

Implementing Optimal Monetary Policy in New-Keynesian Models with Inertia

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Abstract

We consider optimal monetary policy in New Keynesian models with inertia. First order conditions characterizing unconditionally optimal rational expectations equilibrium (REE) are determined and shown numerically to offer significant improvement over the timeless perspective. Implementation of the unconditionally optimal REE is considered via construction of interest-rate rules that are consistent with its associated unique equilibrium. Following Evans and Honkapohja (2004), an expectations based rule is derived that always yields a determinate model and an E-stable equilibrium. Further, the “policy manifold” of all interest-rate rules consistent with the unconditionally optimal REE is classified, and open regions of this manifold are shown to correspond to indeterminate models and unstable equilibria.

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1 Introduction

The benchmark, forward-looking, New-Keynesian models are closed by specifying the policy of the central bank. Preferences over different policy actions are typically modeled using a quadratic loss criterion, which is either taken as generic and so not tied to a particular specification of the economic model, or alternatively, may be derived as a second order approximation to average utility across private agents.¹

Identification of the solution to the government’s optimization problem depends on the surprisingly subtle definition of “optimality.” The subtlety arises because, absent a commitment technology, the fully optimal rule is not time-consistent. To address this issue, Woodford (2003) advises the *timeless perspective*, which informally has policymakers ignore current (initial) conditions, and operationally imposes that policymakers choose inflation to satisfy a time-invariant first order condition (FOC). In effect, the policy maker implements the policy she would have chosen in the distant past.²

The timeless perspective advises setting policy in a time-invariant manner. On the other hand, the fully optimal solution to the government’s problem sets inflation in the current period differently than in future periods: see Woodford (2003), Evans and Honkapohja (2006), and Section 3 below. As Jensen and McCallum (2002) point out, because of the policy maker’s desire to behave differently in the current period, policy from the timeless perspective is not time-consistent, and in this sense, a commitment technology is implicitly assumed.

Given the assumption of a commitment technology capable of implementing the timeless perspective, one is naturally led to wonder whether there is a similar time-invariant policy that offers a superior performance, or even if there is a metric that makes this question meaningful. Noting that under certain conditions a reasonable policy implements an asymptotically stationary economy, Jensen and McCallum (2002), and separately, Blake (2002), suggest evaluating various policies by computing the expected value of the government’s criterion across initial conditions, where the initial conditions are drawn from the associated asymptotic distribution; in this way, these authors are seeking time invariant targeting rules that implement an equi-

¹For a detailed analysis of the New-Keynesian model, its variants, and derivations of the associated welfare functions, see Woodford (2003).

²For a characterization of policy that is optimal from the timeless perspective, see Giannoni and Woodford (2002).

librium that is, in a natural sense, “unconditionally optimal.” Jensen and McCallum (2002) and Blake (2002) have proposed a rule (the same rule, which we call the MJB-alternative) that, using this metric, improves upon Woodford’s timeless perspective; and furthermore, Blake gives an argument showing the proposed rule is the best among rules of a given form.³ Jensen and McCallum, using a calibrated version of the model, show that the MJB-alternate is capable of implementing a 10.6% improvement over the timeless perspective.

The results of Jensen and McCallum (2002) and Blake (2002) identifying the unconditionally optimal REE are specific to the purely forward-looking New Keynesian model; however, substantial levels of inertia in both the AS and IS relations are often present in applied models of the economy. To assess the importance of the MJB-alternative’s potential improvement over the timeless perspective, it is then important to compare their performances in models with inertia: this is the first task of the paper.

Damjanovic, Damjanovic, and Nolan (2008) have recently provided a method for determining the unconditional optimal REE in a general setting. Using their method, we generalize the MJB-alternative to account for inertia. We then analyze its performance for various levels of inertia and find that while under some calibrations of the model’s parameters the improvement over the timeless perspective is insignificant, for other calibrations, and particularly when the government places priority on output stabilization, the improvement may be high – as high as 17% for calibrated models and much higher in case of strong serial correlation in the exogenous shocks. Interestingly, the presence of inertia in the model tends to mitigate this improvement, though the relationship between the degree of inertia and the level of mitigation is complex and non-monotonic.

The timeless perspective and the MJB-alternative are first order conditions which, together with the aggregate supply relation, define different “second best” rational expectations equilibria. As discussed in Section 3.1, some authors suggest taking these first order conditions as specific targeting rules, and thereby implicitly assume that the government’s policy instrument is inflation. While this is reasonable for theoretical results and stylized models, practical implementation requires an alternate policy instrument. To operationalize policy implied by the FOCs, we impose that policymak-

³Jensen and McCallum (2002) provide the rule only as an example of one superior to Woodford’s, and do not explain how they obtained the rule.

ers specify an interest-rate rule, often referred to as a Taylor rule, which is consistent with the FOCs' implementation. The second task of our paper begins with the specification of interest-rate reaction functions designed to be consistent with optimal policy as determined by the timeless perspective or the MJB-alternative.

However, our second task does not end there. It is well-known that closing New-Keynesian monetary models with interest-rate rules may yield indeterminate steady states, and associated to these steady states are multiple sunspot equilibria: see Evans and McGough (2005a) and Evans and McGough (2007). Since, in the presence of multiple equilibria, the equilibrium on which agents ultimately coordinate may exhibit unwanted properties, indeterminacy is undesirable. Therefore it is of considerable interest, when employing interest-rate reaction functions, to ascertain the determinacy properties of the model.

Even if the model has a unique REE under a given interest-rate rule, bad outcomes may obtain if boundedly rational agents are unable to coordinate on it. It is natural, then, for policymakers to desire that the equilibrium be stable under learning. It has been shown that for many models, including the models analyzed in this paper, stability under learning is governed by the notion of E-stability: for details, see Evans and Honkapohja (2001). Therefore it is necessary, when employing interest-rate reaction functions, to ascertain the E-stability properties of the model.

Having specified a collection of rules capable of implementing our FOC, we then examine their stability and determinacy properties. Similar examinations have been conducted in other models. In a purely forward-looking model, Evans and Honkapohja (2006) derive an interest-rate rule (which we call the "EH-rule") exhibiting dependence on agents' expectations, as well as on lagged endogenous variables and current shocks, which is consistent with the timeless perspective, and which always yields a stable and determinate steady state. We generalize this rule by deriving it for both the timeless perspective and the MJB-alternative when inertia is present in the model, and show that the resulting model is always determinate. While analytic results for E-stability are not available, we find numerically that for all calibrations and inertial specifications considered in our paper, as well as for both the timeless perspective and the MJB-alternative, the generalized EH-rule yields an E-stable equilibrium.

