Note

Counting involutory, unimodal, and alternating signed permutations

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Abstract

In this work we count the number of involutory, unimodal, and alternating elements of the group of signed permutations $B_n$, and the group of even-signed permutations $D_n$. Recurrence relations, generating functions, and explicit formulas of the enumerating sequences are given.

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1. Introduction and the type A case

In this work, we consider the enumeration of three classes of signed permutations, which are involutory, unimodal, and alternating. The enumerative results in the case of the symmetric group $S_n$ of degree $n$ (i.e., the type $A$ case) are well established.

An element $\sigma = \sigma_1 \cdots \sigma_n$ of $S_n$ is involutory if $\sigma^2 = e$, the identity element of $S_n$; it is unimodal if there is an index $j$ for which $\sigma_1 < \cdots < \sigma_j > \sigma_{j+1} > \cdots > \sigma_n$; it is alternating (resp., reverse alternating) if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$ (resp., $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$). These definitions apply verbatim in the case of the signed permutation group $B_n$ and the even-signed permutation group $D_n$.

Let $u_{A,n}$, $t_{A,n}$, and $E_n^\Lambda$ be the number of unimodal, involutory, and alternating elements of $S_n$, respectively. By convention, we let $t_{A,0} := u_{A,0} := E_0^\Lambda := 1$. The type $A$ results are summarized as follows.

Theorem 1.1. We have

(i) $t_{A,n+1} = t_{A,n} + nt_{A,n-1}$,
(ii) $I_A(x) := \sum_{n \geq 0} t_{A,n} x^n / n! = \exp \left( x + \frac{1}{2} x^2 \right)$,
(iii) $t_{A,n} = \sum_{k=0}^{[n/2]} \frac{n!}{k!(n-2k)! 2^k}$,
(iv) $u_{A,n} = 2^{n-1}$.

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Theorem 2.1. We have

\[ 2E_n^A = \sum_{j=0}^{n-1} \binom{n-1}{j} E_j^A E_{n-1-j}^A, \]

\[ E^A(x) := \sum_{n \geq 0} E_n^A x^n / n! = \tan x + \sec x. \]

See [1, Exer. III.4.13–14, Proposition 3.70] for a proof of Theorem 1.1(i)–(iii), [2, pp. 150–151] for Theorem 1.1(iv), and [3, pp. 130–131; 4, p. 149] for Theorem 1.1(v)–(vi).

The main purpose of this work is to present the types \( B \) and \( D \) analogues of Theorem 1.1, certain cases of which have already appeared in scattered publications, perhaps in forms different from ours. We believe that, by filling in the missing parts, our unified results will be beneficial to other combinatorialists.

In the sequel, we denote by \( \#S \) the cardinality of a finite set \( S \), by \([a, b]\) the interval of integers \( \{a, a+1, \ldots, b\} \), where \( a, b \in \mathbb{Z} \), and by \([n]\) the interval of integers \( \{1, 2, \ldots, n\} \). The next two sections are devoted to types \( B \) and \( D \) cases, respectively.

2. The type \( B \) case

Denote by \( B_n \) the group of signed permutations, also known as the hyperoctahedral group, which is a Coxeter group of type \( B \) and of rank \( n \). The elements of \( B_n \) are signed permutations, which are bijection \( \sigma: [-n, n] \setminus \{0\} \to [-n, n] \setminus \{0\} \) satisfying \( \sigma(-i) = -\sigma(i), i = 1, 2, \ldots, n \). We shall represent each element \( \sigma \) of \( B_n \) by the word \( \sigma_1 \sigma_2 \cdots \sigma_n \), where \( \sigma_i = \sigma(i), i = 1, 2, \ldots, n \).

2.1. Involutory elements of \( B_n \)

Denote by \( t_{B,n} \) the number of involutory elements of \( B_n \), and let \( I_B(x) := \sum_{n \geq 0} t_{B,n} x^n / n! \), where \( t_{B,0} := 1 \) and that \( t_{B,1} = 2 \).

