On linear quadratic optimal control of linear time-varying singular systems

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Abstract
Linear time-varying singular systems are treated in this paper. We focus on systems with constant-rank $E$ matrices. It is shown that the existence of state feedback for impulse elimination is both sufficient and necessary for the existence of linear-quadratic optimal control. Also, optimal control exists if and only if the corresponding fast subsystem is impulse controllable. The results obtained are extensions of the existing time-invariant theory.

1 Introduction
Mathematical models in the form of linear singular systems, $E(t)\dot{x} = A(t)x + B(t)u$, are essential in describing mechanical multi-body systems and many electrical circuits. They are particularly useful when a nonlinear system subjecting to algebraic constraints is linearized along a nominal trajectory.

Because these systems may exhibit behaviours so different from those of the regular linear systems, theories well established for regular systems are usually inapplicable to singular systems without generalization. In view of this situation, numerous research articles have been dedicated to singular systems in the past three decades. (Liu and Sreeram, 2001), (Chang and Davison, 2001) and (Wang and Liao, 2001) constitute a representative collection of the most recent works.

Finding optimal control subject to linear quadratic cost is crucial in many control problems. For linear systems in regular state-space representation $\dot{x} =
A(t)x + B(t)u, this problem has long been well studied. The quadratic optimal control problem for singular systems, however, remained unexplored until the 1980’s. Two prominent works addressing this issue in the time-invariant setting are (Cobb, 1983) and (Bender and Laub, 1987). Cobb derived a criterion for the existence of optimal control, which condition has close ties to those for state feedback impulse elimination and impulse controllability. The standard Hamilton-Lagrange theory for state-space systems was extended in (Bender and Laub, 1987) to include time-invariant singular systems. Optimal controls can be found by solving Euler-Lagrange equations.

For linear time-varying singular systems, (Kunkel and Mehrmann, 1997) further extended (Bender and Laub, 1987) and showed that finding the linear-quadratic optimal control is equivalent to solving a system of generalized Euler-Lagrange equations. (Wang, 1996) and (Wang and Liao, 2001) presented the conditions for state feedback impulse elimination and impulse controllability, respectively. Based on these results, here we show that the optimal control problem is solvable if and only if there exists state feedback to make the closed-loop system impulse-free, or equivalently, the corresponding fast subsystem is impulse controllable.

The rest of this paper is organized as follows: Section 2 gives the problem formulation. Two preliminary lemmas necessary to our analysis are in Section 3. Our main results are derived and presented in Section 4. In Section 5, we use an example to illustrate our theory. Finally, Section 6 concludes this paper.

2 Problem Formulation

In this paper, we consider the problem of finding a control $u(t)$ to minimize

$$\int_{t_0}^{t_f} x(t)^T Q(t)x(t) + u^T(t)R(t)u(t)dt$$

for linear time-varying singular system

$$E(t)\dot{x} = A(t)x + B(t)u,$$

with

$$E(t_0)x(t_0) = E(t_0)x_0.$$

Here $E$, $A$ and $B$ are $n \times n$, $n \times n$ and $n \times m$ analytic matrix functions, respectively. $t_f$ is finite. $Q$ and $R$ are both positive definite matrix functions. $u : R \to R^m$ is continuous and $x : R \to R^n$ is continuously differentiable on $[t_0, t_f]$. We further assume that $E$ has constant rank and (2) is analytically solvable as described in (Campbell and Petzold, 1983). Note that for $x$ to be continuously differentiable, response for (2) must be impulse-free. Requiring so is by no means restrictive, for impulsive states inevitably incur infinite cost, as has been elegantly shown in (Cobb, 1983).
3 Preliminaries

According to Theorem 2 of (Campbell & Petzold, 1983), if (2) is analytically solvable, (2) can be transformed into

$$\dot{z}_1 = A_1(t)z_1 + G_1(t)u$$  \hspace{1cm} (3)

$$N(t)\dot{z}_2 = z_2 + B_2(t)u,$$  \hspace{1cm} (4)

where the $n_2 \times n_2$ matrix function $N(t)$ is strictly upper triangular for all $t$. (4) is referred to as the fast subsystem of (2). Note that $N$ has constant rank if and only if the rank of $E$ is constant. (3) is the slow subsystem. Because $T_f$ is finite, this paper does not take into account asymptotic behavior of the closed-loop slow subsystem.

