On Modal Components of the $S_4$-logics

Alexei Y. Muravitsky
Louisiana Scholars’ College
Northwestern State University
Natchitoches, LA 71497, U.S.A.
alexeim@nsula.edu

Abstract
We consider the representation of each extension of the modal logic $S_4$ as sum of two components. The first component in such a representation is always included in Grzegorczyk logic and hence contains "modal resources" of the logic in question, while the second one uses essentially the resources of a corresponding intermediate logic. We prove some results towards the conjecture that every $S_4$-logic has a representation with the least component of the first kind.

1 Preliminaries
We consider the intuitionistic propositional logic $\text{Int}$ and modal propositional logic $S_4$, both defined with the postulated rule of substitution, along with the lattices of their normal consistent extensions, $N\text{Ext}\text{Int}$ (intermediate logics) and $N\text{Ext}\text{S}_4$ ($S_4$-logics), respectively. The lattice operations are the set intersection $\cap$ as meet and the deduction closure $\oplus$ as joint. Other logics from $N\text{Ext}\text{Int}$ and $N\text{Ext}\text{S}_4$ will also appear in the sequel.

The mappings $\rho : N\text{Ext}\text{S}_4 \rightarrow N\text{Ext}\text{Int}$ and $\tau : N\text{Ext}\text{Int} \rightarrow N\text{Ext}\text{S}_4$ were defined in [3]. It is well known that the former mapping is a lattice epimorphism and the latter is an embedding; see [3]. Another mapping, $\sigma : N\text{Ext}\text{Int} \rightarrow N\text{Ext}\text{Grz}$, defined by the equality $\sigma L = \text{Grz} \oplus \tau L$ for any $L \in N\text{Ext}\text{Int}$, where $\text{Grz}$ is Grzegorczyk logic, is an isomorphism; see [1] and [2]. This, along with inequalities obtained in [3], implies that

$$\tau \rho M \subseteq M \subseteq \text{Grz} \oplus \tau \rho M,$$

(Blok-Esakia inequality)

for any logic $M \in N\text{Ext}\text{S}_4$.

Thus it can be suggested that any $M \in N\text{Ext}\text{S}_4$ is equal to $M^* \oplus \tau \rho M$ for some logic $M^* \subseteq \text{Grz}$. Indeed, for $M^*$ one can always take $M \cap \text{Grz}$; see [4]. Furthermore, we have the following.

Let $\mathcal{L} = \{\tau L \mid L \in N\text{Ext}\text{Int}\}$. An unspecified element of $\mathcal{L}$ will be denoted by $\tau$. Then any $M \in N\text{Ext}\text{S}_4$ can be represented as $M = M^* \oplus \tau$, where $M^* \subseteq \text{Grz}$, i.e. $\rho M^* = \text{Int}$. In this representation of $M$, the first term, $M^*$, is called the modal component of $M$ and the second term, $\tau$, is its assertoric (or superintuitionistic) component (or $\tau$-component). Such a representation of $M$ we call a $\tau$-representation.

It has been noticed [4] that the assertoric component of $M$ is uniquely determined by $M$ and equals $\tau \rho M$, but its modal component may vary. Given an $S_4$-logic $M$, the modal components of $M$ constitute a dense sublattice of $N\text{Ext}\text{S}_4$ with the top element $M \cap \text{Grz}$. This on-going research aims at proving the conjecture: Every $S_4$-logic has a least modal component.

2 Examples of the modal components of some $S_4$-logics

Below one can see different situations related to modal components of some $S_4$-logics.
• Each logic in $[\text{S4, Grz}]$ itself is its only modal component.
• All logics $\tau \in \mathcal{L}$ have $\text{S4}$ their only modal component.
• If $\text{Grz} \subseteq \text{S4.1} \oplus \tau$ then the logics of $[\text{S4.1, Grz}]$ constitute all the modal components of $\text{Grz} \oplus \tau$.

In the sequel we obtain more examples.