**Theorem 2.1.** We have

(i) \( t_{B,n+1} = 2t_{B,n} + 2nt_{B,n-1} \).

(ii) \( I_B(x) = \exp(x^2 + 2x) \).

**Proof.** Let \( \sigma = \sigma_1 \cdots \sigma_{n+1} \in B_{n+1} \) be involutory. Either \( \sigma_{n+1} = \pm (n+1) \) or \( \pm j \) with \( j \in [n] \). In the first case, \( (n+1) \) or \( (n+1, -(n+1)) \) is in the cycle factorization of \( \sigma \) and \( \sigma_1 \cdots \sigma_n \in B_n \) is involutory. In the second case, \( \sigma = \tau(j, n+1) \), where \( \tau \in B_n \) with \( \tau_j = j \), and whose totality is in bijective correspondence with the involutory elements of \( B_{n-1} \). This proves (i). Multiplying the recurrence relation by \( x^n / n! \), summing over \( n \), and using the initial condition \( I_B(0) = 1 \), we get the initial value problem: \( I_B = 2(1 + x)I_B, I_B(0) = 1 \), which admits the unique solution \( I_B(x) = \exp(x^2 + 2x) \). This proves (ii). \( \square \)

**Proposition 2.2.** We have

\[ t_{B,n} = \sum_{m=0}^{[n/2]} \frac{2^{n-2m} n!}{(n-2m)! m!}. \]

**Proof.** The above formula follows straightforwardly from the generating function \( I_B(x) \) by extracting the coefficients of \( x^n \). We shall, however, prove the formula by direct combinatorial arguments. Every involutory element \( \sigma \) of \( B_n \) can be written as a product of 1-cycles and 2-cycles, and each of them can be signed or unsigned. Now fix \( m \), where \( 0 \leq m \leq [n/2] \). There are \( \binom{n}{n-2m} \) ways of choosing elements of \( \{1, 2, \ldots, n\} \) to form unsigned 1-cycles, and \( 2^{n-2m} \) ways to sign them. The \( 2m \) elements left form \( (2m)!/2^m m! \) unsigned 2-cycles, which can be signed in \( 2^m \) ways. Thus,

\[ t_{B,n} = \sum_{m=0}^{[n/2]} 2^{n-2m} \binom{n}{n-2m} \frac{(2m)!}{2^m m!} \cdot 2^m = \sum_{m=0}^{[n/2]} \frac{2^{n-2m} m!}{(n-2m)! m!}, \]

as desired. \( \square \)
2.2. Unimodal elements of $B_n$

Denote by $u_{B,n}$ be the number of unimodal elements of $B_n$, and let $U_B(x) := \sum_{n \geq 1} u_{B,n} x^{n-1}/(n-1)!$. It is clear that $u_{B,1} = 2$ and $U_B'(x) = \sum_{n \geq 2} u_{B,n} x^{n-2}/(n-2)!$.

**Theorem 2.3.** We have

(i) $u_{B,n} = 2u_{B,n-1} + 2^{2n-2}$,
(ii) $U_B'(x) - 2U_B(x) = 4e^{4x}$,
(iii) $U_B(x) = 2e^{4x}$,
(iv) $u_{B,n} = 2^{2n-1}$.

**Proof.** Let $\sigma = \sigma_1 \ldots \sigma_n \in B_n$ be unimodal. Then either $\sigma_{j+1} = n$ for some $0 \leq j \leq n-1$, or $\sigma_1 = -n$ or $\sigma_n = -n$. In the first case, there are $\binom{n-1}{j}$ ways of choosing $|\sigma_1|, \ldots, |\sigma_j|$, and $2^{n-1}$ ways of signing $|\sigma_j|, \ldots, |\sigma_{j+2}|, \ldots, |\sigma_n|$. Thus, there are

$$\sum_{j \geq 0} \binom{n-1}{j} 2^{n-1} = 2^{2n-2}$$

possibilities. In the second case, there are $2u_{B,n-1}$ possibilities. Combining these two cases, (i) follows. Multiplying (i) by $x^{n-2}/(n-2)!$, followed by summing over $n \geq 2$, (ii) follows. Solving (ii) with $U_B(0) = 2$, we get (iii). Extracting the coefficients of $x^n$, we obtain (iv). \(\square\)

