>
<td>144.290*</td>
<td>278.3983*</td>
</tr>
<tr>
<td>2</td>
<td>0.3069*</td>
<td>5.805*</td>
<td>330.482*</td>
<td>233.8276*</td>
</tr>
<tr>
<td>3</td>
<td>0.3622*</td>
<td>4.076*</td>
<td>48.274*</td>
<td>264.4059*</td>
</tr>
</tbody>
</table>

Table 2: Stylized facts (* indicates a rejection of the null).

To test the absence of autocorrelations we report here the maximum absolute significant autocorrelation of asset returns, where we apply the Bonferroni correction (see Bonferroni, 1936) to take the multiple tests into account. Using this correction, the critical value for testing absence of autocorrelations becomes 0.10 for sample size 1000. The autocorrelations of returns are often empirically found to be small or even insignificant. Hence, the statistic should be small. A value of zero would indicate no significant autocorrelations in returns at all. However, this statistic is sensitive to outliers and hence an inspection of the autocorrelation patterns in Figure 2 panel b might give more insight.

4 Calibration results

This part discusses the calibration of our model parameters. Dawkins et al. (2001) describe calibration as the procedure which aims at finding model parameter values such that the generated data bears the distributional properties of actual data. Initially, simulations with the model are done using parameter values which are not yet validated. However, to understand the real potential of this model and to approach the real world phenomena with it, we investigate to what degree our model adequately matches the real world. We aim at finding parameter values which are “suggested” by actual data and compare them with the values which we use in our model.

Our analysis is done in the following way. First, we run financial market model for a large number of times with different parameter values. After each run we also measure relevant statistical properties which are used for testing the presence of stylized facts in the resulting outputs of such models. We then use nonparametric regressions to get a first impression of the links between reproduced stylized facts, such as excess kurtosis or volatility clustering, as dependent variables and model parameters as independent variables. These nonparametric regressions yield a clear insight how model parameters influence the statistics indicating stylized facts (see Subsection 4.1.3).
Further, we apply linear regression analysis to quantify the relation between these statistics and the model parameters (Subsection 4.2). A sufficiently large number of simulations makes the estimation very accurate. We use the linear regression results and the properties of actual data to calibrate model parameters. We do this by minimizing an appropriately chosen distance between the statistical properties of real-life data and those implied by the relation between the statistical properties of asset returns and the model parameters (Subsection 4.3). We take data on stock market indices, such as S&P500 and FTSE-global as actual observations in our analysis.

4.1 Multiple runs of the market

4.1.1 Values of parameters

Here, as in Section 3, we use a simplified version of our model described in Section 2. We reduce the degree of heterogeneity in our model by setting some of the parameters with common values for all agents in the market, i.e., $\alpha_i = \alpha$, $\theta_i = \theta$, $\delta_i = \delta$, $\psi_{i1} = \{\psi_{m1}, \psi_{f1}\}$, and $\psi_{i2} = \{\psi_{m2}, \psi_{f2}\}$. For this analysis we take a subset of all model parameters, $P = 7$, i.e., we keep the values of parameter of memory length $M_i$ and a risk aversion parameter $\gamma_{it}$ as they are set in Section 3.1. These two parameters have different values for each agent and their values will be a subject of further research. We run our model of the artificial financial market with $J = 3$ assets for $R = 8400$ times, each time with a different set of random values for the model parameters. Table 3 provides the parameter values which we chose to run our market for multiple times. For every run of the market we draw a random number from the uniform distribution on the interval specific to each parameter. The intervals are chosen such that they include the parameter value from Section 3.1. The interval for $\delta$ is chosen following LeBaron (1999).

Table 3: Parameter values are taken from the uniform distributions.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Original value</th>
<th>Value for multiple runs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>3.9</td>
<td>$U(3, 5)$</td>
</tr>
<tr>
<td>$\theta$</td>
<td>1.1</td>
<td>$U(1.05, 2.5)$</td>
</tr>
<tr>
<td>$\psi_{m1}$</td>
<td>0.7</td>
<td>$U(0, 1)$</td>
</tr>
<tr>
<td>$\psi_{m2}$</td>
<td>$10^5$</td>
<td>$U(10^3, 10^6)$</td>
</tr>
<tr>
<td>$\psi_{f1}$</td>
<td>0.7</td>
<td>$U(0, 1)$</td>
</tr>
<tr>
<td>$\psi_{f2}$</td>
<td>$10^9$</td>
<td>$U(10, 10^4)$</td>
</tr>
<tr>
<td>$\delta$</td>
<td>75</td>
<td>$U(25, 250)$</td>
</tr>
</tbody>
</table>

