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Vanishing viscosity limit for incompressible flow inside a rotating circle

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Abstract

In this article we consider circularly symmetric incompressible viscous flow in a disk. The boundary condition is no-slip with respect to a prescribed time-dependent rotation of the boundary about the center of the disk. We prove that, if the prescribed angular velocity of the boundary has finite total variation, then the Navier–Stokes solutions converge strongly in L^2 to the corresponding stationary solution of the Euler equations when viscosity vanishes. Our approach is based on a semigroup treatment of the symmetry-reduced scalar equation.

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1. Introduction

In this article we study the vanishing viscosity limit for incompressible planar and circularly symmetric flow in a circular cylinder, assuming a no-slip boundary condition with respect to an unsteady rotation of the boundary around its axis of symmetry. We prove that, if the angular velocity of the cylinder has finite total variation, then, as viscosity vanishes, the solutions of the Navier–Stokes equations converge to the initial velocity, which can be regarded as a stationary solution of the Euler equations. This work generalizes previous results by S. Matsui, see [17], and by J. Bona and J. Wu, see [3]; see also [11] for a different proof of Matsui's result.

The vanishing viscosity limit for solutions of the Navier–Stokes equations on domains with boundary is a classical problem, with large associated literature and deep physical meaning. In particular, the natural conjecture that the limiting flow satisfies the incompressible Euler equations is an open problem even for smooth bounded domains in the plane, with smooth initial data. The difficulty and relevance of this problem come from the trend that small viscosity flows have to develop large spatial gradients near physical boundaries, a phenomenon called *boundary layer*. Such large gradients are due to the incompatibility between the Dirichlet-type no-slip boundary condition for the Navier–Stokes equations and the material boundary condition for the Euler equations. The analytical treatment of the vanishing viscosity limit is further complicated by the possibility that convection may separate the boundary layer from the material boundary, thereby affecting the interior flow. This is called *boundary layer separation*, and it is an essential feature of the physical modeling of fluid–structure interaction. We refer the reader to [22] for a complete account on the physics of boundary layers.

There are several rigorous convergence results in the literature which should be mentioned. First, K. Asano's [2], and M. Sammartino's and R. Caflisch's work [20,21] show convergence and describe the structure of the error term, i.e. the boundary layer itself, for analytic data, using a modified Cauchy–Kowalewskaya Theorem. Several variants of the vanishing viscosity limit have

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been considered: see [4,14,8] for the less singular problem of the vanishing viscosity limit for Navier–Stokes with Navier friction condition, [24] for the case of non-characteristic boundaries, [9,23,25] for analytic criteria for the convergence of Navier–Stokes to Euler solutions. For rotating fluids, we mention the geophysically motivated work [6,16]. Finally, for the specific problem of rotating flow for 2D Navier–Stokes in the disk, we have the work of Bona and Wu, see [3], where the case of initial velocity compatible with no-slip is considered, using an explicit technique based on Bessel expansions and the work of Matsui [17], who used an approach based on energy and Schauder estimates to treat the situation where the initial velocity is not compatible with the no-slip condition.

We consider planar purely rotational flow on a disk with no-slip condition with respect to a rotating boundary. As in [3,17], the symmetry we are imposing on the flow makes the underlying problem linear, by decoupling pressure and velocity, and planar. Furthermore, the symmetry also precludes boundary layer separation. Considering initial velocity incompatible with the no-slip condition allows us to consider the motion generated by an impulsively stopped rotating cylinder, see the discussion in Section 2. In this sense, the present work is a natural extension of Matsui's work since his result may be regarded as taking the angular velocity of the boundary to be a Heaviside function, which we extend by considering arbitrary BV motions. From the technical point of view, our work differs from the previous treatment of this problem by the use of a semigroup approach. The main point of the present work is to highlight the effect of (finite or infinite) tangential acceleration of the boundary on the production of the boundary layer by means of a highly simplified example.

The remainder of this work is divided into five sections. In Section 2 we introduce rotational flows on the disk, do the symmetry reductions and formulate our main result; in Section 3 we introduce the basic semigroup treatment of the symmetry-reduced equation; in Section 4 we develop the rigorous treatment of the boundary value problem with BV data; in Section 5 we prove our convergence result and in Section 6 we draw conclusions and point the way to future research.

2. Circularly symmetric solutions of 2D Navier–Stokes in the disk

In this section, we present the statement of our main result and discuss symmetry reduction of the Navier–Stokes equations.

Let us denote the unit disk in the plane by $\mathfrak{D}$. We will consider the 2D Navier–Stokes equations in $\mathfrak{D}$, with initial velocity u_0 and we assume that the boundary is rotating with a prescribed angular velocity $\alpha = \alpha(t)$:

$$\begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu = -\nabla p + \nu \Delta u^\nu, & \text{in } \mathfrak{D} \times (0,\infty),\\ \operatorname{div} u^\nu = 0, & \text{in } \mathfrak{D} \times [0,\infty),\\ u^\nu(x,0) = u_0(x) = v_0(|x|)\dfrac{x^\perp}{|x|}, & \text{at } \{t=0\},\\ u^\nu(x,t) = \dfrac{\alpha(t)}{2\pi} x^\perp, & \text{on } |x|=1,\ t\in(0,\infty), \end{cases} \tag{2.1}$$

where $x^\perp = (-x_2, x_1)$, if $x = (x_1, x_2)$.

Our main result is the following theorem.