The generalized EH-rule is not the unique interest-rate reaction function consistent with the timeless perspective or the MJB-alternative. In fact, in

the purely forward-looking model Evans and Honkapohja (2006) also derive what they called the “fundamentals based rule,” which does not depend on agents’ expectations, but is consistent with optimal policy from the timeless perspective. Further, they show that the resulting equilibrium is never E-stable and the associated steady state may be indeterminate. We complete the second task of the paper by extending this result and characterizing the manifold of all policy rules within a given class that are consistent with either the timeless perspective or the MJB-alternative; and numerically, we classify their stability and determinacy properties. We find that open regions of the manifold correspond to instability, indeterminacy, or both, as well as to the presence of sunspot equilibria that are stable under learning. We also show that this finding serves as a caution to policymakers interested in employing rules, like the generalized EH-rule, which are dependent on values of structural parameters: precise knowledge of these parameters may be required to avoid bad outcomes.

This paper is organized as follows: Section 2 reviews the model and theory needed to obtain and discuss the paper’s main points, and in particular, the technical issues regarding determinacy and stability under learning are reviewed; Section 3 obtains, in the New Keynesian model with inertia, the first order conditions necessary to implement optimal policy from the timeless perspective, and presents the proposition describing the MJB-alternative; Section 4 focuses on implementable policy in the form of interest-rate reaction functions that are consistent with the timeless perspective or the MJB-alternative, and then examines the stability and determinacy properties of the New Keynesian model closed with these policy rules; and finally, Section 5 concludes.

2 Background

Here we discuss the necessary theory regarding determinacy and stability under learning, as well as present the hybrid New-Keynesian model of interest. These issues have been examined at length in other papers and so we keep our discussion brief. Because the model under investigation is quite similar, we borrow, in this review section, from Evans and McGough (2007). A more detailed discussion of determinacy and learning as they pertain to these types of monetary models may be found in Evans and McGough (2005a).

2.1 Determinacy

The analysis of this paper is based on the New Keynesian model, which we present in Section 2.3. When closed with a specification of monetary policy – either an interest-rate rule or specific inflation targeting rule – the model yields a reduced form having the following expectational structure:

$$y_t = AE_t y_{t+1} + B y_{t-1} + C \hat{g}_t. \quad (1)$$

Here y_t is a vector of endogenous variables and \hat{g}_t is exogenous fundamental noise, which we assume to be a stationary VAR(1) with damping matrix ρ . A rational expectations equilibrium (REE) is any non-explosive process \bar{y}_t satisfying (1)⁴.

Existence and uniqueness of REE depend on the model’s parameters. As is standard, we say that the model is determinate if there is a unique REE, indeterminate if there are many REE, and explosive otherwise. Methods for assessing whether a model is determinate, indeterminate, or explosive are well known and we refrain from discussing them here: see Evans and McGough (2005a) for a detailed analysis of determinacy in models of the form (1).

In case the model is determinate, the unique equilibrium has the form

$$y_t = \bar{b} y_{t-1} + \bar{c} \hat{g}_t, \quad (2)$$

where the matrices \bar{b} and \bar{c} may be computed using, for example, the method of undetermined coefficients. If the model is indeterminate there will still be at least one REE of the form (2), and we will call them minimal state variable (MSV) solutions.⁵ However, there will also be other REE that exhibit dependence on extrinsic stochastic processes: these processes may be thought of as capturing rational forecast errors, and are sometimes called sunspots. For a detailed discussion of sunspot equilibria in models of the form (1) see Evans and McGough (2005a).

⁴The notion of a non-explosive process can be made precise with several alternate definitions: see Evans and McGough (2005b) for details.

⁵If the model is indeterminate there may be multiple equilibria of the form (2), depending on whether certain eigenvalues are real or complex. Bennett McCallum, who introduced the term “MSV-solution” in McCallum (1983), proposes an additional selection criterion so that the MSV-solution, by his definition, is always unique.

2.2 Learning

If the model is determinate, so that there is a unique REE, it is desirable that the solution be stable under learning. By stability we mean that if agents estimate the parameters of a forecasting model using least squares regression and form their expectations accordingly, then the economy will eventually converge to the equilibrium. Because the models are self-referential, that is, because the evolution of the economy depends on how agents form expectations, the stability of an REE under least squares learning cannot be taken for granted.

To fix ideas, assume the model is determinate and that agents have a forecasting model of the form

$$y_t = a + by_{t-1} + c\hat{g}_t \quad (3)$$

The functional specification of the forecasting model is often referred to as a Perceived Law of Motion (PLM), and is typically taken to have a form consistent with the REE of interest.⁶ Under learning agents obtain least squares estimates (a_t, b_t, c_t) of the PLM's parameters using data through time t , and then use the estimated PLM to form their forecasts $E_t^* y_{t+1}$, which in turn influences the path of y_t through the reduced form model (1).⁷ The question is whether $(a_t, b_t, c_t) \rightarrow (\bar{a}, \bar{b}, \bar{c})$ as $t \rightarrow \infty$. If so, we say that the solution is stable under learning.

⁶The functional form of (3) clearly nests that of the REE (2), but also includes a constant term a . The REE lacks a dependence on a constant only because it has been written in deviation from steady-state form. By including a constant term in agents' PLM, we are imposing the natural assumption that they do not know the value of the steady-state, but rather, they must learn it, which is equivalent to learning that the REE value of the perceived parameter a is zero. It is well known that the stability of a particular REE may hinge on the presence of a constant in the PLM, especially when the model is indeterminate.

⁷The reduced form model (1), when capturing policy determined by an interest-rate rule, incorporates private agents' consumption Euler equation, thus this type of stability analysis is sometimes called *Euler equation learning*. The assumption is that agents form expectations of economic aggregates and then behave in a way that is consistent with (a linearization of) their Euler equation. Alternate methods of stability analysis have been suggested; most notably, Preston (2005) suggests that agents should be modeled as behaving in a way consistent with their lifetime budget constraint. Whether a given REE is stable under learning may depend on the way in which the stability analysis is conducted; for simplicity, we focus on Euler equation learning for this study.

Analysis of stability under learning is usually done using expectational stability (E-stability). This is because, for a wide range of models and solutions, E-stability has been shown to govern the local stability of REE under least squares learning. In many cases this correspondence can be proved, and in cases where it cannot be formally demonstrated the “E-stability principle” has been validated through simulations. For a thorough discussion of E-stability see Evans and Honkapohja (2001).