$^4$When we set the parameter $\theta$ lower than 1.05, the market becomes unstable and the results are unsatisfactory. We are currently working on the multiple runs of our market with $\theta$’s value lower than 1 and the results will be included in the later version of this paper.
4.1.2 Statistical characteristics

We measure 21 statistical characteristic ($S = 21$) of the simulated asset returns. As will become clear later, we will need at least as many statistical characteristics as parameters under consideration, but preferably, we want to measure more statistics to improve our analysis. The characteristics that we use are typically employed to test for the presence of stylized facts in asset returns, such as kurtosis, autocorrelations of asset returns, and ARCH effects in volatility patterns. We investigate the following characteristics:

Excess kurtosis: the fourth central moment, divided by fourth power of its standard deviation.

Skewness: the third central moment.

Average gain/loss: the average gain and the average loss calculated as the average of the positive returns and the absolute value of the average of the negative returns.$^5$

Jarque-Bera statistic: the Jarque-Bera statistic for returns over 1, 4, 12 periods. For instance, in case of weekly frequency, we calculate this statistic for returns over the period of 1, 4, and 12 weeks.$^6$

ARCH test statistic: the statistic of the ARCH test using one lag in the ARCH equation.$^7$

Return-volatility correlation: the correlation between the asset returns and volatility measure. We calculate volatility using rolling windows with sample length of 100 periods.$^8$

Autocorrelations of asset returns: the autocorrelations of asset returns over the lags of 10, 20, 50, and 100 periods. We do it in case of returns, absolute returns and squared returns, thus resulting in $3 \times 4$ autocorrelation measures.

4.1.3 Kernel regressions

To get a first impression of the relations between the statistics of asset returns and each model parameter, we do not impose any parametric structure but use univariate kernel regressions (see Pagan and Ulah, 1999)$^9$. We plot the estimates in Figures 6-14, see Appendix C.$^{10}$ Each figure

$^5$The negative returns are typically found to be higher than the positive returns which is generally reported as gain-loss asymmetry.
$^6$The Jarque-Bera test (see Jarque and Bera, 1987) is applied to test for the Aggregational Gaussianity which means that the distribution of asset returns comes closer to the normal distribution when we increase the period over which we measure the returns. We calculate the statistic using returns over the period of 1, 4 and 12 weeks.
$^7$This test is performed to test for volatility clustering in asset returns. We use Engle’s ARCH test (Engle, 1982) following He and Li (2007).
$^8$This measure is used to test whether returns are negatively correlated with the volatility measures.
$^9$We use the quartic kernel: $K(u) = \frac{15}{16} (1 - u^2)^2$ for $|u| \leq 1$ and 0 otherwise. The bandwidth is chosen after trying out different values and visually inspecting the plots. We want to have sufficiently smooth plots without too many random variations due to noise.
$^{10}$In order not to overload the paper with graphs, we exclude some of the plots which are, nevertheless, available upon request. We only include plots of kernel regressions of asset returns (as well as absolute and squared returns) at lag 10 and leave out the ones with lags 20, 50, and 100. We also leave out plots of kernel regressions of Jarque-Bera statistics measured for asset returns over the periods of 4 and 12 weeks.
with seven panels presents the plots of estimated regression lines and confidence intervals for the case of each estimated statistic. Each panel in every figure presents the effect of each parameter of our model.

The parameter $\alpha$, describing fundamentalist’s behaviour, (see panels $a$ in Figures 6-14) has negative effects on kurtosis, return-volatility correlation, all Jarque-Bera and ARCH statistics and positive effect on autocorrelations of asset returns at lag 10. We find no significant effect on all other statistics.

The parameter $\theta$ (see panels $b$) in most cases has a nonlinear effect on the measured statistics with very narrow confidence intervals. The effect on kurtosis, return-volatility correlation, Jarque-Bera (1 and 4), and ARCH statistics is positive until the parameter approaches value 2, after which the effect diminishes. The effect on skewness, Jarque-Bera (12) statistics and most autocorrelations changes from being positive to being negative (usually this happens between parameter values of 1.5 and 2). What is more, the parameter has a negative effect on average gains and loss but after it reaches value close to 1.2, the effect diminishes.

Panels $g$ in the figures provide estimated regression lines of the statistics on the parameter $\delta$, which indicates the degree of agent’s myopia when evaluating forecasting rules on the basis of past information. This parameter has significant and in most cases highly nonlinear effects on measured characteristics of asset returns. We find no significant effects on autocorrelations of asset returns at lags 50 and 100.