Theorem 2.1. *Let u^ν be the solution of the 2D Navier–Stokes equations in the unit disk (2.1). The boundary of the disk is rotating with angular velocity $\alpha \in BV(\mathbb{R})$. Assume that the initial velocity $u_0 \in L^2(\mathfrak{D})$ has circular symmetry, i.e. $u_0(x) = v_0(|x|)x^\perp$. Then, u_0 is a steady solution of the 2D Euler equation and u^ν converges strongly in $L^\infty([0,T], L^2(\mathfrak{D}))$ to u_0 as $\nu \to 0$.*

The physical situation we wish to consider is that of an infinite circular cylinder filled with fluid. The boundary of the cylinder is a material shell which is assumed to rotate about its center of symmetry with angular velocity $\alpha = \alpha(t)$ (positive rotation is counterclockwise). We restrict our attention to planar and circularly symmetric motions, so that the fluid velocity u is given by

$$u = u(x_1,x_2,x_3,t) = v(r,t)\frac{(-x_2,x_1,0)}{r} = V(x_1,x_2,t)\frac{(-x_2,x_1,0)}{r},$$

with $r = \sqrt{x_1^2 + x_2^2}$. This symmetry assumption is consistent with the Navier–Stokes equations as long as the initial velocity is planar and circularly symmetric, i.e., weak solutions satisfying the symmetry assumptions exist globally in time for symmetric initial data. The proof of this fact is part of our analysis. It should be remarked that the assumption that such flows stay planar for all Reynolds number is unphysical — 3D turbulence is expected if the Reynolds number is large enough. The limit behavior we wish to present is motivated by mathematical considerations.

For these circularly symmetric planar flows, the initial-boundary-value problem for the Navier–Stokes system (2.1) reduces to the scalar problem

$$\begin{cases} V_t^\nu = \nu \Delta V^\nu - \nu \dfrac{V^\nu}{|x|^2}, & \text{in } \mathfrak{D}\times(0,\infty),\\ V^\nu(x,0) = V_0(x) = v_0(|x|), & \text{in } \mathfrak{D},\\ V^\nu(x,t) = \dfrac{\alpha(t)}{2\pi}, & \text{for } |x|=1,\ t\in(0,\infty), \end{cases} \tag{2.2}$$

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in the sense that, if the data were smooth and if V^ν were a smooth solution of the initial-boundary-value problem above then

$$u^\nu = V^\nu \frac{x^\perp}{|x|} \tag{2.3}$$

would be a solution of (2.1).

We will consider $\alpha \in BV(\mathbb{R})$, which includes impulsively accelerated motions. We refer the reader to [5] for the basic facts regarding $BV(\mathbb{R})$. In particular, we recall that BV functions has one-sided limits at every point. Now, the solution of problem (2.2) should not depend on the behavior of α for $t < 0$. Therefore, we may assume that α has the same one-sided limits at $t = 0$, by modifying $\alpha(t)$ for $t < 0$ if necessary. In fact, all information needed on the behavior of α for $t < 0$ should be summarized by the initial data, which, we wish to emphasize, is not assumed to match $\alpha(0)$ or $\alpha(0^+)$ at the boundary. It would be more natural to consider that the initial velocity u_0 is consistent at the boundary with $\alpha(0^-)$, so that it was somehow "generated" by the past motion of the boundary, and retain the possibility that the derivative α' might have a Dirac at $t = 0$. However, this would create a minor technical difficulty in our treatment and still would not influence the Navier–Stokes solution for $t > 0$.

Definition 2.1. We will call a function $\alpha \in BV(\mathbb{R})$ such that $\alpha(0^+) = \alpha(0^-)$ an *adjusted BV function*.

Taking $\nu = 0$ in (2.1) leads to the Euler system, which after symmetry reduction becomes $V_t^0 = 0$, or $V^0(x,t) = V_0(x)$. This expresses the fact that circularly symmetric planar velocities are stationary solutions of the Euler equations. Our main result consists in showing that the solution V^ν of the system (2.2) converges to its initial data when $\nu \to 0$, or, in other words, that the solution u^ν of the Navier–Stokes system (2.1) converges to a solution of the Euler equations when $\nu \to 0$.

The reduction of the Euler and Navier–Stokes equations using circular symmetry is standard. Details regarding the facts presented above can be found, for example, in [15].

We will make a change of variables in (2.2) to obtain an initial-boundary-value problem with homogeneous Dirichlet boundary conditions. Let α be an adjusted BV function and set

$$\overline{V}(x,t) = \frac{\alpha(t)}{2\pi}|x|^2.$$

Set $W^\nu(x,t) = V^\nu(x,t) - \overline{V}(x,t)$. Then, W^ν solves the following problem:

$$\begin{cases} W_t^\nu = \nu \Delta W^\nu - \nu \dfrac{W^\nu}{|x|^2} + \nu \dfrac{3\alpha(t)}{2\pi} - \dfrac{\alpha'(t)}{2\pi}|x|^2, & x \in \mathfrak{D},\ t > 0, \\ W^\nu\lfloor_{t=0}(x) = W_0(x) := V_0(x) - \dfrac{\alpha(0^+)}{2\pi}|x|^2, & x \in \mathfrak{D}, \\ W^\nu(x,t) = 0, & |x| = 1, \end{cases} \tag{2.4}$$

where $\alpha'(t)$ is the distributional derivative of $\alpha(t)$, a finite Radon measure on $\mathbb{R}$. The first equation is a heat equation with singular potential and forcing. We will study it by exploiting the properties of the semigroup generated by (a suitable extension of) the unbounded linear operator $\Delta - \frac{I}{|x|^2}$ with homogeneous Dirichlet boundary conditions on $L^2(\mathfrak{D})$.

3. Semigroups and circular symmetry

One of the key ingredients in the proof of our main theorem is the observation that there exists an extension of the operator $\Delta - \frac{I}{|x|^2}$ which generates a strongly continuous and analytic semigroup on $L^2(\mathfrak{D})$. We refer the reader to e.g. [7] for an introduction to the theory of semigroups and all relevant definitions.

The operator $\Delta - \frac{I}{|x|^2}$ is a Schroedinger operator with singular potential in the sense that the potential is not L^1. Its spectral theory is known (see e.g. [19]). We recall below some facts that are essential to our analysis.