The E-stability technique is based on a mapping from the PLM to the corresponding Actual Law of Motion (ALM) parameters. For the case at hand, if agents believe in the PLM identified by (a, b, c) then their corresponding forecasts are given by $E_t^* y_{t+1} = a + bE_t^* y_t + cE_t^* \hat{g}_{t+1}$. Using $E_t^* y_t = a + by_{t-1} + c\hat{g}_t$, and assuming for convenience that ρ is known so that $E_t^* \hat{g}_{t+1} = \rho\hat{g}_t$, yields

$$E_t^* y_{t+1} = (I_2 + b)a + b^2 y_{t-1} + (bc + c\rho)\hat{g}_t.$$

Inserting $E_t^* y_{t+1}$ into (1) and solving for y_t as a linear function of an intercept, y_{t-1} and \hat{g}_t yields the corresponding ALM:

$$y_t = A(I_2 + b)a + (Ab^2 + B)y_{t-1} + (A(bc + c\rho) + C)\hat{g}_t,$$

which describes the actual evolution of y given the beliefs (a, b, c) . The map from perception to reality, then, is given by

$$a \rightarrow A(I_2 + b)a \tag{4}$$

$$b \rightarrow Ab^2 + B \tag{5}$$

$$c \rightarrow A(bc + c\rho) + C. \tag{6}$$

To simplify notation, combine the regressors into the vector

$$X_t' = (1, y_{t-1}', \hat{g}_t')$$

and write the perceived parameters as $\theta = (a, b, c)'$. Then the PLM can be written as $y_t = \Theta' X_t$, and equations (4)-(6) define a mapping from PLM parameters Θ to the ALM parameter $T(\Theta)$. The REE $\bar{\Theta} = (\bar{a}, \bar{b}, \bar{c})$ is a fixed point of this map and it is said to be E-stable if it is locally asymptotically stable under the differential equation

$$\frac{d\Theta}{d\tau} = T(\Theta) - \Theta. \tag{7}$$

The E-stability principle tells us that E-stable REE are locally learnable for Least Squares and closely related algorithms. That is, if Θ_t is the time t estimate of the coefficient vector Θ , and if Θ_t is updated over time using recursive least squares, then $\bar{\Theta}$ is a possible convergence point, i.e. locally $\Theta_t \rightarrow \bar{\Theta}$, if and only if $\bar{\Theta}$ is E-stable. Computing E-stability conditions is often straightforward, involving computation of eigenvalues of the Jacobian matrices of (7).

The discussion so far has been restricted to the determinate case; however, it is (relatively) straightforward to extend these ideas to sunspot equilibria in case of model indeterminacy. Also, in the indeterminate case, stability under learning of a given equilibrium may depend on the functional form of the forecasting model used by agents when forming their expectations. Because these issues are not central to our work here, we refrain from their discussion and simply refer readers to Evans and McGough (2005a).

2.3 The New Keynesian Model

We study a hybrid version of the New Keynesian monetary model as given by

$$IS : x_t = -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1 - \delta)x_{t-1} + g_t \quad (8)$$

$$AS : \pi_t = \beta(\gamma E_t\pi_{t+1} + (1 - \gamma)\pi_{t-1}) + \lambda x_t + u_t. \quad (9)$$

Here x_t is the proportional output gap, π_t is the inflation rate, and g_t and u_t are independent, exogenous, stationary, zero mean AR(1) shocks with damping parameters $0 \leq \rho_g < 1$ and $0 \leq \rho_u < 1$ respectively.

The first equation is a formulation of the forward-looking IS curve amended to include inertia. This functional form may be obtained from a linearized model of optimization behavior on the part of consumers. In some cases we also allow for an inertial term x_{t-1} , which is present due to habit formation: see for example Smets (2003). The second equation is the forward-looking Phillips curve. When $\gamma = 1$, equation (9) is the pure forward-looking New Keynesian “AS” relationship based on “Calvo pricing,” and employed in Clarida, Gali, and Gertler (1999) and Ch. 3 of Woodford (2003).⁸ Here $0 < \beta < 1$ is the discount factor. Again, this equation is obtained as the linearization around a steady state. The specification of the AS curve in the case $0 < \gamma < 1$ incorporates an inertial term and is similar in spirit to Fuhrer

⁸For the version with mark-up shocks see Woodford (2003) Chapter 6, Section 4.6.

Table 1: Calibrations

Name	ϕ	λ
W	1/.157	.024
CGG	4	.075
MN	.164	.3
MJ	.164	.02

and Moore (1995), the Section 4 model of Galí and Gertler (1999), and the Ch. 3, Section 3.2 model of Woodford (2003), each of which allows for some backward looking elements. Models with $0 < \gamma < 1$ are often called “hybrid” models, and we remark that in some versions, such as Fuhrer and Moore (1995), $\beta = 1$, so that the forward and backward looking components sum to one, while in other versions $\beta < 1$ is possible.

The model may be closed by specifying monetary policy, and we consider alternate optimal specifications below. Once a policy has been specified, the model may be placed in the reduced form (1), and its determinacy and stability properties analyzed. While some analytic results are available, numeric methods must be used in the general case, and this requires assigning values to the model’s parameters. We consider four calibrations of the parameters in the IS-AS curves, as due to Woodford (1999), Clarida, Galí, and Gertler (2000), McCallum and Nelson (1999), and Jensen and McCallum (2002): the relevant parameter values are given in Table 1.

With these calibrations, we consider five inertial specifications, as given by Table 2. Finally, for each calibration and inertial specification, we will consider discount values of $\beta = .98$, $\beta = .99$ and $\beta = 1$.

3 Optimal Policy

We model the preferences of policymakers over alternate equilibrium paths as given by

$$E_t \left(\sum_{k=0}^{\infty} \beta^k (\psi x_{t+k}^2 + \pi_{t+k}^2) \right). \quad (10)$$

For some specifications of the New Keynesian model, a loss criterion of this form may be taken as capturing a second order approximation to aggregate

Table 2: Inertial Specifications

Name	γ	δ
Forward	1	1
Small Lag	.75	.75
Medium Lag	.5	.5
Large Lag	.25	.25
Full Lag	.01	.01

private agent utility, in which case ψ is a function of the model's deep parameters.⁹ Rather than fixing the link between government preferences and the underlying model, we take (10) as a generic loss function parameterized by ψ , which simply captures the relative importance of output stabilization over inflation control. Note we have assumed, for simplicity, that the government and private agents have the same discount factor.