2.3. Alternating elements of $B_n$

Denote by $E_n^B$ the number of alternating elements of $B_n$, and let $E_B(x) := \sum_{n \geq 0} E_n^B x^n/n!$, where $E_0^B := 1$ and $E_1^B = 2$. It is clear that

$$\#\{\sigma \in B_n: \sigma \text{ alternating}\} = \#\{\sigma \in B_n: \sigma \text{ reverse alternating}\},$$

the concerned bijection being $\sigma_1 \ldots \sigma_n \rightarrow \bar{\sigma}_1 \ldots \bar{\sigma}_n$, where $\bar{\sigma}_j := -\sigma_j$.

Let $\sigma = \sigma_1 \ldots \sigma_n \in B_n$ be alternating or reverse alternating. Then $\sigma_j+1 \in \{\pm n\}$ for some $0 \leq j \leq n-1$, and $\sigma_j+2 \ldots \sigma_n \in B_{n-j-1}$ is alternating or reverse alternating according to whether $\sigma_{j+1} = n$ or $\sigma_{j+1} = -n$; $\bar{\sigma}_1 \ldots \bar{\sigma}_j$ is alternating or reverse alternating so as to make $\sigma = \sigma_1 \ldots \sigma_n$ alternating or reverse alternating. There are $\binom{n-1}{j}$ choices for $|\sigma_1|, \ldots, |\sigma_j|$. Thus,

$$2E_n^B = 2 \sum_{j=0}^{n-1} \binom{n-1}{j} E_j^B E_{n-j-1}^B,$$

and we have proved (i) of the following theorem. The proof of (ii), being similar to the proof of Theorem 2.1(ii), is omitted.

**Theorem 2.4.** (i) The numbers $E_n^B$, $n \geq 1$, satisfy

$$E_n^B = \sum_{j=0}^{n-1} \binom{n-1}{j} E_j^B E_{n-j-1}^B.$$

(ii) The formal power series $E^B(x)$ satisfies the differential equation

$$(E^B)'(x) = [E^B(x)]^2 + 1$$

in the ring of formal power series $\mathbb{Q}[[x]]$, and whose solution is $E^B(x) = \tan 2x + \sec 2x$.  

Theorem 3.2. We have that of the usual Euler numbers. The generating function for \( E_n \) has also been computed by other authors. For instance, Steingrímsson [5] had considered indexed permutations, which are a disguised form of the wreath product \( \mathbb{Z}_m \wr \mathfrak{S}_n \), the special case \( m = 2 \) of which is a realization of \( B_n \). The exponential generating function for weakly alternating indexed permutations computed in [5, Theorem 34] coincides with the one computed in the above theorem.

3. The type \( D \) case

Denote by \( D_n \) the group of even-signed permutations, which is a Coxeter group of type \( D \) and of rank \( n \). It is well known that \( D_n \) is a subgroup of \( B_n \) of index 2. Elements \( \sigma = \sigma_1 \cdots \sigma_n \) of \( D_n \) are signed permutations with \( \#i \in [n]: \sigma_i < 0 \) even.

3.1. Involutory elements of \( D_n \)

Denote by \( t_{D,n} \) the number of involutory elements of \( D_n \), and let \( I_D(x) := \sum_{n \geq 0} t_{D,n} x^n/n! \), where \( t_{D,0} := 1 \).