In what follows we denote by (u, v) the standard L^2 inner product on $\mathfrak{D}$ and by $\mathcal{H}^1$ the space

$$\mathcal{H}^1 \equiv H_0^1(\mathfrak{D}) \cap L^2\left(\mathfrak{D}, \frac{\mathrm{d}x}{|x|^2}\right), \tag{3.1}$$

endowed with the inner product

$$((u,v)) := \left(\frac{1}{|x|}u, \frac{1}{|x|}v\right) + (\nabla u, \nabla v),$$

and the corresponding norm

$$|||u|||^2 := ((u.u)) = \|\nabla u\|_{L^2}^2 + \int_D \frac{|u|^2}{|x|^2}\mathrm{d}x.$$

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We next observe that $A = \Delta - \frac{1}{|x|^2} : C_c^\infty(\mathfrak{D} \setminus \{0\}) \to L^2$ is symmetric and semibounded (from below) on $L^2(\mathfrak{D})$, that is,

$$\|u\|_{L^2}^2 \leq (u, -Au) = \|\!|u|\!\|^2, \quad u \in C_c^\infty(\mathfrak{D} \setminus \{0\}). \tag{3.2}$$

It therefore admits a self-adjoint extension on $L^2(\mathfrak{D})$, the Friedrichs extension (for a proof we refer for example to [12, Section 33.3]), which by abuse of notation we continue to call A. This extension has dense domain

$$\mathcal{D}(A) = \left\{ u \in \mathcal{H}^1 | \exists f \in L^2(\mathfrak{D}) \text{ such that } ((u, v)) = (f, v), \ \forall v \in \mathcal{H}^1 \right\}, \tag{3.3}$$

and satisfies $((u, v)) = (-Au, v)$, for all $u \in D(A)$, $v \in \mathcal{H}^1$. Hence, inequality (3.2) continues to hold for this extension, which gives rise to a sectorial operator $-A$ on L^2, being self-adjoint, bounded from below. By Theorem 1.3.4 of [7] we obtain that A generates a strongly continuous and analytic semigroup on $L^2(\mathfrak{D})$.

We summarize these observations in the following proposition.

Proposition 3.1. *Let A be the Friedrichs extension of the operator $\Delta - I/|x|^2$ with dense domain $D(A)$ given in (3.3). Then, A generates a strongly continuous and analytic semigroup e^{tA} on $L^2(\mathfrak{D})$.*

Our next observation is that this semigroup preserves circular symmetry. For each $\theta \in (0, 2\pi)$ we define $R_\theta : \mathfrak{D} \to \mathfrak{D}$ to be the rigid counterclockwise rotation of the disk about its center by the angle θ. Let R_θ^* be the pull-back operator by R_θ, that is, $R_\theta^* W(x) = W(R_\theta(x))$. It is easily checked to be an isometry on $L^2(\mathfrak{D})$. A function W is circularly symmetric if and only if it is invariant under R_θ^*, $R_\theta^* W(x) = W(x)$.

Lemma 3.2. *The semigroup e^{tA} commutes with the operator R_θ^* on $L^2(\mathfrak{D})$. In particular if W_0 is circularly symmetric, then for each $t > 0$ $\mathrm{e}^{tA} W_0$ is circularly symmetric.*

The Lemma is an immediate consequence of the facts that A is analytic, it commutes with rotations, and therefore $[\mathrm{e}^{tA}, R_\theta^*] = 0$.

4. Rotating boundary and Duhamel's principle

The purpose of this section is to prove a Duhamel formula for solutions of (2.4) with α an adjusted BV function.

We begin with an approximation lemma for adjusted BV functions. As observed before, functions of bounded variation of one real variable have one-sided limits at every point t. Additionally, there are at most a countable number of jump discontinuities.

Definition 4.1. Let α be a BV function. The left-continuous BV function $\widetilde{\alpha}$ is given by

(i) $\widetilde{\alpha}(t) = \alpha(t)$ if α is continuous at t;
(ii) $\widetilde{\alpha}(t) = \alpha(t-)$ otherwise.

Observe that when α is adjusted $\widetilde{\alpha}$ is continuous at 0. Recall that $\widetilde{\alpha}' = \alpha'$ in the sense of distributions and, furthermore, there exists a unique Radon measure $\mathrm{d}\widetilde{\alpha}$ on $\mathbb{R}$ such that

$$\int_{[t,+\infty)} \mathrm{d}\widetilde{\alpha} = \widetilde{\alpha}(t) - \widetilde{\alpha}(+\infty).$$

If g is a Borel-measurable function, then g is $\mathrm{d}\widetilde{\alpha}$ measurable and the (Lebesgue–Stieltjes) integral $\int g \mathrm{d}\widetilde{\alpha}$ is well-defined. When g is the characteristic function of the interval $[0, t)$, it follows that:

$$\int_{[0,t)} \mathrm{d}\widetilde{\alpha} = \int_{[0,+\infty)} \mathrm{d}\widetilde{\alpha} - \int_{[t,+\infty)} \mathrm{d}\widetilde{\alpha} = \widetilde{\alpha}(t) - \widetilde{\alpha}(0) = \alpha(t^-) - \alpha(0^+). \tag{4.1}$$

(We refer to [5] for a survey of BV functions and the proofs of their properties.)

We now state and prove our approximation lemma.

Lemma 4.1. *Let $\zeta \in C_c^\infty((-1, 1))$ be such that $\zeta \geq 0$, $\int_{\mathbb{R}} \zeta(t)\mathrm{d}t = 1$. Let $\zeta_n = \zeta_n(t) = n\zeta(nt)$. Let $\alpha \in BV(\mathbb{R})$ and let $\widetilde{\alpha}$ be given by Definition 4.1. Set*

$$\alpha_n = \alpha_n(t) \equiv (\zeta_n * \alpha)\left(t - \frac{1}{n}\right) = \int_{\mathbb{R}} \zeta_n(s)\alpha\left(t - \frac{1}{n} - s\right)\mathrm{d}s.$$

Then:

(1) $\alpha_n \underset{n\to\infty}{\to} \widetilde{\alpha}$ *pointwise and strongly in* $L^p_{loc}(\mathbb{R})$, $1 \leq p < \infty$;
(2) $\alpha_n' \rightharpoonup \alpha' (= \widetilde{\alpha}')$ *weak-$*$ in* $\mathcal{BM}(\mathbb{R})$;