3 $S$-series slicing of $\text{NExtS4}$

We arrange the Scroggs logics as follows:

$$S5 = S_0 \subset \ldots \subset S_2 \subset S_1 = \text{S4} + p \rightarrow \Box p.$$  \hspace{1cm} (S-series)

**Definition 3.1** ($S$-series slicing). A logic $M$ belongs to the $n$th $S$-slice, $n \geq 1$, if $M \subseteq S_n$ and $M \not\subseteq S_{n+1}$. If $M \subseteq S_n$, for all $n \geq 1$, that is to say, $M \in [\text{S4}, S_0]$, then $M$ lies in the 0th $S$-slice. We denote the $n$th $S$-slice by $S_n$, $n \geq 0$.

Thus $\mathcal{S}_0 = [\text{S4, S5}]$. Also, it is obvious that $\{\mathcal{S}_n\}_{n \geq 0}$ is a partition of $\text{NExtS4}$. As well known, $S_n$, $n \geq 1$, is the logic of an $n$-atomic finite interior algebra with only two open elements. We denote such an algebra by $B_n$.

**Definition 3.2** (logics $K_n$). Let $\chi_n$ be the characteristic formula of algebra $B_n$, $n \geq 1$. We define $K_n = \text{S4} + 2\chi_{n+1}$, for $n > 0$, and $K_0 = \text{S4}$.

**Proposition 3.1.** Each $S$-slice is an interval. For $n \geq 1$, logic $S_n$ is the top of the $n$th slice and logic $K_n$ is its bottom. In particular, $\mathcal{S}_1 = [\text{S4.1, S1}]$.

**Corollary 3.1.1.** For each $n \geq 1$, $(K_n, S_{n+1})$ is a splitting pair in $\text{NExtS4}$.

**Proposition 3.2.** Let $\tau \in \mathcal{L}$. All logics from the $n$th $S$-slice having $\tau$ as their $\tau$-component constitute the interval $[K_n \oplus \tau, M_n \oplus \tau]$.

In addition, we prove the following:

- $K_{n+1} \subset K_n$ for any $n \geq 1$; and
- $\bigcap_{n \geq 1} K_n = \text{S4}$.

4 $M$-series slicing of $[\text{S4, Grz}]$

We will be using the following notation:

$$M_0 = \text{Grz} \cap S5 \text{ and } M_1 = \text{Grz}.$$  \hspace{1cm} (M-series)

We note that the interval $[M_0, M_1]$ is ordered by $\subset$ in type $1 + \omega^*$:

$$M_0 \subset \ldots \subset M_2 \subset M_1,$$

where $M_n = M_1 \cap S_n$ and $\bigcap_{n \geq 1} M_n = M_0$. To this, we add the following:

- $M_n \cap S_l$, whenever $M_l \subseteq M_n$ or $S_l \subseteq S_n$;
- $K_n \oplus S_l = S_n$, whenever $S_l \subseteq S_n$;
• $K_n \oplus M_l = M_n$, whenever $M_l \subseteq M_n$;
• $M_n \cap S_{n+1} = M_{n+1}$, for any $n \geq 1$.

**Definition 4.1** (M-series slicing). A logic $M$ from $[\mathbf{S4}, \mathbf{Grz}]$ belongs to the $n$th M-slice if and only if $M$ is in the $n$th S-slice. In other words, the $n$th M-slice equals $[K_n, S_n] \cap [\mathbf{S4}, \mathbf{Grz}]$. We denote the $n$th M-slice, $n \geq 0$, by $E_n$.

We prove that for any $M \in [\mathbf{S4}, \mathbf{Grz}]$ and $n \geq 0$, the following conditions are equivalent:

a) $M \in E_n$;

b) $M \in [K_n, M_n]$;

c) $M \oplus M_0 = M_n$.

For any $n \geq 1$, each of (a) – (c) is equivalent to:

d) $M \subseteq M_n$ and $M \not\subseteq M_{n+1}$.