To construct optimal monetary policy, we begin by identifying the optimal REE, that is the REE which minimizes the government's loss subject to the Phillips curve constraint. In this sense, we are implicitly assuming π_t to be the government's instrument: see Section 3.1 for more discussion. This assumption makes the IS relation superfluous, and the associated reduced form model is obtained by combining aggregate supply with a first order restriction characterizing government behavior. The associated time-path of output and inflation represents the optimal REE consistent with the government's objective and the AS-relation. In Section 4, we will use the IS relation to establish the time-path of the interest-rate consistent with this optimal REE. Interpreting this time-path as an interest-rate rule will then operationalize our optimal policy.

The optimal REE may be identified using Lagrange's method. Let L_t be the time t conditional Lagrangian. Exploiting the law of iterated expectations

⁹The loss criterion associated to models with inertia may include a dependence on lagged endogenous variables.

yields

$$L_t = E_t \left(\sum_{k=0}^{\infty} \beta^k (l(x_{t+k}, \pi_{t+k}) + \omega_{t+k} (\pi_{t+k} - R(x_{t+k}, \pi_{t+k+1}, \pi_{t+k-1}))) \right) \quad (11)$$

where

$$\begin{aligned} l(x_t, \pi_t) &= \psi x_t^2 + \pi_t^2 \\ R(x_t, \pi_{t+1}, \pi_{t-1}) &= \beta\gamma\pi_{t+1} + \beta(1-\gamma)\pi_{t-1} + \lambda x_t + u_t \end{aligned}$$

The associated FOC are given by

$$\omega_{t+k} = \frac{2\psi}{\lambda} x_{t+k} \quad (k \geq 0) \quad (12)$$

$$2\pi_{t+k} = -\omega_{t+k} + \gamma\omega_{t+k-1} + \beta^2(1-\gamma)E_{t+k}\omega_{t+k+1} \quad (k > 0) \quad (13)$$

$$2\pi_t = -\omega_t + \beta^2(1-\gamma)E_t\omega_{t+1} \quad (14)$$

Equations (12) - (14) characterize the fully optimal solution to the government's problem. That this solution is not time consistent is evident from equations (13) and (14), which instruct the policy-maker to behave differently in the initial period than in subsequent periods.

3.1 The Timeless Perspective

To address the inconsistency problem, Woodford advises the timeless perspective, which has policy-makers ignore (14). Equations (12) and (13) may then be combined to obtain the following relation between current inflation and current, lagged and expected future output gap:

$$\pi_t = -\frac{\psi}{\lambda} (x_t - \gamma x_{t-1} - \beta^2(1-\gamma)E_t x_{t+1}). \quad (15)$$

For $\gamma = 1$, equation (15) reduces to the standard first order condition in case of no inertia.

Policy described by (15) is time consistent in the sense that it is time-invariant (unlike the policy described by (12) - (14)) and optimal in the sense that it is the policy an optimizing policy maker from the distant past would have chosen for today. However, it is not time consistent in the sense of Kydland and Prescott (1977): each period policymakers have an incentive

to deviate from the planned policy.¹⁰ For more discussion on the timeless perspective, see Woodford (2003), McCallum (2005), and Dennis (2001).

Until now we have assumed, somewhat unrealistically, that inflation is the government's policy instrument, and that it can be set to satisfy (15). Indeed, some authors suggest taking the first order condition (15) as a specific targeting rule. Interpreted this way, and assuming policymakers can, in some way not explicitly modeled, impose that (15) holds at every point in time, then (9) and (15) comprise a fully specified reduced form model. Furthermore, arguments in Woodford (2003) guarantee this model to be determinate, and hence yield a unique rational expectations equilibrium.¹¹ Whether the unique REE is stable under learning must be determined numerically. We find that for all calibrations and inertial specifications, and with all discount factors, expectational stability always obtains.

Because the specific targeting rule interpretation does not provide a model of policy maker behavior, some ambiguity remains concerning precisely how monetary policy is implemented. We address this issue in Section 4. There we design interest-rate reaction functions consistent with the optimal REE defined by the timeless perspective or the MJB-alternative. However, like Evans and Honkapohja (2006), we find that such policy rules do not necessarily imply (15) is satisfied, and so the issues of determinacy and stability must be revisited.¹² Because of this, we recommend choosing instrument rules consistent with (15) that also result in a determinate model with an E-stable equilibrium.

3.2 The MJB-Alternative

As noted above, the timeless perspective implicitly assumes a commitment technology. Furthermore, while it is optimal from the perspective of a policy maker in the distant past, one is led to wonder how good the timeless perspective is from a current view point, or even if there is a natural metric

¹⁰If the model is determinate and the equilibrium is E-stable, then there is an additional sense in which the timeless perspective is time consistent when agents form their expectations based on least-squares learning: provided policymakers resist following discretionary policy, they do not have to worry about private agents "rationally" anticipating discretionary behavior: see McCallum (1995).

¹¹See Footnote 8 on page 542.

¹²Policy rules consistent with the timeless perspective's first order condition may impart indeterminacy and some of the associated equilibria will not satisfy (15).

that may be used to address this question. Jensen and McCallum (2002), and separately, Blake (2002), suggest considering the average value of the government’s objective, where the average is taken across initial conditions drawn with respect to their asymptotic distribution. These authors then examine, in the purely forward-looking case (i.e. $\gamma = 1$), whether there are rules exhibiting the same linear dependence as (15), (i.e. incorporating the same commitment technology) but which yield lower average losses. Both Jensen and McCallum and Blake show numerically that the relationship

$$\pi_t = -\frac{\psi}{\lambda}(x_t - \beta x_{t-1}) \quad (16)$$

is superior to the timeless perspective: Jensen and McCallum simply state it as an example of an improvement; but Blake proceeds to show that in fact (16) is the optimal rule of that form.

The first goal of our paper is to generalize the FOC (16) to account for inertia in the aggregate supply relation. Fortunately, recent work by Damjanovic, Damjanovic, and Nolan (2008) allows us to do this quite easily. In a general framework, Damjanovic, Damjanovic, and Nolan (2008) provide a simple method for computing the FOC optimizing the average value of a government’s objective subject to a possibly non-linear, possibly forward-looking constraint. They call the associated FOC “unconditionally optimal policy.” We now apply their method to the hybrid New Keynesian model to obtain the FOC analogous to (16) but in the presence of inertia.

