RESPONSE OF NON-LINEAR NON-STATIONARY SYSTEMS SUBJECTED TO PULSE EXCITATION

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The study involves an approximate analysis of non-linear, non-stationary vibrational systems subjected to an arbitrary pulse excitation. The non-stationary system parameters, which may include masses, restoring forces, material properties or damping, are considered to be slowly varying functions of time. A general procedure for obtaining the first and the second order approximate solutions is presented through an application of the Bogoliubov-Mitropolsky technique. Illustrative examples are included and the results are compared with fourth-order Runge-Kutta numerical solutions.

1. INTRODUCTION

The investigation of non-linear vibrational systems is of fundamental importance in many areas of mechanics and engineering design. An important class of these systems are those with time dependent parameters, better known as non-stationary systems. Often such systems are subjected to various types of pulse loading and the analysis of the transient behavior may be extremely important for the purpose of design.

In the past, several authors have investigated the response of non-linear systems subjected to a pulse excitation. Bapat and Srinivasan [1, 2] obtained exact expressions for the periods of the maximum displacements for a class of undamped non-linear systems with constant parameters subjected to a step excitation. They [3] have also applied Ponavko's direct linearization method [4] and Atkinson's superposition method [5] to obtain approximate expressions for the time periods of such systems. Step response of damped non-linear systems with constant parameters has been considered by Sinha and Srinivasan [6], Anderson [7] and Reed [8], among others. These analyses are approximate, since, in general, it is not possible to obtain a closed form exact solution for these problems.

The approximate analysis of response of non-linear systems subjected to arbitrary pulse excitation has also received much attention. Ariaratnam [9] and Bauer [10, 11] have used the perturbation method of Poincaré as modified by Lighthill. Srinagarajan and Srinivasan [12] and Srinagarajan [13] have applied a generalized averaging technique [6, 7] and the well-known Krylov-Bogoliubov-Mitropolsky method [14]. However, these studies have been restricted to systems with constant parameters only. The response of non-linear, non-stationary systems have been considered by Nayfeh and Mook [15], Rubenfeld [16], Evan-Iwanowski [17] and Mitropolsky [18]. However, these studies do not deal with
pulse excitations. Recently, Sinha and Chou [19] analyzed the response of non-linear, non-stationary systems through an application of the generalized averaging technique based on an ultraspherical polynomial expansion. They also obtained an approximate result for the response of such systems when subjected to a step function excitation.

In this study, a general procedure for determining the pulse response of non-linear, non-stationary vibrational systems is presented. The non-stationary system parameters are considered to be slowly varying functions of time (as compared to the “normal” time period of the system) and may include masses, restoring forces, material properties or damping. First and second order approximate solutions are obtained through an application of the Bogoliubov–Mitropolsky technique. The procedure is illustrated through typical examples and the approximate results are compared with numerical solutions.

2. EQUATION OF MOTION AND THE METHOD OF ANALYSIS

Many problems of non-linear vibrations with single degree of freedom and slowly varying system parameters, subjected to an arbitrary pulse excitation, $q(\tau)$, can be reduced to

$$
\frac{d}{d\tau} \left[ m(\tau) \frac{dx}{d\tau} \right] + k(\tau)x + \epsilon f \left( x, \frac{dx}{d\tau} \right) = q(\tau),
$$

(1)

where, $f(x, dx/d\tau)$ is a non-linear function of $x$ and $dx/d\tau$, $\epsilon$ is a small non-linearity parameter, $m(\tau)$ is the variable mass, $k(\tau)$ is the variable stiffness and $\tau$ is the “slow time” defined by $\tau = \epsilon t$. It is assumed that functions $m(\tau), k(\tau), q(\tau)$ and $f(x, dx/d\tau)$, all have the desired number of derivatives with respect to $\tau$, $x$, $dx/d\tau$ for all their finite values and that $m(\tau) > 0$, $k(\tau) > 0$ for all $\tau$ within the time interval under consideration. Without any loss of generality, the initial conditions are taken as

$$
x = dx/d\tau = 0 \quad \text{at} \quad t = 0.
$$

(2)

Many of the available approximate techniques cannot be applied directly to equation (1) if $q(\tau)$ is taken as an arbitrary function of time. Approximate solutions are available [15,18] for the case when $q(\tau)$ is a periodic function of time.