**Proposition 4.1.** Let us fix $n \geq 0$. If a modal logic $M$ lies in $\mathcal{L}_n$, then any its modal component $M^*$ belongs to $E_n$. Conversely, for any modal logic $M^*$ in $E_n$ and any $\tau$ in $\mathcal{L}$, the logic $M^* \oplus \tau$ lies in $\mathcal{L}_n$.

From Proposition 4.1 and some properties mentioned above we derive:

• For any $n \geq 0$, all modal logics of $E_n$ are the modal components of the logic $S_n$.

Also, we obtain the following: Given $\tau \in \mathcal{L}$,

• if $M_n \subseteq K_n \oplus \tau$ then the logics of $[K_n, M_n]$ constitute all the modal components of $M_n \oplus \tau$;

### 5 Least modal components

In this section we will show that the existence of the least modal component of a logic $M$ can be reduced to the question of definability of some function of $M$. The proposition of this section states that the definability of this function should be checked for some logics of the $0$th S-slice.

**Definition 5.1** (Mappings $h_n$, $h_{n0}$, and $g_{n0}$). For any $n \geq 0$, we define: $h_n : M \mapsto M \cap S_n$, where $M \in \text{NEstS4}$. We denote by $h_{n0}$ and by $g_{n0}$ the mapping $h_0$ restricted to $\mathcal{L}_n$ and $E_n$, respectively.

We observe the following:

• $h_{n0}$ a lattice embedding of $\mathcal{L}_n$ into $\mathcal{L}_0$;

• $g_{n0}$ is a lattice embedding of $E_n$ into $E_0$.

**Proposition 5.1.** Given an $\mathbf{S4}$-logic $M \in \mathcal{L}_n$, $M$ has a least modal component if and only if $h_{n0}(M)$ has it.

**Definition 5.2** (difference operation $d(X,Y)$). Given two logics $X$ and $Y$, a logic $C \subseteq X$ is called the difference of the subtraction of $Y$ from $X$, if for any logic $Z$, the following equivalence holds:

$$C \subseteq Z \subseteq X \iff X = Z \oplus Y.$$ 

If such $C$ exists for given $X$ and $Y$, it is obviously unique. We denote it by $d(X,Y)$.
The operation \( d(X,Y) \) is certainly partial. For instance, \( d(M_0, \text{Dum}) \) is undefined, where 
\[
\text{Dum} = S4 + \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow (\Box \Box p \rightarrow p).
\]

Next we define: Given \( M \in NExtS4 \),
\[
d^*(M) = d(M, \tau \rho M).
\]

**Proposition 5.2.** Given \( M \in NExtS4 \), \( d^*(M) \) is defined if and only if \( d^*(M) \) is the least modal component of \( M \).

The next theorem shows that our search for the definability of the \( d^* \) function on \( NExtS4 \) can be reduced to the 0th S-slice.

**Proposition 5.3.** Every \( S4 \)-logic has its least modal component if and only if for any \( M^* \in \lbrack S4, M_0 \rbrack \) and \( \tau \in L \), \( d(M^*, \tau \cap M^*) \) is defined, or, equivalently, \( d^*(M^* \oplus \tau) \) is defined, providing that \( \tau \cap Grz \subseteq M^* \).

### 6 Greatest modal components

We remind the reader that any logic \( M \in NExt(S4) \) has its greatest modal component which is \( M \cap \text{Grz} \). Also, we know from Proposition 3.2 that all \( S4 \)-logics of the \( n \)th S-slice that have a logic \( \tau \in L \) constitute the interval \( \lbrack K_n \oplus \tau, M_n \oplus \tau \rbrack \). The next proposition reads that the greatest modal components of the logics of the last set form an interval.

**Proposition 6.1.** Let \( \tau \in L \). The greatest modal components of all logics from the \( n \)th S-slice having \( \tau \) as their \( \tau \)-component constitute the interval \( \lbrack K_n \oplus (Grz \cap \tau), M_n \rbrack \).

### References