Proposition 3.1. We have

\[
t_{D,n} = \begin{cases} 
\sum_{m=0}^{[n/2]} \frac{2^{n-2m-1}n!}{(n-2m)!m!} & \text{if } n \text{ odd}, \\
\sum_{m=0}^{[n/2]} \frac{2^{n-2m-1}n!}{(n-2m)!m!} + \frac{n!}{2(n/2)!} & \text{if } n \text{ even}.
\end{cases}
\]

Proof. Let \( \sigma \) be an involution in \( D_n \), which can be written as a product of 2-cycles and 1-cycles, each of them can be signed or not, with the number of signed 1-cycles even. Now fix \( m \), where \( 0 \leq m \leq [n/2] \). There are \( \binom{n}{n-2m} \) ways of choosing elements of \( \{1, 2, \ldots, n\} \) to form unsigned 1-cycles, and \( \sum_{k=0}^{[n/2]-m} \binom{n-2m}{2k} \) ways to sign them. A straightforward calculation shows that

\[
\sum_{k=0}^{[n/2]-m} \binom{n-2m}{2k} = \begin{cases} 
2^{n-2m-1} & \text{if } n > 2m \text{ odd}, \\
1 & \text{if } n = 2m \text{ even}.
\end{cases}
\]

For the 2-cycles, the \( 2m \) elements left can form \( (2m)!/2^m m! \) 2-cycles, and can be signed in \( 2^m \) ways. If \( n \) is odd, then

\[
t_{D,n} = \sum_{m=0}^{[n/2]} 2^{n-2m-1} \binom{n}{n-2m} \frac{(2m)!}{2^m m!} \cdot 2^m = \sum_{m=0}^{[n/2]} \frac{2^{n-2m-1}n!}{(n-2m)!m!}.
\]

If \( n \) is even, then

\[
t_{D,n} = \sum_{m=0}^{[n/2]-1} 2^{n-2m-1} \binom{n}{n-2m} \frac{(2m)!}{2^m m!} \cdot 2^m + \frac{(2m)!}{2^m m!} \cdot 2^m
\]

\[
= \sum_{m=0}^{[n/2]-1} \frac{2^{n-2m-1}n!}{(n-2m)!m!} + \frac{n!}{(n/2)!}
\]

\[
= \sum_{m=0}^{[n/2]} \frac{2^{n-2m-1}n!}{(n-2m)!m!} + \frac{n!}{2(n/2)!}.
\]

\( \square \)

Theorem 3.2. We have that \( I_D(x) = e^{x^2} (e^{2x} + 1)/2 \).
Proof. For \( n = 2k + 1 \) odd, so that \( \lfloor n/2 \rfloor = k \), where \( k \geq 0 \),

\[
\sum_{n \geq 0 \text{ odd}} t_{D,n} \frac{x^n}{n!} = \sum_{k \geq 0} \frac{x^{2k+1}}{(2k+1)!} \sum_{m=0}^{k} \frac{2^{2k-2m}(2k+1)!}{(2k-2m+1)!m!}
\]

\[
= \frac{1}{2} \sum_{m \geq 0} \frac{x^{2m}}{m!} \sum_{k \geq m} \frac{(2x)^{2k-2m+1}}{(2k-2m+1)!}
\]

\[
= \frac{e^{x^2}(e^{2x} - e^{-2x})}{4}.
\]

For \( n = 2k \) even, so that \( \lfloor n/2 \rfloor = k \), where \( k \geq 0 \),

\[
\sum_{n \geq 0 \text{ even}} t_{D,n} \frac{x^n}{n!} = \sum_{k \geq 0} \frac{x^{2k}}{(2k)!} \left( \sum_{m=0}^{k} \frac{2^{2k-2m-1}(2k)!}{(2k-2m)!m!} + \frac{(2k)!}{2k!} \right)
\]

\[
= \frac{1}{2} \sum_{m \geq 0} \frac{x^{2m}}{m!} \sum_{k \geq m} \frac{(2x)^{2k-2m}}{(2k-2m)!} + \sum_{k \geq 0} \frac{x^{2k}}{2k!}
\]

\[
= \frac{e^{x^2}(e^{2x} + e^{-2x})}{4} + \frac{e^{x^2}}{2}.
\]