The parameters $\psi$s (panels $c-f$), which play a role in the transition probabilities framework for the herding mechanism in our model, in most cases have less pronounced effects on the measured statistics than the parameters discussed above. For instance, the parameter $\psi_2$ for both types of agents are insignificant, except of $\psi_{m2}$ in cases of kurtosis, Jarque-Bera, ARCH statistics and autocorrelations of asset returns at lag 10. This parameter gives weight to the measure of the rule cost differences in determining transition probabilities for both types of agents. This result implies that the statistical properties of the model output are not affected by the weight of the differences in rule cost but are affected by the weight of the opinion index for both types of agents. The parameters $\psi_1$s have significant effects on the statistics and sometimes with the opposite signs for the two types of agents, i.e., $\psi_{m1}$ and $\psi_{f1}$ are found to effect almost all measured statistics in the opposite directions.

To conclude this part, while inspecting kernel estimates of the univariate regressions we found significant effects of the model parameter under consideration on each measured statistical property of asset returns. Only two parameters, $\psi_{m2}$ and $\psi_{f2}$, are found to have no significant effect on the measured statistics. We use these plots as validation for the results in the linear regres-
sion analysis which we present in the next part. We have shown that these plots provide a good overview of the effects of the model parameters on the statistical characteristics describing the model output. We will further proceed with an analysis of multivariate linear regressions.

4.2 Relating model parameters to stylized facts

Before proceeding with the multivariate linear regression analysis, we pool the simulated data on prices and returns for all three assets in our market.\(^ {11} \) After pooling assets we have time series cross-section data on the statistical characteristics indicating stylized facts. However, model parameter values are the same for the three different assets since they are used for one run of the financial market consisting of the three assets.

The linear regression for each statistical characteristic \( s \) is then the following:

\[
y^s = (1_J \otimes X) \beta^s + \varepsilon^s
\]

where \( y^s \) is a vector of statistical characteristic \( s \) for each asset \( j \) and run \( r \) of the market; \( X = (x_{rp}) \) is a matrix of parameters in the runs of the market (this matrix is the same for the three assets therefore it is replicated \( J \) times). \( \beta^s \) is a vector of coefficients in the regression of statistics \( s \). \( \varepsilon^s \) a vector of error terms \( \sim (0, \Omega_{\varepsilon^s}) \).\(^ {12} \)

The observations for the three assets are taken from the same run of the market. For this reason, by pooling the assets, we encounter contemporaneous correlation in the error terms, \( \varepsilon^s \). In this case, the variance-covariance matrix of residuals, \( \Omega_{\varepsilon^s} \), is not a diagonal matrix. Since OLS estimates are consistent but not efficient, we use feasible generalized LS estimation, instead. We estimate the model in equation (8) using OLS, then we use the residuals to estimate variance-covariance matrix \( \Omega_{\varepsilon^s} \). See Appendix A for the details on the estimation method. The feasible GLS estimator is the following:

\[
\hat{\beta}_{FGLS}^s = \left( X' \hat{\Omega}_{\varepsilon^s}^{-1} X \right)^{-1} X' \hat{\Omega}_{\varepsilon^s}^{-1} y^s
\]

Table 4 gives an overview of the estimation results. Each column in the table presents the estimates of the coefficients in the linear regressions of each statistical characteristic. After comparing

---

\(^ {11} \) This helps to exclude one dimension of the data. What is more, by pooling assets we want to capture the effects which are consistent throughout the assets. The three assets are taken from the same market, thus, they are not structurally different.

\(^ {12} \) We have also estimated the regressions with higher order terms and found that the coefficients are significant for the parameters whose coefficients of the first order terms are significant. However, we chose to exclude the higher order terms from this analysis due to the difficulties with the dimension encountered later in the calibration. This will be the question for the further research.
the significant marginal effects and the results in visual inspection of univariate kernel regression plots in Figures 6-14 (Appendix C), we find that the results in Table 4 are in line with the findings in Subsection 4.1.3.

In the first column of Table 4 we present estimated coefficients of the linear regression of kurtosis on the model parameters under consideration. As it was found earlier, when inspecting univariate kernel regression lines, the parameters $\psi_2s$ do not have significant effects on this statistic. All other parameters are found to have strongly significant (at 1% level) effects with signs in line with the results from the univariate kernel regressions in Subsection 4.1.3.

