Lukasiewicz Modal Operators in Residuated Lattices

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1 Modal Operators in Lukasiewicz 3-valued logic

In Lukasiewicz three-valued logic propositions can have the truth values $T$, $I$, and $F$, where $T$ stands for ‘true’, $I$ means ‘neither true or false’, called also ‘intermediate’, and $F$ denotes false. Additionally, these values may be ordered by $F < I < T$.

The connectives of negation and implication are defined by giving the truth tables:

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>$\rightarrow$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$I$</td>
<td>$I$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Lukasiewicz also introduced modal operations of possibility and necessity (see [4, p. 25]):

<table>
<thead>
<tr>
<th>♦</th>
<th>□</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$I$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

Next we present the essential properties of this three-valued modal logic. The most so-called normal modal logics can be characterized in terms of the following axioms schemes:

(K) $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$
(T) $\Box A \rightarrow A$
(S4) $\Box A \rightarrow \Box \Box A$
(B) $A \rightarrow \Box \Diamond A$
(S5) $\Diamond A \rightarrow \Box \Diamond A$

The following inference rules are usually included in normal modal logics:

(MP) \[ \frac{A \rightarrow B}{B} \]
(RN) \[ \frac{A}{\Box A} \]

A valuation is a mapping $v$ assigning to each proposition variable $p$ its truth value $v(p) \in \{F, I, T\}$. Evaluation $v$ extends canonically to all formulas by

\[ v(\neg A) = \neg v(A) \]
\[ v(A \rightarrow B) = v(A) \rightarrow v(B) \]
\[ v(\Diamond A) = \Diamond v(A) \]
\[ v(\Box A) = \Box v(A) \]

Furthermore, a formula $A$ is valid, if $v(A) = T$ for all valuations $v$. The next theorem can be easily verified by applying truth-tables (see [2], for further details).
Theorem 1 (Soundness). Lukasiewicz three-valued modal logic is sound with respect to axioms (K)–(S5), that is, they are valid formulas. Additionally, (MP) and (RN) preserve validity.

It is also easy to observe that □ and ♦ are mutually dual, that is, □A ≡ ¬♦¬A and ♦A ≡ ¬□¬A for all formulas A, where φ ≡ ψ means that φ and ψ are semantically equivalent, that is, v(φ) = v(ψ) for all valuations v.

As noted in [2], in this framework, ♦ and □ can be defined in terms of ¬ and →. Namely,

\[ ♦A \overset{\text{def}}{=} ¬A → A \quad \text{and} \quad □A \overset{\text{def}}{=} ¬(A → ¬A). \]  

(1)

We conclude this section by noting that if these definitions are done in classical two-valued logic, then ♦A ≡ A ≡ □A for all formulas A. In the next section we study the properties of the connectives (1) in residuated lattices and Heyting algebras generalizing the above three-valued system.

2 Modal Operators in Residuated Lattices and Heyting Algebras

A residuated lattice (see [1], for instance) is an algebra

\[(L, ∨, ∧, ⊙, →, 0, 1)\]

with four operations and two constants such that

(i) The algebra \((L, ∨, ∧, 0, 1)\) is a lattice with the greatest element 1 and the least element 0 (with respect to the lattice ordering ≤ of \(L\)).

(ii) The algebra \((L, ⊙, 1)\) is a commutative semigroup with the unit element 1, that is, ⊙ is commutative, associative, and \(1 ⊙ x = x\) for all \(x\).

(iii) The operations ⊙ and → form an adjoint pair, that is, for all \(x, y, z\),

\[ z ≤ (x → y) ⇐⇒ x ⊙ z ≤ y. \]

Notice that in a residuated lattice,

\[ x → (y → z) = (x ⊙ y) → z, \]

for all \(x, y, z\). We may define negation \(¬\) on \(L\) by setting for all \(x ∈ L\),

\[ ¬x = x → 0. \]

It is now easy to see that \(¬0 = (0 → 0) = 1\) and \(¬1 = 1 → 0 = 1 ⊙ (1 → 0) ≤ 0\), that is, \(¬1 = 0\). Note also that \(x ⊙ x ≤ x, x ⊙ ¬x = 0,\) and \(x ≤ y ⇐⇒ x → y = 1\). In addition, \(x ≤ ¬¬x\) and \(¬x = ¬¬¬x\) for any \(x\).