The method of Damjanovic, Damjanovic, and Nolan (2008) may be employed as follows. Act on the time t conditional Lagrangian L_t , (11), by the unconditional expectations operator, multiplied by $(1 - \beta)$:

$$\begin{aligned} & (1 - \beta)EL_t \\ &= (1 - \beta)E \left(\sum_{k=0}^{\infty} \beta^k (l(x_{t+k}, \pi_{t+k}) + \omega_{t+k} (\pi_{t+k} - R(x_{t+k}, \pi_{t+k+1}, \pi_{t+k-1}))) \right) \\ &= E (\psi x_t^2 + \pi_t^2 + \omega_t (\pi_t - \beta\gamma\pi_{t+1} - \beta(1 - \gamma)\pi_{t-1} - \lambda x_t - u_t)) \\ &= E (\psi x_t^2 + \pi_t^2 + \omega_t \pi_t - \beta\gamma\omega_t \pi_{t+1} - \beta(1 - \gamma)\omega_t \pi_{t-1} - \lambda\omega_t x_t - \omega_t u_t) \\ &= E (\psi x_t^2 + \pi_t^2 + \omega_t \pi_t - \beta\gamma\omega_{t-1} \pi_t - \beta(1 - \gamma)\omega_{t+1} \pi_t - \lambda\omega_t x_t - \omega_t u_t) \end{aligned}$$

where the second and last equalities are obtained by exploiting the stationarity of the various processes: $E(\omega_t \pi_{t+1}) = E(\omega_{t-1} \pi_t)$, etc. This manipulation highlights the key difference between the timeless perspective and unconditional optimal policy: the former takes ω_{t-1} as given whereas the latter

accounts for its average impact. Differentiating this last expression with respect to π_t and x_t gives the FOC, which we summarize in the following proposition:¹³

Proposition 1 *The unconditionally optimal policy for the model determined by (10) and (9) is given by*

$$\pi_t = -\frac{\psi}{\lambda} (x_t - \beta\gamma x_{t-1} - \beta(1 - \gamma)E_t x_{t+1}). \quad (17)$$

The reduced form model given by (9) and (17) is necessarily determinate, and we call its unique bounded solution the “unconditionally optimal REE.” Analytic results for E-stability of the this REE are not available, but we find numerically that for all calibrations and inertial specifications, it is stable under learning.¹⁴

Notice that in case $\gamma = 1$ so that there is no inertia in the Phillips curve, (17) reduces to (16), the condition suggested by McCallum and Jensen, and established as optimal by Blake. Notice, too, that for $\beta = 1$, (17) and the timeless perspective are identical as expected. We call (17) the “MJB-alternative.”

3.3 Comparing the Timeless Perspective and the MJB-Alternative

The results of the previous section indicate that the MJB-alternative (17) at least weakly dominates the timeless perspective (15); however, the MJB-alternative is simply a small continuous transformation of the timeless perspective, and so it is quite natural to wonder how much improvement is obtained. This question has been considered by Jensen and McCallum (2002) in the non-inertial case, with $\lambda = .02$, for varying values of ψ , β , and ρ . They obtain a maximum improvement of 10.26%. We conducted a similar analysis, and some of the results we obtained are reported in Table 3.

¹³We obtained the FOC identified in this proposition independently, before we were aware of the work of Damjanovic, Damjanovic, and Nolan (2008). Because our proof is complicated and applies only to the hybrid New Keynesian model, we omit it, and instead rely on their result.

¹⁴Here we are interpreting (17) as a specific targeting rule, and again, concerns regarding its implementation apply. We address these concerns by constructing explicit interest-rate rules in Section 4.

As expected, the MJB-alternative always yields an improvement over the timeless perspective, and in the forward-looking case, with heavy weight place on output stabilization, the improvement may be quite significant. Interestingly, the presence of inertia seems to mitigate the second-best nature of the timeless perspective: for a hybrid model with $\gamma = .5$, the maximum improvement across calibrations is only .06%. Further examination of this mitigating effect indicates a complex, non-monotonic pattern. Consider Figure 1, which plots the percent improvement of the MJB-alternative over the timeless perspective for the Woodford calibration and varying γ . As γ increases to .5, which corresponds to the Medium Lag inertial specification, the improvement drops from 1.5% to near zero before rising again. This pattern is qualitatively the same as seen across calibrations, with the minimum improvement obtaining near $\gamma = .5$.

Fig 1 here

The magnitude of the serial correlation in the markup shock also impacts the improvement of the MJB alternative over the timeless perspective, with the numerically obtained qualitative result being that for most calibrations and inertia specifications, larger serial correlation leads to larger improvement; however, for inertial specifications with γ near .5, the relationship between serial correlation and improvement is again non-monotonic: see Figure 2 as an example.

Fig 2 here

For high serial correlation, the improvement may be very large: for example, if the MJ calibration is used under the forward inertial specification, then with $\beta = .99$, $\psi = 10$ and $\rho = .999$ the improvement is 1882%. We conclude that the a-priori minor modification to the timeless perspective advocated by the MJB-alternative can yield significantly superior results in terms of average government loss, and so must be taken seriously.

4 Implementing Optimal Policy

We have characterized the optimal REE via first order conditions, taking the AS relation as a constraint. Our goal now is to investigate how the optimal REE can be implemented. That is, we look for interest-rate reaction

Table 3: TP/MJB Comparison (Forward Model and Medium Lag, $\beta = .98$)

ψ	Forward Model				Medium Lag			
	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	2.42%	.45%	.04%	3.08%	.04%	.01%	.00%	.04%
1	8.76%	2.46%	.30%	10.16%	.06%	.04%	.01%	.05%
10	16.12%	8.81%	1.79%	17.08%	.01%	.05%	.03%	.01%

functions that, when combined with the AS and IS relations, result in the optimal REE attaining according to determinacy and E-stability. To this end, we “nest” the timeless perspective and the MJB-alternative in the following generalized FOC:

$$\pi_t = -\frac{\psi}{\lambda} \left(x_t - \xi\gamma x_{t-1} - \frac{\beta^2}{\xi}(1 - \gamma)E_t x_{t+1} \right), \quad (18)$$

where $\xi = 1$ in case of the timeless perspective and $\xi = \beta$ in case of the MJB-alternative.

4.1 The Fundamentals Rule

As noted above, for β near 1, the system of expectational difference equations (9) and (18) is determinate for either value of ξ . Letting $y = (x, \pi)'$, the unique stationary equilibrium can be written

$$y_t = by_{t-1} + cy_t. \quad (19)$$

Now recall the IS-relation (8), repeated here for convenience:

$$x_t = -\phi(i_t - E_t\pi_{t+1}) + \delta E_t x_{t+1} + (1 - \delta)x_{t-1} + g_t. \quad (8)$$

To create an interest-rate rule consistent with the optimal REE, we follow Evans and Honkapohja (2006) (EH) and assume agents form expectations using (19). Imposing these expectations into the IS relation (8), we may then solve for i_t , thus obtaining an interest-rate rule of the form

$$i_t = \tilde{\alpha}^L y_{t-1} + \tilde{\alpha}^{\hat{g}} \hat{g}_t, \quad (20)$$

where $\hat{g} = (g, u)'$. We call (20) the “fundamentals rule.” The values of the 1×2 matrices $\tilde{\alpha}^L$ and $\tilde{\alpha}^{\hat{g}}$ depend on the model’s reduced form parameters as well as which FOC is used.