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(3) *For any function* $\phi \in C^0(\mathbb{R})$ *and for any* $t > 0$ *we have:*

$$\lim_{n\to\infty} \int_{[0,t)} \phi(s)\alpha_n'(s)\mathrm{d}s = \int_{[0,t)} \phi(s)\mathrm{d}\widetilde{\alpha}(s).$$

Proof. We start by showing pointwise convergence in (1). Fix $t \in \mathbb{R}$ and estimate

$$\begin{aligned} |\alpha_n(t) - \widetilde{\alpha}(t)| &= \left| \int \zeta_n(s)[\alpha(t - 1/n - s) - \widetilde{\alpha}(t)]\mathrm{d}s \right| \\ &\le \sup_{\{-1/n \le s \le 1/n\}} |\alpha(t - 1/n - s) - \widetilde{\alpha}(t)| \underset{n\to\infty}{\longrightarrow} 0, \end{aligned}$$

since $\widetilde{\alpha}(t) = \alpha(t-)$. Next, we note that $\{\alpha_n\}$ is bounded in L^∞ so that strong convergence in L^p, $1 \le p < \infty$ follows by the Dominated Convergence Theorem.

To see (2), we first observe that strong convergence in L^p from (1) immediately gives convergence $\alpha_n' \rightharpoonup \widetilde{\alpha}'$ in the sense of distributions (in fact in the Sobolev space $W^{-1,p}$). Since $\{\alpha_n'\}$ is bounded in $\mathcal{BM}$, a subsequence converges weakly to a Radon measure, which therefore must agree with the distribution $\widetilde{\alpha}'$ by uniqueness of the limit.

Finally, we address (3). We pick a function $\varphi \in C^1(\mathbb{R})$ and we use an integration by part formula (Theorem 3.36 in [5] appropriately modified for left-continuous BV functions):

$$\int_{[0,t)} \varphi(s)\mathrm{d}\widetilde{\alpha}(s) + \int_{[0,t)} \varphi'(s)\widetilde{\alpha}(s)\mathrm{d}s = \varphi(t)\widetilde{\alpha}(t) - \varphi(0)\widetilde{\alpha}(0). \tag{4.2}$$

Usual integration by parts also gives

$$\int_{[0,t)} \phi(s)\alpha_n'(s)\mathrm{d}s = -\int_{[0,t)} \phi'(s)\alpha_n(s)\mathrm{d}s + \phi(t)\alpha_n(t) - \phi(0)\alpha_n(0).$$

Passing to the limit $n \to \infty$, using convergence in part (1),

$$\begin{aligned} \lim_{n\to\infty} \int_{[0,t)} \phi(s)\alpha_n'(s)\mathrm{d}s &= -\int_{[0,t)} \phi'(s)\widetilde{\alpha}(s)\mathrm{d}s + \phi(t)\widetilde{\alpha}(t) - \phi(0)\widetilde{\alpha}(0) \\ &= \int_{[0,t)} \phi(s)\mathrm{d}\widetilde{\alpha}(s), \end{aligned}$$

where we employed (4.2) in the last step. (3) now follows by approximating $\phi \in C^0(\mathbb{R})$ with C^1 functions ϕ_k and by noting that the limits in n and k can be exchanged, since α_n' is uniformly bounded in $\mathcal{BM}$. ■

Let $W_0 \in L^2(\mathfrak{D})$ and fix α an adjusted BV function. Consider

$$W^\nu(x,t) = \mathrm{e}^{\nu t A} W_0(x) + \nu \int_0^t \frac{3\widetilde{\alpha}(\tau)}{2\pi} \mathrm{e}^{\nu(t-\tau)A}[1]\mathrm{d}\tau - \int_{[0,t)} \frac{\mathrm{e}^{\nu(t-\tau)A}[|x|^2]}{2\pi} \mathrm{d}\widetilde{\alpha}(\tau), \tag{4.3}$$

where the first integral is a Bochner integral, while the second is a Lebesgue–Stieltjes–Bochner integral owing to the integrability of α and the strong continuity in t of the semigroup e^{tA} on $L^2(\mathfrak{D})$. Here, we are using the notation $\mathrm{e}^{\nu s A}[f]$ to denote the operator $\mathrm{e}^{\nu s A}$ evaluated at the function $f \in L^2(\mathfrak{D})$.

We wish to show that the function W^ν, defined through the expression above, is a weak solution to problem (2.4). We require the following technical result. Its proof is standard and will be omitted.

Lemma 4.2. *Let* $\Omega \subset \mathbb{R}^N$ *be such that* $|\Omega| < \infty$. *Let* $\{f_n\}$ *be a sequence of functions bounded in* $L^p(\Omega)$, *for some* $1 < p \le \infty$, *and assume that* $f_n \to f$ *a.e. in* Ω. *Then* $f \in L^p$ *and* $f_n \to f$ *strongly in* $L^1(\Omega)$ *and weakly in* $L^p(\Omega)$.

Proposition 4.3. *Let* W^ν *be defined through* (4.3) *with* $W_0 \in L^2(\mathfrak{D})$ *and* $\alpha \in BV(\mathbb{R})$. *Then* W^ν *is a weak solution in* $L^\infty((0,+\infty); L^2(\mathfrak{D}))$ *of the problem*

$$\begin{cases} W_t^\nu = \nu A W^\nu + \nu \dfrac{3\alpha(t)}{2\pi} - \dfrac{\alpha'(t)}{2\pi}|x|^2, & x \in \mathfrak{D},\ t > 0, \\ W^\nu\lfloor_{t=0}(x) = W_0(x), & x \in \mathfrak{D}, \\ W^\nu(x,t) = 0, & |x| = 1. \end{cases} \tag{4.4}$$

Proof. Fix $W_0 \in L^2(\mathfrak{D})$ and $\alpha \in BV(\mathbb{R})$. Then there exist $\{W_{n,0}\} \subset \mathcal{D}(A)$ such that $W_{n,0} \to W_0$ strongly in $L^2(\mathfrak{D})$ and also pointwise a.e. $\mathfrak{D}$. Consider the approximating sequence $\{\alpha_n\}$ given by Lemma 4.1.