To mention briefly the overall results in Table 4, the parameters $\alpha$ and $\delta$ are found to negatively affect most of the statistical characteristics, except for autocorrelations of asset returns at lag 10. The parameter $\theta$ affects the statistics in both directions. The parameters $\psi_1s$ for two types of agents have an opposite effect in half of all regressions.

The consistency of the significant effects, found in both univariate kernel and multivariate linear regressions, of model parameters on statistical characteristics is valid throughout most regressions (with the exception of the parameter $\delta$ in case of skewness, return-volatility correlation and Jarque-Bera statistics). We find that the results of nonparametric regressions validates the results of the linear regression analysis. We conclude that the parameters in our model affect the reproduction of stylized facts in the asset returns and can be quantified using linear regressions.

\footnote{Although, it was not straightforward whether we can reject the hypothesis that the parameter $\psi_{m2}$ has no significant effect on kurtosis when visually inspecting the plots in panel d of Figure 6, we find that estimated coefficient in the multiple linear regression is insignificant.}
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Kurtosis</th>
<th>Skewness</th>
<th>Corr($r_{vol}$)</th>
<th>AvgGain</th>
<th>AvgLoss</th>
<th>J-B1</th>
<th>J-B4</th>
<th>J-B12</th>
<th>ARCHst</th>
<th>AC($r_{10}$)</th>
<th>AC($r_{20}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>3.2488***</td>
<td>0.3924***</td>
<td>-0.0040***</td>
<td>2.1647***</td>
<td>1.8916***</td>
<td>-215.52***</td>
<td>-221.34***</td>
<td>30.26***</td>
<td>59.72***</td>
<td>0.3809***</td>
<td>-0.1007***</td>
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<tr>
<td></td>
<td>(0.0734)</td>
<td>(0.0216)</td>
<td>(0.0010)</td>
<td>(0.0150)</td>
<td>(0.0148)</td>
<td>(23.11)</td>
<td>(44.11)</td>
<td>(7.90)</td>
<td>(2.67)</td>
<td>(0.0156)</td>
<td>(0.0074)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>-0.0509***</td>
<td>0.0062</td>
<td>-0.0013***</td>
<td>0.0070**</td>
<td>-0.0114***</td>
<td>-16.14***</td>
<td>-19.61**</td>
<td>-5.42***</td>
<td>-3.41***</td>
<td>0.0166***</td>
<td>-0.0052***</td>
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<td></td>
<td>(0.0139)</td>
<td>(0.0041)</td>
<td>(0.0002)</td>
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<td>(0.0028)</td>
<td>(4.39)</td>
<td>(8.37)</td>
<td>(1.50)</td>
<td>(0.0569)</td>
<td>(0.0030)</td>
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<tr>
<td>$\theta$</td>
<td>1.3281***</td>
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<td>0.0138***</td>
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<td>-0.0660***</td>
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<td>(0.0038)</td>
<td>(0.0038)</td>
<td>(5.93)</td>
<td>(11.32)</td>
<td>(2.03)</td>
<td>(0.0653)</td>
<td>(0.0040)</td>
<td>(0.0019)</td>
</tr>
<tr>
<td>$\psi_{m1}$</td>
<td>-0.1067***</td>
<td>-0.1212***</td>
<td>0.0005</td>
<td>-0.1554***</td>
<td>-0.1207***</td>
<td>-66.76***</td>
<td>-150.15***</td>
<td>-18.78***</td>
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<td>(0.0276)</td>
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<td>(0.0056)</td>
<td>(0.0056)</td>
<td>(8.70)</td>
<td>(16.60)</td>
<td>(2.97)</td>
<td>(1.0051)</td>
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<td>(0.0028)</td>
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<td>0.0000</td>
<td>0.0000</td>
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<td>(0.0000)</td>
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<td>(0.0000)</td>
<td>(0.0000)</td>
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</tr>
<tr>
<td>$\psi_{f1}$</td>
<td>0.8098***</td>
<td>-0.0352***</td>
<td>0.0048***</td>
<td>0.0254***</td>
<td>227.65***</td>
<td>105.97***</td>
<td>9.84***</td>
<td>-13.44***</td>
<td>0.0291***</td>
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<td>(8.79)</td>
<td>(16.77)</td>
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<td>(0.0152)</td>
<td>(0.0059)</td>
<td>(0.0028)</td>
</tr>
<tr>
<td>$\psi_{f2}$</td>
<td>-0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>-0.0020***</td>
<td>-0.0002***</td>
<td>-0.0000***</td>
<td>-0.0029***</td>
<td>-0.0017***</td>
<td>-0.2359***</td>
<td>0.5172***</td>
<td>-0.0371***</td>
<td>-0.0001***</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0000)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0001)</td>
<td>(0.0000)</td>
<td>(0.0000)</td>
</tr>
</tbody>
</table>