Upon introducing the transformation

$$
x(t) = y(t) + p(t),
$$

(3)

equation (1) becomes

$$
\frac{d}{d\tau} \left[ m(\tau) \frac{dy}{d\tau} \right] + \frac{d}{d\tau} \left[ m(\tau) \frac{dp}{d\tau} \right] + k(\tau)y + k(\tau)p + \epsilon f \left( y + p, \frac{dy}{d\tau} + \frac{dp}{d\tau} \right) = q(\tau).
$$

(4)

Choosing $p(t)$ such that

$$
\frac{d}{d\tau} \left[ m(\tau) \frac{dp}{d\tau} \right] + k(\tau)p = q(\tau),
$$

(5)

yields

$$
\frac{d}{d\tau} \left[ m(\tau) \frac{dy}{d\tau} \right] + k(\tau)y + \epsilon f \left( y + p, \frac{dy}{d\tau} + \frac{dp}{d\tau} \right) = 0.
$$

(6)

The initial conditions (2) transform to

$$
y(t) = -p(t), \quad dy/d\tau = -dp/d\tau \quad \text{at} \quad t = 0.
$$

(7)
Although equation (5) is linear, its coefficients are time dependent and, in general, there are no exact solutions for such equations. If \( p(t) \) is known, then some standard approximate techniques [15, 18] can be used to determine \( y(t) \) and the desired solution can be found. It is also observed from equation (6) that the solutions of first \((O(\varepsilon))\) and second \((O(\varepsilon^2))\) order approximations require the evaluation of \( p(t) \) independent of \( \varepsilon \) and up to the order of \( \varepsilon \), respectively. At this point, note that equation (5) can be rewritten as

\[
e^2m(\tau) \frac{d^2p}{d\tau^2} + \frac{dm(\tau)}{d\tau} \frac{dp}{d\tau} + k(\tau)p = q(\tau).
\]

(8)

Therefore, for first and second order approximations, it suffices to use an approximate expression for \( p(t) \) given by

\[ p(t) = \frac{q(\tau)}{k(\tau)}. \]

Once \( p(t) \) is known, equation (6) takes the standard form

\[
\frac{d}{dt} \left[ m(\tau) \frac{dy}{dt} \right] + k(\tau)y + \varepsilon f \left( \tau, y, \frac{dy}{dt} \right) = 0.
\]

(9)

Following the approach suggested by Bogoliubov and Mitropolsky [14, 18], a solution of equation (10) is sought in the form

\[
y = a \cos \psi + \varepsilon u_1(\tau, a, \psi) + \varepsilon^2 u_2(\tau, a, \psi) + \cdots,
\]

(11)

where, \( u_1(\tau, a, \psi), u_2(\tau, a, \psi), \cdots \), are periodic functions of \( \psi \) and

\[
\frac{da}{dt} = \varepsilon A_1(\tau, a) + \varepsilon^2 A_2(\tau, a) + \cdots,
\]

(12)

\[
\frac{d\psi}{dt} = \omega(\tau) + \varepsilon B_1(\tau, a) + \varepsilon^2 B_2(\tau, a) + \cdots,
\]

(13)

\[
\omega(\tau) = \left[ \frac{k(\tau)}{m(\tau)} \right]^{1/2}.
\]

(14)

It can be shown [18] that

\[
A_1(\tau, a) = -a \frac{d[m(\tau)\omega(\tau)]}{d\tau} + \frac{1}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f_0(\tau, a, \psi) \sin \psi \, d\psi,
\]

(15)

\[
B_1(\tau, a) = \frac{1}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f_0(\tau, a, \psi) \cos \psi \, d\psi,
\]

(16)

\[
A_2(\tau, a) = -\frac{1}{2\omega(\tau)} \left[ \frac{\partial B_1}{\partial a} A_1 + 2A_1 B_1 + a \frac{d[m(\tau)B_1]}{d\tau} \right]
\]

\[
+ \frac{1}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f_1(\tau, a, \psi) \sin \psi \, d\psi,
\]

(17)

\[
B_2(\tau, a) = \frac{1}{2\omega(\tau)} \left[ \frac{\partial A_1}{\partial a} A_1 - a B_1^2 + \frac{1}{m(\tau)} \frac{d[m(\tau)A_1]}{d\tau} \right]
\]

\[
+ \frac{1}{2\pi m(\tau)\omega(\tau)} \int_0^{2\pi} f_1(\tau, a, \psi) \cos \psi \, d\psi,
\]

(18)

\[
u_1(\tau, a, \psi) = -\frac{1}{2\pi k(\tau)} \sum_{n \neq 0} \frac{1}{1 - n^2} \int_0^{2\pi} f_1(\tau, a, \psi) e^{-in\psi} \, d\psi,
\]