We may now consider the full reduced form model, assuming monetary policy is implemented by following the fundamentals rule. The model is given by (9), (8), and (20). Results concerning a special case of this model are known. Evans and Honkapohja (2006) obtained the fundamentals rule in case $\gamma = 1$ and the timeless perspective is assumed. They showed analytically that the model may be indeterminate, and the equilibria are always unstable under learning. Similarly, we find

Proposition 2 *In case $\gamma = \delta = 1$ and the MJB-alternative is used, under the fundamentals based rule, the economy may be indeterminate, and the equilibria are always unstable under learning.*

The proof of this proposition mimics the proof by EH, and we suppress the details.

Similar work can be done in case of inertia; however, we must proceed numerically as analytic results are unavailable. Some of our results are collected in Tables 4 and 5. Here $\beta = .99$ and the MJB-alternative is used. To identify the stability and determinacy properties, we use the notation SD (stable determinacy), UD (unstable determinacy), SI (stable indeterminacy, that is, E-stable sunspot equilibria), and UI (unstable indeterminacy). Similar results obtain for other calibrations and inertial specifications and for the timeless perspective.

Interpreting the first order conditions given by the timeless perspective or the MJB-alternative as specific targeting rules, and assuming that policymakers are in some unspecified way able to achieve the specific targeting rule so that only the AS relation is considered as a restriction, both the timeless perspective and the MJB-alternative resulted in stable determinacy. The work of Evans and Honkapohja (2006) and the results here show that simply assuming RE and using the IS relation to design an interest-rate rule consistent with the associated optimality condition can be destabilizing and hence ill-advised.

4.2 The EH-rule

The results reported in Proposition 2 and Tables 4 and 5 warn against the use of the fundamentals rule. Evans and Honkapohja, in case $\gamma = 1$ and

Table 4: Fundamentals Rule (Small Lag and Medium Lag)

	Small Lag				Medium Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	UI	UI	SD	UI	UI	UI	SD	SD
1	UD	UD	UI	UI	UI	UI	SD	UI
10	UD	UD	UI	UI	UI	UD	UI	UI

Table 5: Fundamentals Rule (Large Lag and Full lag)

	Large Lag				Full Lag			
ψ	W	CGG	MN	MJ	W	CGG	MN	MJ
.1	SD	UI	SD	SD	SD	SD	SD	SD
1	UI	UI	SD	SD	SD	SD	SD	SD
10	UI	UI	SD	SD	SD	SD	SD	SD

under the timeless perspective, faced a similar problem, and proposed the following solution. Instead of forming a rule by first computing expectations under RE, EH suggest taking expectations as given. Specifically, combine the general FOC (18) and the Phillips curve (9) and solve for x_t to obtain

$$x_t = \frac{\lambda}{\psi + \lambda^2} \left(\frac{\psi\xi\gamma}{\lambda} x_{t-1} + \frac{\psi\beta^2(1-\gamma)}{\lambda\xi} E_t x_{t+1} - \beta\gamma E_t \pi_{t+1} - \beta(1-\gamma)\pi_{t-1} - u_t \right).$$

This equation may then be combined with the IS relation (8) to obtain a rule of the form

$$i_t = \hat{\alpha}^f E_t y_{t+1} + \hat{\alpha}^L y_{t-1} + \hat{\alpha}^{\hat{g}} \hat{g}_t \quad (21)$$

where

$$\hat{\alpha}^f = \left(\frac{\delta\xi(\psi + \lambda^2) - \psi\beta^2(1-\gamma)}{\xi\phi(\psi + \lambda^2)}, 1 + \frac{\lambda\beta\gamma}{\phi(\psi + \lambda^2)} \right) \quad (22)$$

$$\hat{\alpha}^L = \left(\frac{(1-\delta)(\psi + \lambda^2) - \psi\gamma\xi}{\phi(\psi + \lambda^2)}, \frac{\lambda\beta(1-\gamma)}{\phi(\psi + \lambda^2)} \right) \quad (23)$$

$$\hat{\alpha}^{\hat{g}} = \left(\frac{1}{\phi}, \frac{\lambda}{\phi(\psi + \lambda^2)} \right). \quad (24)$$

We call this the EH-rule and note that in case $\gamma = 1$, $\delta = 1$ and the timeless perspective is assumed, (21) reduces to the rule obtained by Evans and Honkapohja.

Evans and Honkapohja showed analytically that their rule always resulted in determinacy, and that the unique REE was always stable under learning. Similarly,

Proposition 3 *The reduced form model given by the IS and AS relations (8) and (9) and closed with the EH-rule (21) is determinate for both $\xi = 1$ and $\xi = \beta$.*

The proof of this proposition follows from the observation that solutions to the system (8), (9) and (21) are bijective with those of (9) and the relevant FOC. This proposition extends the determinacy result of Evans and Honkapohja to models with inertia, and to rules implementing either the timeless perspective or the MJB-alternative. Analytic results on stability are not obtainable for us, but numerically, we find the same result as EH. In particular, for all permutations of calibrations and inertial specifications, and under both the timeless perspective and the MJB-alternative, the EH-rule results in stability.

4.3 The Optimal Policy Manifold

Equations (20) and (21) indicate that there are at least two rules of the form

$$i_t = \alpha^f E_t y_{t+1} + \alpha^L y_{t-1} + \alpha^{\hat{g}} \hat{g}_t \quad (25)$$

consistent with the optimal REE (19), and further show that rules implementing the optimal REE may not impart the same stability and determinacy properties. We are then led to wonder if the collection of all rules of the form (25) can be classified, and how their associated stability and determinacy properties may vary. We turn to these issues now.