Table 4: Results from regressions of statistics indicating stylized facts on model parameters using FGLS. Note: standard errors are in parenthesis; ***, **, * indicates the significance level of 1%, 5%, and 10%, respectively.
4.3 Relating simulated asset returns with actual data

We take daily and weekly data on S&P500 and FTSE-global for the period of 1995-2012 from Datastream for actual observations in our analysis.\textsuperscript{14} The choice of FTSE-global index stems from the fact that there are three assets in our market and we pool them to do parameter analysis. Therefore, we would also like to include an index which combines assets from different markets. Table 5 provides statistical properties (Subsection 4.1.2) of both indices and the asset returns of our market in Section 3.

Table 5: Statistical characteristics of S&P500 and FTSE-global index at different frequencies for the period of 1992-2012.

<table>
<thead>
<tr>
<th></th>
<th>Kurtosis</th>
<th>Skewness</th>
<th>Corr(r,vol)</th>
<th>AvgGain</th>
<th>AvgLoss</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>5.805</td>
<td>0.127</td>
<td>0.0095</td>
<td>1.706</td>
<td>1.728</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>daily</td>
<td>12.05</td>
<td>-0.046</td>
<td>-0.004</td>
<td>0.784</td>
<td>0.777</td>
</tr>
<tr>
<td>weekly</td>
<td>8.48</td>
<td>-0.492</td>
<td>-0.006</td>
<td>1.690</td>
<td>1.783</td>
</tr>
<tr>
<td>FTSE-gl</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>daily</td>
<td>10.38</td>
<td>-0.009</td>
<td>-0.012</td>
<td>0.711</td>
<td>0.761</td>
</tr>
<tr>
<td>weekly</td>
<td>8.792</td>
<td>-0.658</td>
<td>0.002</td>
<td>1.740</td>
<td>1.863</td>
</tr>
</tbody>
</table>

(continued)

|     | JB1     | JB4     | JB12    | ARCHst | AC(r_{10}) | AC(|r_{10}|) | AC(r_{10}^2) |
|-----|---------|---------|---------|--------|------------|------------|-------------|
|     | 329.87  | 52.22   | 1.211   | 50.53  | -0.015     | 0.131      | 0.143       |
|     | 17811   | 7202    | 5553    | 187.9  | 0.025      | 0.283      | 0.246       |
|     | 21183   | 12054   | 5251    | 78.59  | 0.0084     | 0.120      | 0.058       |
|     | 7470    | 2589    | 1856    | 156.4  | 0.002      | 0.263      | 0.302       |
|     | 19511   | 6993    | 12989   | 39.91  | -0.0025    | 0.105      | 0.031       |

We use the estimated coefficients from the linear regression analysis to calibrate the values of the model parameters. We do this by minimizing the quadratic distance between the measured statistics (Subsection 4.1.2) of actual data and the calculated counterparts using estimates from Table 4 and model parameters. We use rolling windows with a sample size of 400 time periods to measure the statistical properties of the returns of both stock market indices. Thus, we generate 4055 samples in case of daily observations and 498 samples in case of weekly observations which result into so many observations of the statistical properties of the returns. This way, we later obtain a distribution of calibrated values for each of the model parameters.

Using data on stock market index a and sample w, the calibrated values of the model parameters are

\textsuperscript{14}Since FTSE global index has been recorder starting 1995, we choose this period of data. According to www.ftse.com: “The FTSE Global Equity Index Series covers over 7,400 securities in 48 different countries and captures 98% of the world’s investable market capitalisation - covering every equity and sector relevant to international investors needs.”
found by solving the following problem:

$$
\min_{q^{aw}} \left( y^{aw} - \hat{B}' q^{aw} \right)' \left( y^{aw} - \hat{B}' q^{aw} \right) \tag{10}
$$

where $y^{aw}$ is a vector of measured statistical characteristics of asset $a$ from sample $w$, $\hat{B} = \left( \hat{\beta}_{FGLS} \right)$ is the $(P \times S)$ matrix of estimated coefficients using equation (9) (see results in Table 4), $q^{aw}$ is a vector of the calibrated model parameters using data of asset $a$ from the sample $w$. See Appendix B for a more detailed derivations.