In the sequel we will need the following lemma.

Lemma 2. Let \((L, ∨, ∧, ⊙, →, 0, 1)\) be a residuated lattice. Then for all \(x, y ∈ L\),

\[ ¬¬x ⊙ ¬¬y ≤ ¬¬(x ⊙ y). \]

Proof. For all \(x, y ∈ L\),

\[
¬¬x ⊙ ¬¬y → ¬¬(x ⊙ y) = ¬¬x → (¬¬y → ¬¬(x ⊙ y))
= ¬¬x → (¬(x ⊙ y) → ¬y)
= ¬(x ⊙ y) → (¬¬x → ¬y)
= ¬(x ⊙ y) → (y → ¬x)
= ¬(x ⊙ y) ⊙ y → ¬x
= ¬(x ⊙ y) ⊙ y → (x → 0)
= ¬(x ⊙ y) ⊙ (y ⊙ x) → 0
= 0 → 0
= 1.
\]

This means that \(¬¬x ⊙ ¬¬y ≤ ¬¬(x ⊙ y)\). □
Let us define for a residuated lattice \((L, \lor, \land, \to, 0, 1)\) the operators \(\Diamond\) and \(\Box\) as in (1), that is, for all \(x \in L\),
\[
\Diamond x \overset{\text{def}}{=} \neg \neg (x \circ x) \quad \text{and} \quad \Box x \overset{\text{def}}{=} \neg (x \to \neg x).
\]

Now we can write the following proposition.

**Proposition 3.** Let \((L, \lor, \land, \circ, \to, 0, 1)\) be a residuated lattice. Then for all \(x, y \in L\),

(i) \(\Box x = \neg \neg (x \circ x)\).
(ii) \(\Diamond 0 = \Box 0 = 0\) and \(\Diamond 1 = \Box 1 = 1\).
(iii) \(x \to \Diamond x = 1\), \(\neg \neg x \to \Diamond x = 1\), and \(\Box x \to \neg x = 1\).
(iv) \(x \to y = 1\) implies \(\Diamond x \to \Diamond y = 1\) and \(\Box x \to \Box y = 1\).
(v) \(\Box (x \to y) \to (\Box x \to \Box y) = 1\).

**Proof.** (i) Obviously, \(\neg (x \circ x) = (x \circ x) \to 0 = x \to (x \to 0) = x \to \neg x\). This implies \(\Box x = \neg (x \to \neg x) = \neg (x \circ x)\).
(ii) Now \(\Diamond 0 = 1 \to 0 = 0\) and \(\Box 0 = \neg (0 \to 1) = \neg 1 = 0\). Similarly, \(\Diamond 1 = 0 \to 1 = 1\) and \(\Box 1 = \neg (1 \to 0) = \neg 0 = 1\).
(iii) For all \(x \in L\), \(x \to \Diamond x = x \to (\neg x \to x) = (x \circ \neg x) \to x = 0 \to x = 1\). Similarly, \(\neg \neg x \to \Diamond x = \neg \neg x \to (\neg x \to x) = (\neg \neg x \circ \neg x) \to x = 0 \to x = 1\). Because \(x \circ x \leq x\), \(\Box x = \neg (x \circ x) \leq \neg \neg x\). Thus, \(\Box x \to \neg x = 1\).
(iv) Let \(x \to y = 1\), that is, \(x \leq y\). Thus, \(\Diamond x = \neg x \to x \leq \neg x \to y \leq \neg y \to y = \Diamond y\). Moreover, \(x \leq y\) implies \(x \circ x \leq x \circ y \leq y \circ y\). So, \(\Box x = \neg (x \circ x) \leq \neg (y \circ y) = \Box y\).
(v) By Lemma 2,
\[
\Box (x \to y) \to \Box x = \neg (x \circ x) \to \neg ((x \to y) \circ (x \to y)) \to \neg ((x \circ x) \circ (x \to y))
\]
\[
\leq \neg ((x \to y) \circ (x \to y) \circ x \circ x)
\]
\[
= \neg (\neg (x \to y) \circ (x \to y) \circ x \circ (x \to y))
\]
\[
\leq \neg (y \circ y)
\]
\[
= \Box y.
\]

This means that \(\Box (x \to y) \leq \Box x \to \Box y\), that is, \(\Box (x \to y) \to (\Box x \to \Box y) = 1\).