(19)

where

\[
f_0(\tau, a, \psi) = f(\tau, a \cos \psi, -a\omega \sin \psi),
\]

(20)
\[ f_i(\tau, a, \psi) = u_i f_x(\tau, a \cos \psi, -a \omega \sin \psi) \]

\[ + [A_1 \cos \psi - a B_1 \sin \psi + \left( \frac{\partial u_i}{\partial \psi} \right) \omega(\tau)] f_y(\tau, a \cos \psi, -a \omega \sin \psi) \]

\[ -2m(\tau)\omega(\tau) \left[ \frac{\partial^2 u_i}{\partial \tau^2} + \frac{\partial^2 u_i}{\partial \psi \partial \psi} A_1 + \frac{\partial^2 u_i}{\partial \psi^2} B_1 \right] \frac{\partial u_i}{\partial \psi} \frac{d[m(\tau)\omega(\tau)]}{d\tau}. \]  

(21)

The symbols \( f_x \) and \( f_y \) denote partial derivatives of \( f \) with respect to \( y \) and \( dy/dt \), respectively.

For the special case of the first order approximation, which is always of considerable interest, the solution of \( x(t) \) is given by

\[ x(t) = a(t) \cos \psi(t) + q(\tau)/k(\tau), \]  

(22)

with

\[ \frac{da}{dt} = \varepsilon A_1(\tau, a), \quad \frac{d\psi}{dt} = \omega(\tau) + \varepsilon B_1(\tau, a), \]  

(23)

where \( A_1 \) and \( B_1 \) are determined from equations (15) and (16) respectively. The initial conditions on \( a(0) \) and \( \psi(0) \) can be obtained from equations (7) and (22) as

\[ a(0) = \left[ (q(0)/k(0)) + \{ \omega(0)^{-1} (d/dt)[q(\tau)/k(\tau)] \}_{\tau=0} \right]^{1/2}, \]  

(24)

\[ \psi(0) = \arctan \left( \frac{k(0)}{\omega(0)q(0)} \right). \]  

(25)

In the following section, two examples are considered to illustrate the procedure.

3. ILLUSTRATIVE EXAMPLES

3.1. EXAMPLE 1

First, consider a non-stationary system with non-linear stiffness and viscous damping subjected to a step function excitation. The equation of motion may be written as

\[ \frac{d}{dt} \left[ m(\tau) \frac{dx}{dt} \right] + k(\tau)x + \varepsilon \left( ax^3 + \beta \frac{dx}{dt} \right) = F_0 h(\tau), \]  

(26)

where, \( m(\tau), k(\tau), \alpha, \beta > 0 \), and \( h(\tau) \) is the unit step function defined by

\[ h(\tau) = \begin{cases} 0 & \text{if } \tau \leq 0^- \\ 1 & \text{if } \tau \geq 0^+ \end{cases}. \]  

(27)

The first order approximation is given by

\[ x = a \cos \psi + F_0 h(\tau)/k(\tau). \]  

(28)

The expressions for \( da/dt \) and \( d\psi/dt \) are obtained from equations (15), (16) and (23) as

\[ \frac{da}{dt} = \frac{\varepsilon a}{2m(\tau)\omega(\tau)} \frac{d[m(\tau)\omega(\tau)]}{d\tau} - \frac{\varepsilon \beta a}{2m(\tau)}, \]  

(29)

\[ \frac{d\psi}{dt} = \omega(\tau) + \frac{3\varepsilon \alpha a^2}{8m(\tau)\omega(\tau)} + \frac{3\varepsilon \alpha F_0^2}{2k^2(\tau)}. \]  

(30)

Equation (29) can be integrated to yield

\[ a = a(0) \left[ \frac{m(0)\omega(0)}{m(\tau)\omega(\tau)} \right]^{1/2} \exp \left[ -\frac{\varepsilon \beta}{2} \int_{0}^{\tau} \frac{dt}{m(\tau)} \right]. \]  

(31)
Substituting this in equation (30) gives

$$
\psi = \int_0^t \left\{ \omega(\tau) + \frac{3\varepsilon \omega^2(0)}{8m(\tau)\omega(\tau)} \left[ \frac{m(0)\omega(0)}{m(\tau)\omega(\tau)} \right] \exp \left[ -\varepsilon \beta \int_0^t \frac{dt}{m(\tau)} \right] + \frac{3\varepsilon F_0^2}{2k^2(\tau)} \right\} dt + \psi(0).
$$

(32)

\(a(0)\) and \(\psi(0)\) are easily obtained from equations (24) and (25), respectively. For \(m(\tau) = k(\tau) = \text{const.}\), the results given by equations (31) and (32) are consistent with those obtained by Sinha and Srinivasan [6].