We provide the results using data at daily and weekly frequencies of the returns of both stock market indices. As mentioned before, using rolling windows we measure the statistics in different time periods of the sample and, using the solution to equation (10), get a distribution of the calibrated parameter values. We use density functions to make a comparison between our model parameter values and the calibrated parameter values. This allows us to concentrate on the calibrated values with a higher likelihood. Using the series of the calibrated parameter values, we estimate density functions nonparametrically and mark the 95% confidence intervals. Figures 15-18 in Appendix D present estimated density functions of calibrated values in case of daily and weekly data of both S&P500 and FTSE-global. Panels $a-g$ in these figures present plots for every parameter under consideration. To compare our values and calibrated values of the model parameters, we also mark the intervals from which we drew random values to do multiple runs of our market (Table 3).

Table 6: The model parameters whose values, used for the multiple runs of our market, do not overlap with the 95% confidence interval of the calibrated values. We use actual observations of S&P500 and FTSE-global index at daily and weekly frequencies.

<table>
<thead>
<tr>
<th></th>
<th>Considering all 7 parameters</th>
<th>Excluding $\psi_{m2}$ and $\psi_{f2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S&amp;P500</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>daily</td>
<td>$\psi_{f1}, \psi_{m1}, \psi_{f2}$</td>
<td>$\theta, \psi_{f1}$</td>
</tr>
<tr>
<td>weekly</td>
<td>$\psi_{f1}, \psi_{f2}, \delta$</td>
<td>$\theta, \psi_{m1}$</td>
</tr>
<tr>
<td><strong>FTSE-gl</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>daily</td>
<td>$\psi_{m1}, \psi_{f1}, \psi_{f2}$</td>
<td>$\theta, \psi_{f1}$</td>
</tr>
<tr>
<td>weekly</td>
<td>$\psi_{f2}, \delta$</td>
<td></td>
</tr>
</tbody>
</table>

We make conclusions about validity of our model parameters by comparing the 95% confidence intervals of the calibrated parameter values and the intervals of the parameter values used to run our model (see Table 3). In Table 6 we report the parameters whose range of values used for the multiple runs of our financial market do not have an overlapping region with the 95% confidence interval of the calibrated values. This implies that the values outside the confidence interval are very unlikely to result in the calibration when using daily and weekly data on S&P500 and FTSE-global for the period of 1995-2012.

When we consider all 7 parameters (see the first part of Table 6), we find that our values of most parameters, which are found to have significant effects on stylized facts, have an overlapping region with

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15 We also did this analysis with data at a monthly frequency. The results showed that the range of parameters which we use for the multiple runs of the market are distant from the calibrated values using monthly returns. Thus, we conclude that our model cannot be validated using monthly returns of S&P500 and FTSE-global.

16 The estimate is based on a normal kernel function using bandwidth as a function of data points.
the 95% confidence interval of the calibrated parameter values. For data on S&P500, only values of the parameters $\psi_f$, $\psi_m$, and $\psi_f$ (with data at daily frequency) and the parameters $\psi_f$, $\psi_f$, and $\delta$ (with data at weekly frequency) do not have an overlapping region with the 95% confidence interval of calibrated values. For data on FTSE-global, values of the parameters $\psi_m$, $\psi_f$, and $\psi_f$ (with data at daily frequency) and values of parameters $\psi_f$ and $\delta$ (with data at weekly frequency) do not have an overlapping region. Since dividends in our artificial financial market are modeled as a weekly process, our emphasis is on the calibrated values when using weekly data on both indices. We find that the parameter $\psi_f$, which is found to have insignificant effects on the statistical characteristics of asset returns, and the parameter $\delta$ do not have an overlapping region with the calibrated values.

The estimation results in Subsection 4.2 showed that parameters $\psi_m$ and $\psi_f$ have no significant effect on statistical characteristics of asset returns in our financial market (except for the parameter $\psi_f$ in the regressions of autocorrelations of asset returns, absolute, and squared returns at lag 20), see Table 4. For this reason, we also did our analysis with estimations (as in Subsection 4.2) and calibration while excluding the parameters $\psi_m$ and $\psi_f$. We report the kernel density functions in Figures 19-22 in Appendix D. The second part of Table 6 shows the parameters whose range of values do not have an overlapping region with the 95% confidence interval of the calibrated values. We find that after excluding these two parameters from the analysis, using weekly data on FTSE-global all other parameter values have an overlapping region with the 95% confidence interval of calibrated values.

5 Conclusions

Heterogeneous agent-based models of financial markets deviate from the traditional approach with a representative agent. These agent-based models provide ways to study the interactions and learning in groups of heterogeneous agents. However, to understand the real potential of this model and to approach the real world phenomena with it, we need to answer a question to what degree our model adequately matches the real world.