Denote by Ω the set of all $\alpha = (\alpha^f, \alpha^L, \alpha^{\hat{g}})'$ so that (25) is consistent with (19). We call Ω the optimal policy manifold, and note that it is a subset of \mathbb{R}^6 , and furthermore is non-empty, as it contains both the fundamentals rule (20) and the EH-rule (21). Now define the matrix $\hat{c} = (0, c)$, so that the optimal REE may be written $y_t = b y_{t-1} + \hat{c} \hat{g}_t$. Forming expectations with respect to this REE and imposing them into (25) yields

$$i_t = (\alpha^f b^2 + \alpha^L) y_{t-1} + (\alpha^{\hat{g}} + \alpha^f (b\hat{c} + \hat{c}\rho)) \hat{g}_t,$$

which must be the same as the fundamentals rule in an REE. Thus $\alpha \in \Omega$ provided

$$\alpha^f b^2 + \alpha^L = \tilde{\alpha}^L \quad (26)$$

$$\alpha^f (b\hat{c} + \hat{c}\rho) + \alpha^{\hat{g}} = \tilde{\alpha}^{\hat{g}}. \quad (27)$$

Equations (26) and (27) characterize the optimal policy manifold. Furthermore, notice that we may trivially solve (26) and (27) for α^L and $\alpha^{\hat{g}}$ as linear functions of α^f . Thus the optimal policy manifold is a two-dimensional hyperplane in \mathbb{R}^6 , given by

$$\Omega = \{ \alpha \in \mathbb{R}^6 : \exists \alpha^f \in \mathbb{R}^2 \text{ with } \alpha^L = \tilde{\alpha}^L - \alpha^f b^2 \text{ and } \alpha^{\hat{g}} = \tilde{\alpha}^{\hat{g}} - \alpha^f (b\hat{c} + \hat{c}\rho) \}.$$

The stability and determinacy properties of rules associated to elements of Ω can be characterized numerically. As an example, consider Figure 3. Here the Woodford calibration is used together with the Lag inertial specification, $\beta = .98$, $\psi = 10$ and the timeless perspective is assumed. To create the figure, a $(-1, 2) \times (-1, 2)$ lattice was imposed over the $(\alpha_\pi^f, \alpha_x^f)$ -space, and at each point on the lattice, the associated element of Ω was determined and the stability and determinacy properties were recorded.¹⁵ The center of the large gray dot indicates the location of the EH-rule, which lies in the interior of the stable determinate region.

Figure 3 Here

We have chosen this admittedly extreme case to emphasize that while the EH-rule necessarily lies in the region of stable determinacy, it may be very near a boundary and surrounded by all types of bad outcomes: unstable determinacy and stable and unstable indeterminacy are all possibilities. We conducted similar analysis on a $(-5, 5) \times (-5, 5)$ lattice in $(\alpha_\pi^f, \alpha_x^f)$ -space for all combinations of calibration, inertia specification, β -value, ψ -value, and FOC-type, and we found the results of Figure 3 to be qualitatively universal, with the caveat that the MN-calibration with various permutations of the remaining parameters may yield relatively large regions of stable determinacy, and may house the EH-rule far from any problem region.

¹⁵For some points in the region of UI, there were an insufficient number of real eigenvalues for a CF-representation to exist. For details on this and other issues regarding common factor representations in monetary models, see Evans and McGough (2005a).

Analysis of the policy manifold indicates that arbitrarily choosing an optimal interest-rate rule among those available in Ω is unwise, for instability or indeterminacy may result. These results also suggest that relying on an unconstrained numerical algorithm to search for the optimal rule is ill-advised for such an algorithm will be unable to distinguish between points on the manifold – all points yield the same value of the government’s objective. These searches must be constrained to those regions in policy space corresponding to stable determinacy.¹⁶

4.4 Model Uncertainty

While analysis of the policy manifold indicates potential problems for optimal interest-rate rules, one may wonder about the relevance of these concerns given the existence of the EH-rule, which yields stable determinacy for all calibrations. And indeed if the structural parameters of the AS and IS relations are known with precision, the EH-rule can be implemented and no concern over potential indeterminacy or instability problems is warranted. On the other hand, model uncertainty, which here takes the form of uncertainty about the true values of the model’s structural parameters, may imply bad outcomes even when the EH-rule is employed. To argue this point, notice that the set of all policy rules (25) may be identified with \mathbb{R}^6 . The optimal policy manifold is precisely the subset of \mathbb{R}^6 coinciding with those rules which implement the relevant FOC. A simple “continuity of eigenvalues” argument shows that since the optimal policy manifold has subsets corresponding to instability, indeterminacy, and stable sunspots, which are non-empty and open in the *relative* topology, there must be corresponding non-empty open subsets of \mathbb{R}^6 whose associated policies induce indeterminacy, instability or stable sunspots. Now suppose a policy maker chooses to use the EH-rule by estimating the structural parameters of the model and setting policy accordingly. The policy maker thinks she is choosing a point on the optimal policy manifold and in a region corresponding to stable determinacy. However, if her estimates of the structural parameters are off, it is very likely she is not on the manifold, and, much more importantly, because the location of the open sets in \mathbb{R}^6 corresponding to stable determinacy are not where she thinks they are, the associated model may be unstable, indeterminate, or have stable sunspot equilibria. As a concrete example of this phenomenon, we note

¹⁶This point was emphasized for other interest-rate rules in Evans and McGough (2007).

that if the policy maker with $\psi = .1$ thinks the MN calibration with medium lag prevails and sets policy according to the associated EH-rule under the timeless perspective, but if in fact the true calibration is W or CGG then the model may exhibit either unstable indeterminacy, stable indeterminacy, or explosiveness depending on the inertial specification.

These problems indicate that, in case of model uncertainty, the optimal rule should be chosen to have nice stability and determinacy properties across possible model specifications, as well as to maximize some measure of welfare. A technique for determining these types of rules is provided in Evans and McGough (2007).

5 Conclusion

To avoid the problem of time-inconsistency, Woodford offers the timeless perspective, which induces an invariant linear restriction characterizing the associated “optimal” REE. Jensen and McCallum, and separately, Blake, argue that assuming the commitment technology implicit in the timeless perspective, an alternate linear restriction, which we call the MJB-alternative, is available that provides, on average, a smaller loss. We use the method of Damjanovic, Damjanovic, and Nolan (2008) to extend the results of these authors to more realistic models that include lags in their structural equations, and find that while the improvement over the timeless perspective provided by the MJB alternative may be quite large, inertia in the model may mitigate this effect.

When constructing the timeless perspective and the MJB-alternative, the policy instrument is somewhat unrealistically taken to be inflation. In the second part of the paper, we then turn to issues of implementing these first order conditions by deriving appropriate interest-rate rules. We extend the results of Evans and Honkapohja to find an expectations based rule, which we call the EH-rule, that always produces a stable and determinate outcome. However, in characterizing all possible optimal policy rules, we find that many are associated to instability, indeterminacy, or stable sunspots. These findings suggest that in the presence of structural parameter estimation error, bad outcomes may obtain even in case the EH-rule is employed. This serves as a strong caution to policymakers and suggests policy that is robust to parameter uncertainty and other types of model uncertainty is important.