In particular, if \(m(\tau)\) and \(k(\tau)\) vary according to the linear laws

\[ m(\tau) = m_0 + m_1 \tau, \quad k(\tau) = k_0 + k_1 \tau, \]

(33)
equation (31) can be further simplified to

\[ a = a(0) \left[ \frac{m_0}{m_0 + m_1 \tau} \right]^{(\beta/2m_1 + 1/4)} \left[ \frac{k_0}{k_0 + k_1 \tau} \right]^{1/4}. \]

(34)

For \(m(\tau)\) and \(k(\tau)\) given by equation (33), a closed form expression for \(\psi\) can also be obtained from equation (32). However, it is rather lengthy and is therefore omitted.

It is observed from equation (34) that the amplitude growth or decay is a power law type function, in contrast to systems with constant parameters where the amplitude is an exponential function of time. Results of the analysis are shown in Figures 1 and 2 for some typical values of system parameters. In the first case, the stiffness \(k(\tau)\) is taken as an increasing function of time, while for the second case, \(k_1\) is negative and \(k(\tau)\) decreases with time. For both cases, \(m(\tau)\) decreases with time. Results obtained from a fourth order Runge–Kutta numerical scheme are also presented for a comparison. A second order approximation is not considered for this example problem for reasons which will become obvious after the consideration of example 2.

3.2. EXAMPLE 2

Next, the pulse response of a pendulum of constant mass but slowly varying length is considered. Many practical problems can be reduced to this simple model. With the angular displacement denoted by \(x\), the acceleration due to gravity by \(g\), the mass of the...
pendulum by \( m \), the slowly varying length by \( l(\tau) \), and the arbitrary pulse loading by \( q(\tau) \), the differential equation of motion takes the form

\[
\frac{d}{d\tau} \left[ m l^2(\tau) \frac{dx}{d\tau} \right] + m g l(\tau) \sin x = q(\tau).
\]  

(35)

When the amplitude of vibration is not too large, \( \sin x \) may be replaced by the first two terms of its power series expansion

\[
\sin x = x - x^3/6.
\]  

(36)

Equation (35) may then be written as

\[
\frac{d}{d\tau} \left[ m l^2(\tau) \frac{dx}{d\tau} \right] + m g l(\tau) x + ef(\tau, x) = q(\tau),
\]  

(37)

where

\[
ef(\tau, x) = -m g l(\tau) x^3/6.
\]  

(38)

When the procedure outlined in section 2 is followed, \( p(t) \) is given by

\[
p(t) = q(\tau)/m g l(\tau),
\]  

(39)

while the equation for \( y(t) \) takes the form

\[
\frac{d}{d\tau} \left[ m g l^2(\tau) \frac{dy}{d\tau} \right] + m g l(\tau) y - \frac{m g l(\tau)}{6} \left[ a \cos \psi + \frac{q(\tau)}{m g l(\tau)} \right]^3 = 0.
\]  

(40)

The first as well as the second order approximations are reported for this problem.

3.3. FIRST ORDER APPROXIMATION

After performing the necessary integrations in equations (15) and (16), the amplitude and the phase equations are obtained from equations (23) as

\[
\frac{da}{d\tau} = \frac{3 l'(\tau)}{4 l(\tau)} a, \quad \frac{d\psi}{d\tau} = \omega(\tau) \left[ 1 - \frac{\epsilon a^2}{16} - \frac{\epsilon q^2(\tau)}{4 k^2(\tau)} \right],
\]  

(41a,b)
where \( \omega(\tau) = \left[ g/l(\tau) \right]^{1/2} \) and \( k(\tau) = mgl(\tau) \), and the prime denotes differentiation with respect to \( \tau \). These can be easily integrated to yield

\[
a = a(0)\left[ l(0)/l(\tau) \right]^{3/4},
\]

\[
\psi = \int_0^\tau \omega(\tau) \left( 1 - \frac{a^2(0)}{16} \left[ l(0)/l(\tau) \right]^{3/2} - \varepsilon \frac{q^2(\tau)}{4k^2(\tau)} \right) dt + \psi(0).
\]