The following are two preliminary lemmas essential to our analysis. We first rephrase Theorem 2 and Theorem 3 of (Kunkel and Mehrmann, 1997) according to our problem formulated in section 2.

Lemma 1 (Kunkel and Mehrmann, 1997) The linear-quadratic optimal control problem (1) and (2) has a minimizing solution $u(t)$ if and only if there exists $\lambda(t)$ such that

$$\begin{bmatrix}
E(t) & 0 & 0 & 0 \\
0 & -E(t)^T & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}(t) \\
\dot{\lambda}(t) \\
\dot{u}(t)
\end{bmatrix}
= \begin{bmatrix}
A(t) & 0 & B(t) \\
Q(t) & A^T(t) + \dot{E}(t)^T & 0 \\
0 & B^T(t) & R(t)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\lambda(t) \\
u(t)
\end{bmatrix},$$  \hspace{1cm} (5)

with

$$E(t_0)x(t_0) = E(t_0)x_0.$$  

Next, our earlier result on state feedback impulse elimination.

Lemma 2 (Wang, 1996) There exists an analytic feedback $L(t)$ such that the natural response of

$$E(t)\dot{x} = [A(t) + B(t)L(t)]x$$ is impulse-free

if and only if

$E(t)$ has constant rank

and

$$\text{Im}(E(t)) + A(t)\text{Ker}(E(t)) + \text{Im}(B(t)) = \mathbb{R}^n$$ for all $t$.  


4 Results

We now present the results of this paper.

**Theorem 1** The linear-quadratic optimal control problem (1) and (2) has a minimizing solution \( u(t) \) if and only if there exists an analytic state feedback \( L(t) \) such that the natural response of

\[
E(t)\dot{x} = [A(t) + B(t)L(t)]x
\]

is impulse-free.

**Proof:** It has been shown in (Bunse-Gerstner et al., 1991) that the analytic singular value decomposition exists and \( E(t) \) can be decomposed into

\[
P(t)\Sigma(t)U^T(t),
\]

where \( P \) and \( U \) are both orthogonal and analytic. Let the rank of \( E(t) \) be \( r \) for all \( t \). Thus, \( \Sigma(t) = \begin{bmatrix} \gamma(t) & 0 \\ 0 & 0 \end{bmatrix} \) with \( \gamma \) being a \( r \times r \) diagonal, nonsingular matrix function. Let \( z(t) = U^T(t)x(t), \xi(t) = P^T(t)\lambda(t) \) and denote \( (U(t)^T)^{-1} \) by \( V(t) \), \( (P(t)^T)^{-1} \) by \( W(t) \). For notation brevity, we drop the time dependence in the following equations. (5) can be rewritten as

\[
\begin{bmatrix}
\Sigma & 0 & 0 & 0 \\
0 & -\Sigma^T & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{z} \\
\xi \\
u
\end{bmatrix} =
\begin{bmatrix}
W^TAV + \Sigma \dot{U}^TV \\
V^TQV \\
0
\end{bmatrix}
\begin{bmatrix}
0 \\
\dot{\xi}^T + V^T\dot{U}\Sigma^T + V^T\dot{A}^TW \\
B^TW
\end{bmatrix}
\begin{bmatrix}
R \\
w
\end{bmatrix}
\begin{bmatrix}
z \\
u
\end{bmatrix}.
\]