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For each n let

$$W_n^\nu = \mathrm{e}^{\nu t A} W_{n,0}(x) + \nu \int_0^t \frac{3\alpha_n(\tau)}{2\pi} \mathrm{e}^{\nu(t-\tau)A}[1]\mathrm{d}\tau - \int_0^t \frac{\alpha_n'(\tau)}{2\pi} \mathrm{e}^{\nu(t-\tau)A}[|x|^2]\mathrm{d}\tau. \tag{4.5}$$

Since α_n is a C^1 function and since $W_{n,0} \in \mathcal{D}(A)$, it follows that W_n^ν is the unique mild solution of (4.4) with data $W_{n,0}$ and α_n. This is also a classical solution, and hence a weak solution (see [18], Chapter 4, Corollary 2.5).

Fix $T > 0$ and let $\phi \in C_c^1([0,T), \mathcal{D}(A))$; we use again the notation $(\cdot,\cdot)$ for the L^2 inner product. We have that:

$$\int_0^T (\phi_t, W_n^\nu)\mathrm{d}t + \int_0^T (A\phi, W_n^\nu)\mathrm{d}t + \int_0^T \frac{3\nu\alpha_n}{2\pi}(\phi, 1)\mathrm{d}t - \int_0^T \frac{\alpha_n'}{2\pi}(\phi, |x|^2)\mathrm{d}t = -(\phi(0), W_{n,0}). \tag{4.6}$$

Note that, from expression (4.5), we may conclude that $\{W_n^\nu\}$ is uniformly bounded in $L^\infty((0,T); L^2(\mathfrak{D})) \subset L^2((0,T)\times\mathfrak{D})$. On the other hand, given the convergence properties of $W_{n,0}$ and of α_n, it follows that W_n^ν converges pointwise to the function W^ν defined by (4.3). We find, using Lemma 4.2 that W_n^ν converges weakly in $L^2((0,T)\times\mathfrak{D})$ to W^ν and hence weak-$*$ in $L^\infty((0,T); L^2(\mathfrak{D}))$. Passing to the limit in (4.6) we see that W^ν satisfies:

$$\int_0^T (\phi_t, W^\nu)\mathrm{d}t + \int_0^T (A\phi, W^\nu)\mathrm{d}t + \int_0^T \frac{3\nu\widetilde{\alpha}}{2\pi}(\phi, 1)\mathrm{d}t - \int_{[0,T)} \frac{1}{2\pi}(\phi, |x|^2)\mathrm{d}\widetilde{\alpha}(t) = -(\phi(0), W_0). \tag{4.7}$$

Clearly we have

$$\int_0^T \frac{3\nu\widetilde{\alpha}}{2\pi}(\phi, 1)\mathrm{d}t = \int_0^T \frac{3\nu\alpha}{2\pi}(\phi, 1)\mathrm{d}t.$$

Furthermore, if $\phi \in C_c^\infty((0,T), \mathcal{D}(A))$ then it can be easily seen that

$$\int_{[0,T)} \frac{1}{2\pi}(\phi, |x|^2)\mathrm{d}\widetilde{\alpha}(t) = \int_0^T \frac{1}{2\pi}(\phi, |x|^2)\alpha'(t)\mathrm{d}t,$$

where α' is understood in the sense of distributions. We have hence shown that W^ν is a distributional solution of the partial differential equation in problem (4.4) which assumes its initial data W_0 as described in (4.7). This concludes the proof. ■

5. Main theorem

In this section we establish the convergence of the viscous flow toward the inviscid Euler limit. The first step is to rigorously establish the connection between the symmetry-reduced system (2.4) and the Navier–Stokes system, in the case of non-smooth initial and boundary data.

Proposition 5.1. *Let $W_0 \in L^2(\mathfrak{D})$ and $\alpha \in BV(\mathbb{R})$. If W_0 is circularly symmetric then*

$$u^\nu = u^\nu(x,t) = \left(W^\nu + \frac{\alpha(t)}{2\pi}|x|^2\right)\frac{(-x_2, x_1)}{\sqrt{x_1^2 + x_2^2}} \tag{5.1}$$

is the unique weak solution of the 2D Navier–Stokes initial-boundary-value problem (2.1) in $L^\infty((0,\infty); L^2(\mathfrak{D})) \cap L^2_{loc}((0,\infty); H^1(\mathfrak{D}))$. The function W^ν above is given by expression (4.3).

Proof. Fix $W_0 \in L^2(\mathfrak{D})$ and $\alpha \in BV$ and assume that W_0 is circularly symmetric. From Lemma 3.2 we obtain that W^ν, given by (4.3), is also circularly symmetric, since the forcing term in (4.4)

$$F(t) \equiv \nu\frac{3\alpha(t)}{2\pi} - \frac{\alpha'(t)}{2\pi}|x|^2$$

is circularly symmetric. Indeed, to see that W^ν is circularly symmetric use Lemma 3.2 on each term of the expression for W^ν. As a consequence, $V^\nu = W^\nu + \alpha(t)|x|^2/(2\pi)$ is also circularly symmetric.

Consider the approximations α_n given by Lemma 4.1 and let $W_{n,0} \in \mathcal{D}(A)$ be circularly symmetric and such that $W_{n,0} \to W_0$ in $L^2(\mathfrak{D})$ (the approximations $W_{n,0}$ can be obtained, for instance, by solving the heat equation in the disk with homogeneous Dirichlet boundary condition). Let W_n^ν be defined through (4.5). We have shown, in the proof of Proposition 4.3, that

$$W_n^\nu \to W^\nu, \tag{5.2}$$

as $n \to \infty$, where the convergence is a.e.-$((0,\infty)\times\mathfrak{D})$ and weak in $L^2((0,T)\times\mathfrak{D})$ for any $T > 0$.