### 3.4. SECOND ORDER APPROXIMATION

After some simple but rather lengthy calculations, the expressions for \( A_2, B_2, \) and \( u_3 \) are obtained from equations (17), (18) and (19), respectively. Then equations (11), (12) and (13) yield the following results for the second order approximation:

\[
x = a \cos \psi - \frac{\varepsilon q(\tau)}{12mgl(\tau)} \cos 2\psi - \frac{\varepsilon a^3}{192} \cos 3\psi + \frac{q(\tau)}{mgl(\tau)},
\]

\[
d\psi/dt = \omega(\tau) \left( 1 - \frac{\varepsilon a^2}{16} - \frac{\varepsilon q^2(\tau)}{4k^2(\tau)} \right) + \frac{\varepsilon^2}{2\omega(\tau)} \left[ - \frac{3l^2(\tau)}{16l^2(\tau)} + \frac{5l(\tau)l''(\tau)}{4l^2(\tau)} \right]
\]

\[
+ \frac{\omega^2(\tau)q^4(\tau)}{32k^4(\tau)} - \frac{\omega^2(\tau)q^4(\tau)}{72k^4(\tau)} + \frac{\omega^2(\tau)q^2(\tau)a^2}{32k^2(\tau)} + \frac{5\omega^2(\tau)a^4}{1536},
\]

where

\[
\alpha(\tau) = -\varepsilon \left[ \frac{3l'(\tau)}{4l(\tau)} + \frac{e l'(\tau)q^2(\tau)}{8l(\tau)k^2(\tau)} - \frac{e q(\tau)q'(\tau)}{4k^2(\tau)} \right], \quad \beta(\tau) = \frac{3\varepsilon^2 l'(\tau)}{64l(\tau)}.
\]

It is observed that equation (43b) is a special case of a Bernoulli equation and can be transformed to the linear form

\[
d a^*/dt + 2a(\tau)a^* = -2\beta(\tau), \quad a^* = a^{-2}.
\]

The above equation can be integrated to yield

\[
a^* = \exp \left[ -2 \int \alpha(\tau) \, dt \right] \left\{ -2\beta(\tau) \exp \left[ 2 \int \alpha(\tau) \, dt \right] \right\} dt + C_0 \exp \left[ -2 \int \alpha(\tau) \, dt \right],
\]

where \( C_0 \) is a constant.

It is observed that the equations of second order approximation are quite complicated and even for a simple form of \( l(\tau) \), it may be a formidable task to obtain a closed form solution. Under these circumstances, a simple numerical scheme may be employed to integrate the amplitude and the phase equations. The initial value \( a(0) \) and \( \psi(0) \) are obtained from equation (2), where \( x \) is given by equation (43a). This leads to two simultaneous non-linear algebraic equations which must be solved through an iteration procedure.

Two different types of excitations are considered. Figures 3 and 4 show the response due to an exponentially decreasing pulse and a cosine pulse, respectively. In both cases, the length of the pendulum is assumed to be a linear function of time. It is seen that the approximate results agree well with the numerical solutions generated by a fourth-order Runge-Kutta method. However, it is rather disappointing to note that the results of second order approximation show very little improvement over the first order solutions.
It is so insignificant that the two approximations are indistinguishable in both Figures 3 and 4.

4. DISCUSSION AND CONCLUSIONS

An approximate analysis of non-linear, non-stationary vibrational systems subjected to an arbitrary pulse excitation is presented. The non-linearity is assumed to be small. The non-stationary system parameters and the forcing function may vary slowly with respect to the "normal" period of the system. First as well as second order approximate solutions are presented through an application of the well-known Bogoliubov–Mitropolsky method. Results of the illustrative examples show that there is a close agreement between the approximate and the numerical solutions. The first order expressions are simple, and, in most cases, the amplitude and the phase can be evaluated in closed forms. At the most, a simple numerical quadrature, such as Simpson's rule,
may be needed to integrate the phase equation. However, the calculations for the second order approximation are quite cumbersome and the expressions for $da/dt$ and $d\psi/dt$ turn out to be rather complicated. For most practical problems, each equation must be integrated numerically, which seems to defeat the very purpose of an approximate analytical analysis.

In conclusion, the Bogoliubov–Mitropolsky technique can be successfully used to study the pulse response of non-linear systems with slowly varying parameters. The first order approximate solutions are simple, accurate and may prove to be useful in some practical situations. A similar analysis has been carried out for systems with multiple degrees of freedom and is reported in a companion paper.

REFERENCES