Express (6) as

\[
\begin{bmatrix}
\gamma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\gamma & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\xi_1 \\
\xi_2 \\
u_1 \\
u_2
\end{bmatrix} =
\begin{bmatrix}
\alpha_{11} & \alpha_{12} & 0 & 0 & B_{11} & B_{12} & z_1 \\
\alpha_{21} & \alpha_{22} & 0 & 0 & B_{21} & B_{22} & z_2 \\
Q_{11} & Q_{12} & C_{11} & C_{12} & 0 & 0 & \xi_1 \\
Q_{21} & Q_{22} & C_{21} & C_{22} & 0 & 0 & \xi_2 \\
0 & 0 & B_{11}^T & B_{12}^T & R_{11} & R_{12} & u_1 \\
0 & 0 & B_{12}^T & B_{22}^T & R_{21} & R_{22} & u_2
\end{bmatrix}.
\]
with initial condition \( z_1(t_0) = [I_{r \times r} \ 0] U(t_0) x_0 \), the above equation can be reordered as

\[
\begin{bmatrix}
\gamma & 0 & 0 & 0 & 0 \\
0 & -\gamma & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{z}_1 \\
\dot{\xi}_1 \\
\dot{\xi}_2 \\
\dot{z}_2 \\
\dot{u}_1 \\
\dot{u}_2
\end{bmatrix}
= \begin{bmatrix}
\alpha_{11} & 0 & 0 & \alpha_{12} & B_{11} & B_{12} \\
Q_{11} & C_{11} & C_{12} & Q_{12} & 0 & 0 \\
\alpha_{21} & 0 & 0 & \alpha_{22} & B_{21} & B_{22} \\
Q_{21} & C_{21} & C_{22} & Q_{22} & 0 & 0 \\
0 & B_{21} & B_{21} & 0 & R_{11} & R_{12} \\
0 & B_{22} & B_{22} & 0 & R_{21} & R_{22}
\end{bmatrix}
\begin{bmatrix}
z_1 \\
\xi_1 \\
\xi_2 \\
z_2 \\
u_1 \\
u_2
\end{bmatrix}.
\] (7)

Since \( \gamma \) is nonsingular, (7) is solvable if and only if

\[
\begin{bmatrix}
0 & \alpha_{22} & B_{21} & B_{22} \\
C_{22} & Q_{22} & 0 & 0 \\
B_{21} & 0 & R_{11} & R_{12} \\
B_{22} & 0 & R_{21} & R_{22}
\end{bmatrix}
\]

is nonsingular. Recall that \( Q \) and \( R \) are positive definite, hence

\[
\begin{bmatrix}
Q_{22} & 0 & 0 \\
0 & R_{11} & R_{12} \\
0 & R_{21} & R_{22}
\end{bmatrix}
\]

is nonsingular. Moreover, the \((i, j)\)th element for \( \dot{\Sigma}^T \) are all zero for \( i = r + 1, \cdots, n \), \( j = r + 1, \cdots, n \); therefore, \( C_{22} = \alpha_{22}^T \). Now, we only need to show that

\[
\begin{bmatrix}
\alpha_{22} & B_{21} & B_{22}
\end{bmatrix}
\]

has full rank for all \( t \)

if and only if

\[
\text{Im}(E(t)) + A(t) \text{Ker}(E(t)) + \text{Im}(B(t)) = \mathbb{R}^n \text{ for all } t.
\]

Note that

\[
\text{Im}(E(t)) + A(t) \text{Ker}(E(t)) + \text{Im}(B(t)) = \mathbb{R}^n \text{ for all } t
\]

if and only if

\[
\text{Im} \left( \begin{bmatrix} \gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) + \text{Im} \left( \begin{bmatrix} \alpha_{12} & \alpha_{22} \\ \alpha_{22} & \alpha_{22} \end{bmatrix} \right) + \text{Im} \left( \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \right) = \mathbb{R}^n \text{ for all } t; \quad (8)
\]

(8) is equivalent to saying \( \begin{bmatrix} \alpha_{22} & B_{21} & B_{22} \end{bmatrix} \) has full rank for all \( t \), because \( \gamma \) is nonsingular. We have thus proved this theorem. \( \square \)
We have indicated in Remark 4 of (Wang and Liao, 2001) that the impulse controllability for (4) is equivalent to the existence of state feedback for impulse elimination in (2), provided that \( N \), or equivalently, \( \mathcal{E} \) has constant rank. Theorem 2 below states this result.