Let us next consider such residuated lattices that \(x = \neg \neg x\) for all \(x\).

**Proposition 4.** If \((L, \lor, \land, \circ, \to, 0, 1)\) is a residuated lattice such that \(x = \neg \neg x\) for all \(x \in L\),
then for all \(x, y \in L\),

(i) \(\Box x = x \circ x\).
(ii) \(\Box x = x \to 1\).
(iii) \(\neg \neg x = \Diamond x\) and \(\Diamond x = \neg \neg x\).
(iv) \(\Diamond (x \to y) = \Box x \to \Diamond y\)

**Proof.** Cases (i) and (ii) are obvious by Proposition 3.
(iii) \(\neg \neg x = \neg (\neg x \to \neg x) = \neg (\neg x \to x) = \neg \Diamond x\) and \(\Diamond x \to \Box \neg x = \Diamond (\Diamond \neg x) = \Diamond \neg x\).
(iv) Because \(x \to y = \neg (x \circ y)\), we have \(\neg (x \to y) = \neg \neg (x \circ y) = x \circ y\). Hence,
\[
\Diamond (x \to y) = \neg \Box (x \to y)
\]
\[
= \neg (\neg (x \to y) \circ (x \to y))
\]
\[
= \neg ((x \circ \neg y) \circ (x \circ \neg y))
\]
\[
= \neg ((x \circ \neg y) \circ (\neg y \circ \neg y))
\]
\[
= \neg ((\Box x \circ \Box y) \circ \Box y)
\]
\[
= \neg (\Box x \circ \neg \Box y)
\]
\[
= \neg (\Box x \circ \neg (\Diamond y))
\]
\[
= \Box x \to \Diamond y
\]

\(\Box\)
In addition, if \((L, \lor, \land, \circ, \rightarrow, 0, 1)\) is a residuated lattice such that \(x = \lnot \lnot x\) for all \(x \in L\), then
\[
\Box x \rightarrow \Box \Box x = 1 \iff x \circ x = x \circ x \circ x \circ x.
\]
for all \(x \in L\).

A Heyting algebra \((L, \lor, \land, \rightarrow, 0, 1)\) can be defined as a residuated lattice \((L, \lor, \land, \circ, \rightarrow, 0, 1)\) such that \(\circ\) coincides with \(\land\). It is well known that Heyting algebras are always distributive lattices. Note also that a Heyting algebra is a Boolean algebra if and only if \(\lnot x = x\) for all \(x\). For further properties of Heyting algebras, see [3], for example.

Since the \(\circ\) and \(\land\) are equal in Heyting algebras and the meet operation is idempotent, we have for all \(x \in L\),
\[
\Box x = \lnot \lnot x.
\]
This implies that
\[
\Box x = \lnot \lnot x = \lnot \lnot \lnot \lnot x = \Box \Box x.
\]

We end this extended abstract by the following proposition stating that in Heyting algebras \(\Box\) and \(\Diamond\) behave in a similar manner than in Boolean algebras: there is no difference between modalities, they are both equal to the double negation.

\textbf{Proposition 5.} Let \((L, \lor, \land, \rightarrow, 0, 1)\) be a Heyting algebra. Then for all \(x \in L\),
\[
\Diamond x = \Box x = \lnot \lnot x.
\]

\textit{Proof.} It is enough to prove that (i) \(\lnot \lnot x \rightarrow (\lnot x \rightarrow x) = 1\) and (ii) \((\lnot x \rightarrow x) \rightarrow \lnot \lnot x = 1\).

(i) \(\lnot \lnot x \rightarrow (\lnot x \rightarrow x) = (\lnot \lnot x \land \lnot x) \rightarrow x = 0 \rightarrow x = 1\).

(ii) \((\lnot x \rightarrow x) \rightarrow \lnot \lnot x = (\lnot x \rightarrow x) \rightarrow (\lnot x \rightarrow 0) = ((\lnot x \rightarrow x) \land \lnot x) \rightarrow 0 = (\lnot x \land x) \rightarrow 0 = 1\).

By the previous proposition, \(x \leq \Box \Diamond x\) and \(\Diamond x = \Box \Diamond x\) for all \(x \in L\).

\textbf{References}