We present a model with fundamentalists and MV investors trading risky assets. Agents form demand for risky assets based on their forecasts about market fundamentals or asset returns. Market prices are determined by a market maker in a temporary Walrasian equilibrium while a regulator controls the orderly functioning of the financial market. Both the fundamentalists and the MV investors have heterogeneous expectations, characterized by the amount of data and the model they use to forecast the relevant statistics. In each trading period agents are allowed to switch between the two forecasting rules based on the forecasting performance of each rule and the majority opinion about the rules in the market. We show that a market with agents switching between the forecasting rules generates the stylized facts, such as heavy tails, and volatility clustering.

Since these models are often highly parameterized, we analyze how model parameters affect the reproduction of stylized facts and later calibrate them using statistical properties of real-life data. Our analysis
combined several steps. We first ran our financial market for a large number of times with different set of model parameter values each time. We then measured relevant statistical properties of asset returns from the resulting output of our model. Further, we used the linear regression results and statistical characteristics of actual data to calibrate model parameters. We did this by minimizing an appropriately chosen distance between the statistical properties of actually observed data and the properties implied by the estimated relation between the model parameters and model output. To make conclusions about the validity of our model we compared model parameter values with the calibrated counterparts using kernel estimates of density functions. We proposed a simple and effective way to calibrate model parameters. The results showed that most of our model parameter values fall in the 95% confidence interval of calibrated parameter values when using data on weekly returns of S&P500 and FTSE-global.

References


## A  FGLS estimation

We measure 21 statistical characteristic, thus we have 21 regression models. The regression model for each statistics is the following:

\[ y_s = (1_J \otimes X) \beta^s + \varepsilon^s \]

where \( y_s = (y_{srj}) \) is a \((RJ \times 1)\) vector of the statistical characteristic \( s \) for each asset \( j \) and run \( r \) of the market; \( X = (x_{rp}) \) is a \((R \times P)\) matrix of each parameter \( p \) in the run \( r \) of the market (this matrix is the same for the three assets therefore it is replicated \( J \) times). \( \beta^s \) is a vector of coefficients in the regression of statistics \( s \). \( \varepsilon^s \) is a \((RJ \times 1)\) vector of error terms, assumed to be \( \sim (0, \Omega^s) \).

We can also rewrite equation above as:

\[ y_{srj}^s = x_{rp} \beta_{pj}^s + \varepsilon_{srj}^s \]

The observations for the three assets are taken from the same run of the market. After pooling the assets, we encounter contemporaneous correlation in the error terms. For assets \( i \) and \( j \) the following holds:

\[ \mathbb{E} (\varepsilon_{sri}^s \varepsilon_{srij}^s) \neq 0, \quad \text{for} \quad i \neq j \]
thus, $\Omega_{e^s}$ is the matrix:

$$
\Omega_{e^s} = V^s \otimes I_T;
$$

$$
V^s = (v^s_{ij}) = (\sigma^2_{ij}) = E(\varepsilon^s_i \varepsilon^s_j);
$$

$$
\Omega_{e^s} = \begin{pmatrix}
\sigma^2_{11} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \sigma^2_{11} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \sigma^2_{J1} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \sigma^2_{J1} & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \sigma^2_{JJ} & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \sigma^2_{JJ} & 0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
$$

Due to this, OLS estimates are consistent but not efficient. Therefore, the feasible generalized LS estimation is used, instead. We estimate the model using OLS and use the residuals to estimate the variance covariance matrix $\Omega_{e^s}$:

$$
\hat{\Omega}_{e^s} = \hat{V}^s \otimes I_T;
$$

$$
\hat{\varepsilon}^s_{ij} = \hat{\sigma}^2_{ij} = \frac{\sum_{t=1}^T \varepsilon^s_{it} \varepsilon^s_{jt}}{T}
$$

After taking into account contemporaneous correlation in the error terms and estimating variance-covariance matrix $\Omega$, the feasible GLS estimator is the following:

$$
\hat{\beta}_{FGLS} = \left( X' \hat{\Omega}_{OLS}^{-1} X \right)^{-1} X' \hat{\Omega}_{OLS}^{-1} y^s
$$