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Figure 1: MJB/TP Improvement for Varying γ
Woodford calibration, $\psi=1$

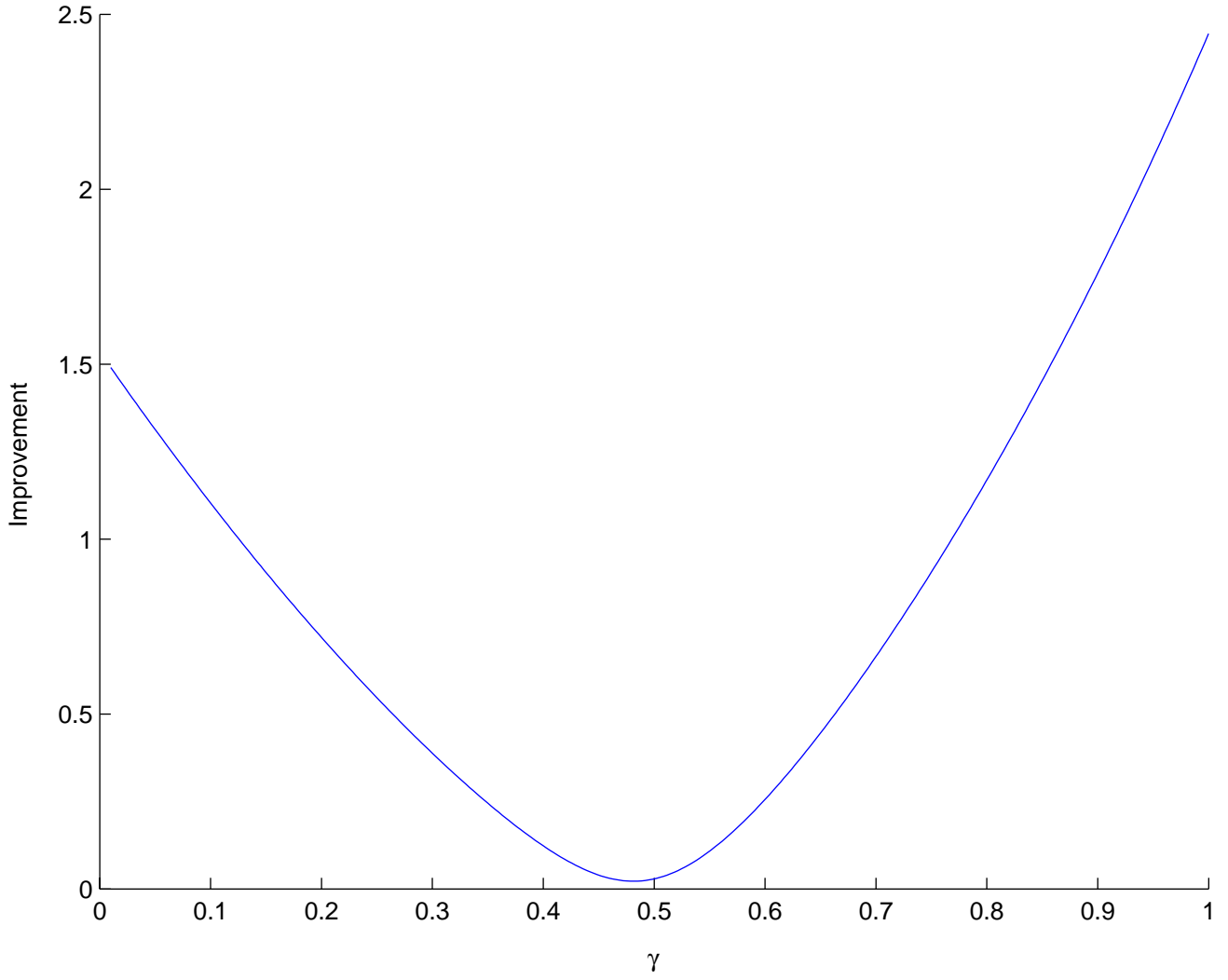


Figure 2: MJB/TP Improvement:
Woodford calibration, $\psi=1$

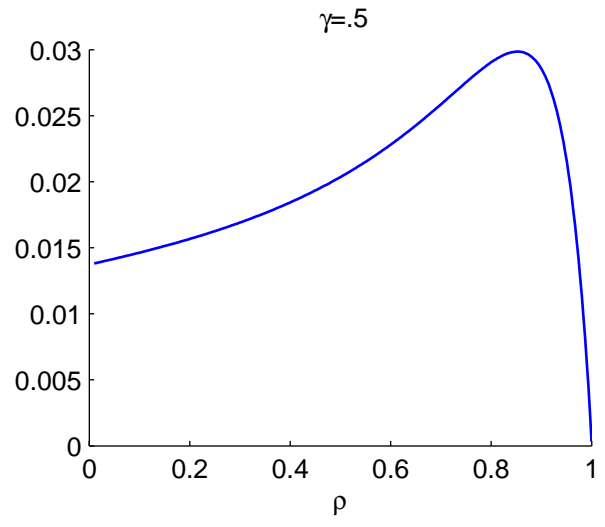
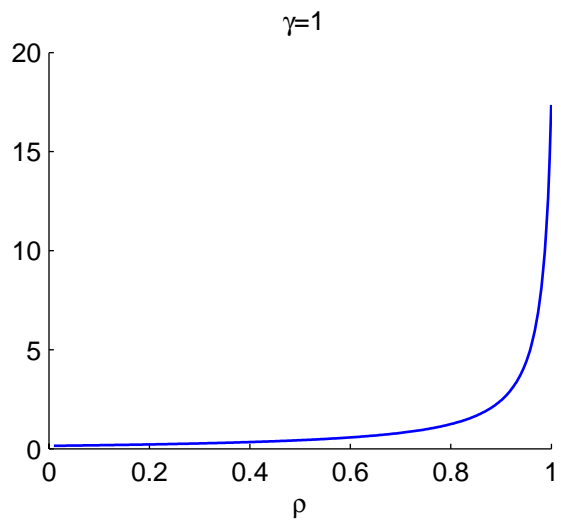


Figure 3: Policy Manifold, Timeless Perspective
 $\phi=6.37, \lambda=.024, \gamma=.5, \delta=.5, \beta=.98, \psi=10$