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Since $W_{n,0}$ was assumed to be circularly symmetric, we have that W_n^ν is also circularly symmetric, and so, of course, is the corresponding

$$V_n^\nu \equiv W_n^\nu + \frac{\alpha_n}{2\pi}|x|^2.$$

We introduce $u_n^\nu(x,t) = \left(W_n^\nu + \frac{\alpha_n(t)}{2\pi}|x|^2\right)\frac{(-x_2,x_1)}{\sqrt{x_1^2+x_2^2}}$.

We will first verify that u_n^ν is the weak solution of the Navier–Stokes equations. By Lemma 3.2.1 and Theorem 3.2.2 in [7], $W_n^\nu \in C([0,\infty); L^2) \cap C^1((0,\infty); L^2)$. Furthermore, $W_n^\nu(\cdot,t) \in \mathcal{D}(A)$ for all $t > 0$, which allows us to perform a standard energy estimate. Fix $t > 0$ and multiply equation (4.4) by W_n^ν, integrating in $\mathfrak{D} \times (0,t)$ to obtain

$$E[W_n^\nu](t) \equiv \int_{\mathfrak{D}} |W_n^\nu|^2(x,t)\mathrm{d}x + \int_0^t \int_{\mathfrak{D}} \left(\frac{|W_n^\nu(x,s)|^2}{|x|^2} + |\nabla W_n^\nu(x,s)|^2\right)\mathrm{d}x\mathrm{d}s$$
$$= \int_{\mathfrak{D}} |W_{n,0}(x)|^2\mathrm{d}x + \int_0^t \int_{\mathfrak{D}} \left(\frac{3\nu\alpha_n(s)}{2\pi}W_n^\nu(x,s) - \frac{\alpha_n'(s)}{2\pi}W_n^\nu(x,s)|x|^2\right)\mathrm{d}x\mathrm{d}s.$$

Let $T > 0$. We estimate the right-hand side of the identity above to obtain that, for every $0 < t < T$,

$$E[W_n^\nu](t) \le \|W_{n,0}\|^2_{L^2(\mathfrak{D})} + C\|\alpha_n\|^2_{BV} + \frac{1}{2}\|W_n^\nu\|^2_{L^\infty((0,T);L^2(\mathfrak{D}))}.$$

Taking the supremum of E in $(0,T)$ we find that

$$\frac{1}{2}\|W_n^\nu\|^2_{L^\infty((0,T);L^2(\mathfrak{D}))} + \int_0^T \int_{\mathfrak{D}} \left(\frac{|W_n^\nu(x,s)|^2}{|x|^2} + |\nabla W_n^\nu(x,s)|^2\right)\mathrm{d}x\mathrm{d}s \le \|W_{n,0}\|^2_{L^2(\mathfrak{D})} + C\|\alpha_n\|^2_{BV},$$

which, by construction, is uniformly bounded.

We conclude that, for each $T > 0$,

$$W_n^\nu \in L^\infty((0,T); L^2(\mathfrak{D})) \cap L^2((0,T); H_0^1(\mathfrak{D}) \cap L^2(\mathfrak{D}, \mathrm{d}x/|x|^2)), \tag{5.3}$$

and that the sequence $\{W_n^\nu\}$ is uniformly bounded in each of the spaces above.

Consequently, fixing $T > 0$,

$$V_n^\nu \in C([0,T]; L^2) \cap C^1((0,T); L^2) \cap L^2((0,T); H^1 \cap L^2(\mathrm{d}x/|x|^2)). \tag{5.4}$$

A straightforward calculation using (4.6) allows us to conclude that

$$\partial_t V_n^\nu = \nu\Delta V_n^\nu - \nu\frac{V_n^\nu}{|x|^2},$$

as distributions in $\mathfrak{D} \times (0,T)$. As the left-hand side of the identity above belongs to L^2, pointwise in $(0,T)$, it follows that $(\nu\Delta V_n^\nu - \nu V_n^\nu/|x|^2)(\cdot,t) \in L^2(\mathfrak{D})$, for all $0 < t < T$.

Let ϕ be a test vector field in $C_c^\infty(\mathfrak{D} \setminus \{0\})$. We will first show that

$$\left(\partial_t u_n^\nu, \phi\right) = (\nu\Delta u_n^\nu, \phi) \equiv \left(\nu V_n^\nu, \frac{x^\perp}{|x|}\cdot\Delta\phi\right). \tag{5.5}$$

To see this, note that

$$(\partial_t u_n^\nu, \phi) = \left(\partial_t\left(V_n^\nu\frac{x^\perp}{|x|}\right), \phi\right) = \left(\partial_t V_n^\nu, \frac{x^\perp}{|x|}\cdot\phi\right)$$
$$= \left(\nu\Delta V_n^\nu - \nu\frac{V_n^\nu}{|x|^2}, \frac{x^\perp}{|x|}\cdot\phi\right) = \left(\nu V_n^\nu, \Delta\left(\frac{x^\perp}{|x|}\cdot\phi\right) - \frac{x^\perp}{|x|^3}\cdot\phi\right).$$

Therefore, to show (5.5) it is enough to show that

$$\left(\nu V_n^\nu, \frac{x^\perp}{|x|}\cdot\Delta\phi - \left[\Delta\left(\frac{x^\perp}{|x|}\cdot\phi\right) - \frac{x^\perp}{|x|^3}\cdot\phi\right]\right) = 0.$$

An easy calculation yields

$$\frac{x^\perp}{|x|}\cdot\Delta\phi - \left[\Delta\left(\frac{x^\perp}{|x|}\cdot\phi\right) - \frac{x^\perp}{|x|^3}\cdot\phi\right] = \operatorname{div}\left(2\frac{x\cdot\phi}{|x|^3}x^\perp\right).$$