The desired result now follows upon summing the odd and even parts of \( I_D(x) \) just computed. \( \square \)

Corollary 3.3. We have that, for \( n \geq 2 \),

\[
t_{D,n+2} = 2t_{D,n+1} + (4n + 2)t_{D,n} - 4nt_{D,n-1} - 4n(n-1)t_{D,n-2}.
\]

Proof. Recall that \( I_B(x) = \exp(x^2 + 2x) \), so that differentiation yields \( [d/dx - 2(x + 1)]I_B(x) = 0 \). Now rewrite the expression of \( I_D(x) \) as

\[
2I_D(x) - I_B(x) = e^{x^2},
\]

differentiation of which yields

\[
2I'_D(x) - I'_B(x) = 2xe^{x^2}.
\]

Eliminating \( e^{x^2} \) from the two preceding equations yields

\[
I'_D(x) - 2xI_D(x) = I_B(x).
\]

Applying \( d/dx - 2(x + 1) \) to both sides, we get

\[
\left[ \frac{d}{dx} - 2(x + 1) \right] \left[ \frac{d}{dx} - 2x \right] I_D(x) = 0,
\]

which on expanding becomes

\[
I''_D(x) - (4x + 2)I'_D(x) - (4x^2 + 4x - 2)I_D(x) = 0.
\]

Now extract the coefficients of \( x^n \) to obtain the recurrence relation. \( \square \)
3.2. Unimodal elements of $D_n$

Denote by $u_{D,n}$ the number of unimodal elements of $D_n$, and let $U_D(x) := \sum_{n \geq 1} u_{D,n} x^{n-1}/(n-1)!$. It is clear that $u_{D,1} = 1$ and $U'_D(x) = \sum_{n \geq 2} u_{D,n} x^{n-2}/(n-2)!$.

**Theorem 3.4.** We have

(i) $u_{D,n} = 2u_{D,n-1} + 2^{2n-3}$,
(ii) $U'_D(x) - 2U_D(x) = 2e^{4x}$,
(iii) $U_D(x) = e^{4x}$,
(iv) $u_{D,n} = 4^{n-1}$.

**Proof.** We first note that for $n \geq 1$,

$$\sum_{k=0}^{[(n-1)/2]} \binom{n}{2k+1} = 2^{n-1} = \sum_{k=0}^{[n/2]} \binom{n}{2k}.$$ 

It follows that $\#(\sigma \in B_n \setminus D_n \colon \sigma \text{ unimodal}) = u_{D,n}$. Also note that

$$\#(\sigma \in D_n \colon \sigma \text{ increasing}) = 2^{n-1} = \#(\sigma \in B_n \setminus D_n \colon \sigma \text{ increasing}),$$

where $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ is increasing if $\sigma_1 < \sigma_2 < \cdots < \sigma_n$. Now let $\sigma = \sigma_1 \cdots \sigma_n \in D_n$ be unimodal. Then either

1. $\sigma_{j+1} = n$ for some $0 \leq j \leq n - 1$, or
2. $\sigma_1 = -n$ or $\sigma_n = -n$.

In case (1), there are $\binom{n-1}{j}$ ways of choosing $|\sigma_1|, \ldots, |\sigma_j|$, and $2^{n-2}$ ways of even signing $|\sigma_1|, \ldots, |\sigma_j|, |\sigma_{j+2}|, \ldots, |\sigma_n|$. Thus there are $\sum_{j \geq 0} \binom{n-1}{j} 2^{n-2} = 2^{2n-3}$ possibilities. In case (2), there are $2u_{D,n-1}$ possibilities. Combining the contributions from both cases, (i) follows. Multiplying (i) by $x^{n-2}/(n-2)!$, followed by summing over $n \geq 2$, we get (ii). Solving (ii) with the initial condition $U_D(0) = 1$, we get (iii). Extracting the coefficient of $x^n$ in (iii), we have (iv). □

3.3. Alternating elements of $D_n$

Denote by $E^D_n$ the number of alternating elements of $D_n$, also called the $n$th Euler number of type $D$. Let $E^D_n(x) := \sum_{n \geq 0} E^D_n x^n/n!$, where $E^D_0 := 1$. Note also that $E^D_n$ is equal to $E^D_n$.