**B Calibration**

To calibrate the model parameters we use the fact that an agent-based model of the financial market, preferably, has to replicate the statistical features of the real-life data. After measuring the 21 statistical characteristic of both simulated asset returns in our artificial financial market and real-life data (returns of stock market index), we minimize the distance between the two values taking into account the relation between model parameters and those characteristics. Hence, the calibrated values of the model parameters are found by using data on stock market index $a$ and sample $w$ and solving the following problem:

$$
\min_{q^{aw}} \left( y^{aw} - \hat{B} q^{aw} \right)' \left( y^{aw} - \hat{B} q^{aw} \right)
$$

33
where $y^w$ is a $(S \times 1)$ vector of measured statistical characteristics of asset $a$ from sample $w$, $\hat{B} = \left( \hat{\beta}_{FGLS} \right)$ is a $(P \times S)$ matrix of estimated coefficients using equation (9) (see results in Table 4), $q^w$ is a $(P \times 1)$ vector of the calibrated model parameters using data of asset $a$ from sample $w$.

\[
y^w = \begin{pmatrix}
y^w_1 \\
\vdots \\
y^w_S
\end{pmatrix} ;
q^w = \begin{pmatrix}
q^w_1 \\
\vdots \\
q^w_P
\end{pmatrix} ;
\hat{B} = \begin{pmatrix}
\hat{\beta}_1 \\
\vdots \\
\hat{\beta}_P
\end{pmatrix}
\]

After applying the same principle as in the OLS estimation, we find the values for the model parameters:

\[
q^w = (\hat{B}\hat{B}')^{-1}\hat{B}y^w.
\]
C Kernel regressions

Figure 6: Univariate kernel regressions of Kurtosis on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.
Figure 7: Univariate kernel regressions of skewness on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.

(a) $\alpha$

(b) $\theta$

(c) $\psi_{m1}$

(d) $\psi_{m2}$

(e) $\psi_{f1}$

(f) $\psi_{f2}$

(g) $\delta$
Figure 8: Univariate kernel regressions of return-volatility correlation on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.

(a) $\alpha$

(b) $\theta$

(c) $\psi_{m1}$

(d) $\psi_{m2}$

(e) $\psi_{f1}$

(f) $\psi_{f2}$

(g) $\delta$

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Figure 9: Univariate kernel regressions of average loss on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.
Figure 10: Univariate kernel regressions of Jarque-Bera (1) statistics on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.
Figure 11: Univariate kernel regressions of ARCH statistics on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.
Figure 12: Univariate kernel regressions of autocorrelation of $r_{10}$ on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.

(a) $\alpha$

(b) $\theta$

(c) $\psi_{m1}$

(d) $\psi_{m2}$

(e) $\psi_{f1}$

(f) $\psi_{f2}$

(g) $\delta$
Figure 13: Univariate kernel regressions of $|r_{10}|$ on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.

(a) $\alpha$

(b) $\theta$

(c) $\psi_{m1}$

(d) $\psi_{m2}$

(e) $\psi_{f1}$

(f) $\psi_{f2}$

(g) $\delta$
Figure 14: Univariate kernel regressions of autocorrelation of $r_{T0}^2$ on the model parameters. Solid line: estimated regression line; dotted lines: confidence intervals.

(a) $\alpha$
(b) $\theta$
(c) $\psi_m1$
(d) $\psi_m2$
(e) $\psi_f1$
(f) $\psi_f2$
(g) $\delta$
D Kernel densities of calibrated parameter values

Figure 15: Calibrated (using S&P500 daily data) and modelling values of parameters. Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.
Figure 16: Calibrated (using S&P500 weekly data) and modelling values of parameters. Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.
Figure 17: Calibrated (using FTSE-global daily data) and modelling values of parameters. Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.
Figure 18: Calibrated (using FTSE-global weekly data) and modelling values of parameters. Solid lines: kernel densities; semi-dashed lines: modelling values; doted lines: 95% confidence interval of calibrated values.
Figure 19: Calibrated (using S&P500 daily data) and modelling values of parameters (excluding parameters $\psi_{m2}$ and $\psi_{f2}$ from the analysis). Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.
Figure 20: Calibrated (using S&P500 weekly data) and modelling values of parameters (excluding parameters $\psi_m$ and $\psi_f$ from the analysis). Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.
Figure 21: Calibrated (using FTSE-global daily data) and modelling values of parameters (excluding parameters $\psi_{m2}$ and $\psi_{f2}$ from the analysis). Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.
Figure 22: Calibrated (using FTSE-global weekly data) and modelling values of parameters (excluding parameters $\psi_m^2$ and $\psi_f^2$ from the analysis). Solid lines: kernel densities; semi-dashed lines: modelling values; dotted lines: 95% confidence interval of calibrated values.