As V_n^ν is circularly symmetric we obtain

$$\left(\nu V_n^\nu,\ \mathrm{div}\left(2\frac{x\cdot\phi}{|x|^3}x^\perp\right)\right)=0,$$

since ∇V^ν is parallel to x. This establishes (5.5). Since $C_c^\infty(\mathfrak{D}\setminus\{0\})$ is dense in $L^2(\mathfrak{D})$ we find

$$\partial_t u_n^\nu = \nu\Delta u_n^\nu. \tag{5.6}$$

Next we will show that

$$(u_n^\nu\cdot\nabla u_n^\nu,\Phi)=0, \tag{5.7}$$

Q1 for any divergence-free vector field $\Phi\in C_c^\infty(\mathfrak{D})$ and for each $t>0$. Indeed,

$$\begin{aligned}(u_n^\nu\cdot\nabla u_n^\nu,\Phi) &= -\int_{\mathfrak{D}} u_n^\nu\cdot(u_n^\nu\cdot\nabla\Phi)\mathrm{d}x = -\int_{\mathfrak{D}}\frac{(V_n^\nu)^2}{|x|^2}x^\perp\cdot(x^\perp\cdot\nabla\Phi)\mathrm{d}x\\ &= -\int_{\mathfrak{D}}\frac{(V_n^\nu)^2}{|x|^2}x^\perp\cdot(\nabla(x^\perp\cdot\Phi)+\Phi^\perp)\mathrm{d}x = -\int_{\mathfrak{D}}\frac{(V_n^\nu)^2}{|x|^2}[x^\perp\cdot(\nabla(x^\perp\cdot\Phi))+x\cdot\Phi]\mathrm{d}x\\ &= -\int_{\mathfrak{D}}\frac{(V_n^\nu)^2}{|x|^2}\,\mathrm{div}\,[x^\perp(x^\perp\cdot\Phi)]\mathrm{d}x-\int_{\mathfrak{D}}\frac{(V_n^\nu)^2}{|x|^2}x\cdot\Phi\mathrm{d}x\equiv I_1+I_2.\end{aligned}$$

Both I_1 and I_2 vanish. Indeed,

$$I_1=\int_{\mathfrak{D}}\nabla\left(\frac{(V_n^\nu)^2}{|x|^2}\right)\cdot[x^\perp(x^\perp\cdot\Phi)]\mathrm{d}x=0,$$

since $(V_n^\nu)^2/|x|^2$ is circularly symmetric, which implies that its gradient is proportional to x. Also, since $V_n^\nu\in L^2(\mathfrak{D},\mathrm{d}x/|x|^2)$ we find that

$$\frac{(V_n^\nu)^2}{|x|^2}x=\nabla\left[\int_0^{|x|}\frac{(V_n^\nu)^2(r,t)}{r}\mathrm{d}r\right],$$

where we have abused the notation by considering V_n^ν as a function of r. Therefore, given that Φ is divergence-free and compactly supported, it follows that $I_2=0$. This concludes the proof of (5.7).

Next we put together (5.6) and (5.7) and integrate in time, using the continuity in time of u_n^ν at $t=0$, given by (5.4). We obtain that, for any divergence-free test vector field $\Phi\in C_c^\infty([0,\infty)\times\mathfrak{D})$,

$$\int_0^\infty\int_{\mathfrak{D}}\Phi_t\cdot u_n^\nu+(u_n^\nu\cdot\nabla\Phi)\cdot u_n^\nu\mathrm{d}x\mathrm{d}t+\int_{\mathfrak{D}}\Phi(x,0)\cdot u_n^\nu(x,0)\mathrm{d}x=\nu\int_0^\infty\int_{\mathfrak{D}}\Delta\Phi\cdot u_n^\nu\mathrm{d}x\mathrm{d}t, \tag{5.8}$$

which is precisely the weak formulation of the Navier–Stokes equations in the interior of the disk.

We now wish to pass to the limit $n\to\infty$ in (5.8). First note that, by (5.2) and by Lemma 4.1, we have

$$u_n^\nu\to\left(W^\nu+\frac{\widetilde{\alpha}(t)}{2\pi}|x|^2\right)\frac{(-x_2,x_1)}{\sqrt{x_1^2+x_2^2}}\equiv u^\nu\quad\text{a.e. }(0,\infty)\times\mathfrak{D}.$$

Next, we go back to the observation following (5.3), from which we obtain that, for each $T>0$, $\{V_n^\nu\}$ is uniformly bounded in $L^\infty((0,T);L^2(\mathfrak{D}))\cap L^2((0,T);H^1(\mathfrak{D})\cap L^2(\mathfrak{D},\mathrm{d}x/|x|^2))$. Trivially, $\{u_n^\nu\}$ is uniformly bounded in $L^\infty((0,T);L^2(\mathfrak{D}))$ and a bound for u_n^ν in $L^2((0,T);H^1(\mathfrak{D}))$ follows from using both the H^1 and $L^2(\cdot,\mathrm{d}x/|x|^2)$ bounds for V_n^ν. These estimates allow us to pass to the limit in identity (5.8) and obtain that the limit u^ν is a weak solution of the initial-value problem for the 2D Navier–Stokes equations. The fact that u^ν is a weak solution of problem (2.1) may be deduced from the fact that u^ν satisfies the boundary condition $u=\alpha(t)x^\perp/(2\pi)$, in the trace sense, on $|x|=1$, since $W^\nu(\cdot,t)\in H_0^1(\mathfrak{D})$, for almost all time $t>0$. Uniqueness can be established in a standard way, using the available energy estimate. ■

We are now ready to prove our main result concerning strong convergence to the corresponding stationary solution of the Euler equation as $\nu\to 0^+$.

Theorem 5.2. *Let $u_0\in L^2(\mathfrak{D})$ be a circularly symmetric vector field and let α be an adjusted BV function. Consider u^ν as the unique weak solution to the Navier–Stokes system (2.1) with initial data u_0 and boundary data α, given by Proposition 5.1. Then, for any $T>0$, u^ν converges strongly in $L^\infty((0,T);L^2(\mathfrak{D}))$ to u_0 as $\nu\to 0$.*

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Proof. Fix $T > 0$. We will show that

$$u^\nu \to u_0 = u_0(x) = V_0(x)\frac{x^\perp}{|x|} = \left(W_0(x) + \frac{\widetilde{\alpha}(0)|x|^2}{2\pi}\right)\frac{x^\perp}{|x|},$$

as $\nu \to 0$ in $L^\infty((0,T); L^2(\mathfrak{D}))$.