It is clear that the cardinalities of the following sets are all equal to $E^D_n$:

- $\{\sigma \in D_n \colon \sigma \text{ alternating}\}$,
- $\{\sigma \in B_n \setminus D_n \colon \sigma \text{ alternating}\}$,
- $\{\sigma \in D_n \colon \sigma \text{ reverse alternating}\}$,
- $\{\sigma \in B_n \setminus D_n \colon \sigma \text{ reverse alternating}\}$.

Let $\sigma = \sigma_1 \cdots \sigma_n \in D_n$ be alternating or reverse alternating. Then $\sigma_{j+1} \in \{\pm n\}$ for some $0 \leq j \leq n - 1$.

If $j = 0$, then $\sigma_2 \cdots \sigma_n$ is in $D_{n-1}$ alternating, or in $B_{n-1} \setminus D_{n-1}$ reverse alternating, according to whether $\sigma_1 = n$ or $\sigma_1 = -n$.

If $j = n - 1$, then $\sigma_n = n$ or $\sigma_n = -n$. In the first case, $\sigma_1 \cdots \sigma_{n-1} \in D_{n-1}$ is reverse alternating or alternating according to whether $n$ is odd or even; in the second case, $\sigma_1 \cdots \sigma_{n-1} \in B_{n-1} \setminus D_{n-1}$ is alternating or reverse alternating according to whether $n$ is odd or even.

If $1 \leq j \leq n - 2$, then either $(\sigma_1 \cdots \sigma_j, \sigma_{j+2} \cdots \sigma_n) \in D_j \times D_{n-j-1}$ or $(B_j \setminus D_j) \times (B_{n-j-1} \setminus D_{n-j-1})$ with $\sigma_{j+2} \cdots \sigma_n$ alternating if $\sigma_{j+1} = n$, or $(\sigma_1 \cdots \sigma_j, \sigma_{j+2} \cdots \sigma_n) \in D_j \times (B_{n-j-1} \setminus D_{n-j-1})$ or $(B_j \setminus D_j) \times D_{n-j-1}$ with $\sigma_{j+2} \cdots \sigma_n$ alternating if $\sigma_{j+1} = n$. Thus there are $2u_{D,n-1}$ alternating or reverse alternating possibilities.
reverse alternating if \( \sigma_{j+1} = -n \), where \( \sigma_1 \cdots \sigma_j \) is alternating or reverse alternating so as to make \( \sigma_1 \cdots \sigma_n \) alternating or reverse alternating. There are \( \binom{n-1}{j} \) choices for \( |\sigma_1|, \ldots, |\sigma_j| \). Thus,

\[
2E_n^D = 2E_{n-1}^D + 4 \sum_{j=1}^{n-2} \binom{n-1}{j} E_j^D E_{n-j-1}^D + 2E_{n-1}^D,
\]

and we have proved (i) of the following theorem. The proof of (ii) is omitted.

**Theorem 3.5.** (i) The sequence of type D Euler numbers \( \{E_n^D\} \) satisfies

\[
E_n^D = 2 \sum_{j=0}^{n-1} \binom{n-1}{j} E_j^D E_{n-j-1}^D - 2E_{n-1}^D \quad \text{for } n \geq 2.
\]

(ii) The formal power series \( E^D(x) \) satisfies the differential equation

\[
(E^D)'(x) = 2[E^D(x)]^2 - 2E^D(x) + 1
\]

in \( \mathbb{Q}[x] \) and whose solution is

\[
E^D(x) = (\tan 2x + \sec 2x + 1)/2.
\]

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