**Theorem 2** The linear-quadratic optimal control problem (1) and (2) has a minimizing solution \( u(t) \) if and only if (4), the corresponding fast subsystem for (2), is impulse controllable.

**Proof:** (4) is impulse controllable if and only if
\[
\text{Im}(N(t)) + \text{Ker}(N(t)) + \text{Im}(B_2(t)) = \mathbb{R}^{n_2} \text{ for all } t.
\]
The condition above is equivalent to
\[
\text{Im}(\mathcal{E}(t)) + A(t)\text{Ker}(\mathcal{E}(t)) + \text{Im}(B_2(t)) = \mathbb{R}^n \text{ for all } t
\]
as noted in Theorem 4.1 of (Wang, 1996), and is sufficient and necessary for the existence of optimal control according to Lemma 2 and Theorem 1.

5 Example

Consider \( \mathcal{E} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \), \( A(t) = \begin{bmatrix} 0 & \sin(t) \\ 1 & 1 \end{bmatrix} \), \( B(t) = \begin{bmatrix} 0 \end{bmatrix} \), \( x_1(t_0) = 1 \), \( R = 1 \), \( Q = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \). Clearly, there exists state feedback to eliminate the impulse; \( Q \) and \( \mathcal{R} \) are both positive definite. Reorder the Euler-Lagrange equations, we have
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{\lambda}_1 \\
\dot{\lambda}_2 \\
\dot{x}_2 \\
\dot{u}
\end{bmatrix}
= \\
\begin{bmatrix}
0 & 0 & 0 & \sin(t) & 0 \\
-1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 \\
0 & -\sin(t) & -1 & -2 & 0 \\
0 & 0 & -1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\lambda_1 \\
\lambda_2 \\
x_2 \\
u
\end{bmatrix},
\]
or equivalently,
\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{x}_1 \\
\dot{\lambda}_1 \\
\dot{\lambda}_2 \\
\dot{x}_2 \\
\dot{u}
\end{bmatrix}
= \\
\begin{bmatrix}
-\frac{1}{3} \sin(t) & -\frac{1}{2} \sin^2(t) & 0 & 0 & 0 \\
-\frac{5}{3} & -\frac{5}{3} \sin(t) & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 \\
0 & -\sin(t) & -1 & -2 & 0 \\
0 & 0 & -1 & 0 & -1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\lambda_1 \\
\lambda_2 \\
x_2 \\
u
\end{bmatrix}.
With \( x_1(t_0) = 1 \), \( x_1 \) and \( \lambda_1 \) can be determined from

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{\lambda}_1
\end{bmatrix} =
\begin{bmatrix}
-\frac{1}{3} \sin(t) & -\frac{1}{3} \sin^2(t) \\
-\frac{2}{3} & \frac{2}{3} \sin(t)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\lambda_1
\end{bmatrix}.
\]

Subsequently,

\[
\begin{bmatrix}
\lambda_2 \\
x_2 \\
u
\end{bmatrix} =
\begin{bmatrix}
0 & -1 & -1 \\
-1 & -2 & 0 \\
-1 & 0 & -1
\end{bmatrix}^{-1}
\begin{bmatrix}
1 & 0 \\
0 & \sin(t) \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\lambda_1
\end{bmatrix}.
\]

The linear-quadratic optimal control problem is solvable.

6 Conclusions

For linear time varying singular systems with constant-rank \( E \) matrices, we showed that the existence of linear-quadratic optimal control is equivalent to the existence of state feedback for impulse elimination, as well as the impulse controllability of the corresponding fast subsystem. The results obtained are straightforward generalizations of existing ones for time-invariant singular systems.

7 References