We begin by using Proposition 5.1 to write

$$u^\nu = u^\nu(x,t) = \left(W^\nu + \frac{\widetilde{\alpha}(t)|x|^2}{2\pi}\right)\frac{x^\perp}{|x|}.$$

Note that we substituted α for $\widetilde{\alpha}$ in (5.1), which we are entitled to do since α and $\widetilde{\alpha}$ differ on a set of measure zero and the equality above is understood in $L^\infty((0,T); L^2(\mathfrak{D})) \cap L^2((0,T); H^1(\mathfrak{D}))$.

In order to prove our result it is enough to show that

$$W^\nu \to W_0 - \frac{\widetilde{\alpha}(t) - \widetilde{\alpha}(0)}{2\pi}|x|^2,$$

since, whenever α is adjusted, we have $W_0(x) + \widetilde{\alpha}(0)|x|^2/(2\pi) = V_0(x)$. We recall the expression (4.3) for W^ν:

$$W^\nu = W^\nu(x,t) = \mathrm{e}^{\nu t A}W_0(x) + \nu\int_0^t \frac{3\widetilde{\alpha}(\tau)}{2\pi}\mathrm{e}^{\nu(t-\tau)A}[1]\mathrm{d}\tau - \int_{[0,t)}\frac{\mathrm{e}^{\nu(t-\tau)A}[|x|^2]}{2\pi}\mathrm{d}\widetilde{\alpha}(\tau) \equiv I_1^\nu + I_2^\nu - I_3^\nu.$$

Since $\mathrm{e}^{\nu t A}W_0 \in C([0,T]; L^2(\mathfrak{D}))$, it follows that $I_1^\nu \to W_0$, as $\nu \to 0$, strongly in $L^\infty([0,T]; L^2(\mathfrak{D}))$.

Next we observe that $\|\mathrm{e}^{\nu s A}[1]\|_{L^2(\mathfrak{D})} \le C$, for all $0 \le s \le T$. Therefore, by Minkowski's inequality for integrals, we have as $\nu \to 0$

$$\sup_{\{0\le t\le T\}} \|I_2^\nu\|_{L^2} \le \frac{3C}{2\pi}\nu\int_0^T |\widetilde{\alpha}|\mathrm{d}\tau \to 0.$$

Finally, with an argument analogous to what we used for I_1^ν, together with Dominated Convergence, we find

$$I_3^\nu \to \frac{|x|^2}{2\pi}\int_{[0,t)}\mathrm{d}\widetilde{\alpha}(\tau),$$

as $\nu \to 0$, strongly in $L^\infty((0,T); L^2(\mathfrak{D}))$, and by (4.1)

$$\frac{|x|^2}{2\pi}\int_{[0,t)}\mathrm{d}\widetilde{\alpha}(\tau) = (\widetilde{\alpha}(t) - \widetilde{\alpha}(0))\frac{|x|^2}{2\pi},$$

This concludes the proof. ■

6. Conclusions and further problems

Let us first observe that, for the sake of simplicity, we have not considered forcing, except for the boundary forcing already present in our problem. It is, in fact, easy to rewrite our argument to include circularly symmetric forcing f^ν in (2.1), as long as $f^\nu \to f$, strongly in $L^1((0,\infty); L^2(\mathfrak{D}))$. This is how the main result in [17] was written and only after this modification would our work become, strictly speaking, an extension of [17].

Imposing $\alpha \in BV$ appears naturally in considering the mild formulation (4.3), which requires to give meaning to α' as a measure. However, a different formulation can be used that allows more irregular driving motions. In [13], we have extended the results of this paper to include angular velocities $\alpha \in L^p[0,T]$, $1 \le p < \infty$. That work shows also that BV regularity is necessary to study the behavior of vorticity in the vanishing viscosity limit.

In [25], Xiaoming Wang generalized Kato's convergence criterion for the inviscid limit, proving that the vanishing viscosity limit (with convergence in $L^\infty((0,T); L^2(\Omega))$) of Navier–Stokes solutions is an Euler solution if and only if the energy dissipation of the viscous flow due to tangential derivatives of velocity in a suitable boundary layer are vanishingly small. This result holds both in two and three space dimensions and in general domains with boundary. For circularly symmetric flows, the tangential derivatives vanish identically and therefore, the energy dissipation criterion is valid, implying strong convergence in $L^\infty((0,T); L^2(\mathfrak{D}))$. It would appear, therefore, that all the known convergence results for circularly symmetric flows follow as immediate corollaries of Wang's work. Now, Wang's proof is based on constructing an approximate boundary layer and estimating the error directly using energy estimates, and his argument requires that the tangential velocity at the boundary have square-integrable time derivatives. In our setting this means imposing that $\alpha \in H^1_{loc}(\mathbb{R})$, and also that the initial data and the boundary data be compatible. Therefore neither Matsui's work in the non-compatible case nor our result is included in Wang's result. Wang's theorem does imply Bona's and Wu's result, and it also implies our result if we assume that $\alpha \in H^1$. The other convergence criteria, Kato's in [9], Temam's and

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Wang's in [23], and Kelliher's in [10], do not apply to our problem because they assume the no-slip condition for the Navier–Stokes system with respect to a stationary boundary.

We mention a few questions which arise naturally from the work presented here. A natural question is to consider the vanishing viscosity limit for 3D flows generated by a rotating sphere, or inside other bodies of revolution. In this case the flow is axisymmetric with swirl, and the nonlinearity does not disappear. It would be very interesting to examine this same problem for some specific flow where boundary layer separation *does* occur, for instance, flows induced by initial vorticity configurations on the disk which are odd with respect to a diameter, single signed on each side of the disk and steady with respect to the Euler dynamics.

Uncited references

[1].

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