Universal augmentation schemes for network navigability

- Overcoming the $\sqrt{n}$-barrier

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Network navigability: the small world effect

- Very short paths exist.
- People are able to discover them locally.

Milgram 1967
Kleinberg model (2000)

- Mesh: global geographical knowledge
- Red random arcs: local and private knowledge

$Pr(u \rightarrow v) \propto \frac{1}{|u-v|^2}$
Navigability in Kleinberg model

A routing algorithm is claimed decentralized if:

1. it knows all links of the mesh,

2. it discovers locally the extra random links.
Navigability in Kleinberg model

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2. it discovers locally the extra random links.

Greedy routing computes paths of expected length $O(\log^2 n)$ between any pair in this model.
Augmented graphs
\(f(n)\)-navigability

Problem:

- A graph \(G\) + one random link/node
- Which graph and which distribution s.t. greedy routing computes paths of length \(f(n)\)?
Augmented graphs $f(n)$-navigability

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- A graph $G$ + one random link/node
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Ex: $d$-dimensional meshes are $O(\log^2 n)$-navigable (with $d$-harmonic distribution of links).
Polylog(n)-navigability

- Bounded growth graphs [DHLS 05]
Polylog($n$)-navigability

- Bounded growth graphs [DHLS 05]
- Bounded treewidth graphs [Fraigniaud 05]
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- Graphs excluding a fixed minor [Abraham&Gavoille 06]
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**BUT**: not all graphs can be augmented.
Polylog(n)-navigability

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**BUT**: not all graphs can be augmented.

For some graphs, greedy paths are of length at least \( \Omega(n^{1/\sqrt{\log n}}) \) for any augmentation. [FLL 06]
Navigability of arbitrary graphs

- Lower bound: $\Omega(n^{1/\sqrt{\log n}})$.

- Upper bound: $O(n^{1/2})$ with uniform augmentation.
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Improvement of the upper bound to $\tilde{O}(n^{1/3})$

**Theorem:**

Any graph can be augmented by one link/node s.t. greedy routing computes paths of expected length $\tilde{O}(n^{1/3})$ between any pair.
Improvement of the upper bound to $\tilde{O}(n^{1/3})$

Theorem:
Any graph can be augmented by one link/node s.t. greedy routing computes paths of expected length $\tilde{O}(n^{1/3})$ between any pair.

- **Augmentation process:**
  1. Node $u$ picks a level $k$ in $0...\log n$ (u.a.r.)
  2. Node $u$ picks a node $v$ in $B(u,2^k)$ (u.a.r)
Proof idea: $O(n^{2/5})$

Ex:
1. With proba. 1/2 pick $v$ u.a.r. in $G$
2. With proba. 1/2 pick $v$ u.a.r. in $B(u,n^{2/5})$

Ex: set of size $n^{3/5}$

Diagram:
- Target set of size $n^{3/5}$
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Diagram:
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- Target
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![Diagram showing shortest path and set of size $n^{3/5}$]
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$\Pr(u \rightarrow \text{blue}) \geq \frac{1}{2} \times \frac{n^{2/5}/2}{B_u(n^{2/5})} = \Omega(1/n^{1/5})$
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Pr($u \rightarrow$ blue) \geq (1/2) \times (n^{2/5}/2) / B_u(n^{2/5})
= \Omega(1/n^{1/5})

At most $n^{1/5}$ blue intervals
Proof idea: $O(n^{2/5})$

**Ex:**
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At most $n^{1/5}$ blue intervals

$E(\# \text{ steps}) \leq O(n^{2/5}) + O(n^{1/5}) \times n^{1/5} = O(n^{2/5})$. 
Another perspective: matrix augmentation

- A gap remains between $\tilde{O}(n^{1/3})$ and $\Omega(n^{1/\sqrt{\log n}})$.

- A new perspective to augment arbitrary graphs: a priori augmentation by giving a matrix of links distribution.
An augmentation matrix

$p_{i,j} =$ probability that the link of node i is node j

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$M = (p_{i,j})_{1 \leq i, j \leq 6}$
An augmentation matrix

$p_{i,j} = \text{probability that the link of node } i \text{ is node } j$

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$M = (p_{i,j})_{1 \leq i, j \leq 6}$

$\sum \leq 1$

Nodes destinations of the links

Diagram: Nodes 1, 2, 3, 4, 5, 6 with links and probabilities.
Name-independent matrix augmentation

- Distribution of links given by the matrix without looking at the graph.
- Without further information, what improvement can be hoped?
Name-independent matrix augmentation

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Theorem:

If the matrix is given independently from the graph labeling, the uniform matrix is optimal.
**Lemma:** in any augmentation matrix, there is a set of $\sqrt{n}$ indices s.t. $\sum p_{i,j} < 1$ on this set.
Name-independent matrix augmentation

**Lemma:** in any augmentation matrix, there is a set of $\sqrt{n}$ indices s.t. $\sum p_{i,j} < 1$ on this set.

- **BUT:** $\sum p_{i,j}$ on a set of indices $I$ is the expected number of links going out from $I$ into $I$. 
Name-independent matrix augmentation

**Lemma:** in any augmentation matrix, there is a set of $\sqrt{n}$ indices s.t. $\Sigma p_{i,j} < 1$ on this set.
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$\sqrt{n}$

Source

Target
Lemma: in any augmentation matrix, there is a set of $\sqrt{n}$ indices s.t. $\sum p_{i,j} < 1$ on this set.

- An adversary can label an interval with the bad set of indices.
- The expected number of shortcuts is $<1$ inside the interval: $\Omega(\sqrt{n})$ greedy steps.
Matrix augmentation with labeling

• The idea: keep the a priori augmentation (given matrix) but associate a proper labeling scheme.
Matrix augmentation with labeling

- **The idea**: keep the a priori augmentation (given matrix) but *associate a proper labeling scheme*.

Ex: the matrix fits well with the labels, paths $O(\log^2 n)$
Matrix augmentation with labeling

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Ex: the matrix fits well with the labels, paths $O(\log^2 n)$

Here: different labels, paths may rise up to $\Omega(\sqrt{n})$
Matrix augmentation with labeling

- The key of efficient augmentations: find good separators in the graphs to distribute the links hierarchically.
Matrix augmentation with labeling

Matrix augmentation with labeling:
Matrix augmentation with labeling

- Matrix augmentation with labeling:

  1. Build an augmentation matrix with "hierarchical" distribution among indices,
Matrix augmentation with labeling

- **Matrix augmentation with labeling:**
  
  1. Build an augmentation matrix with “hierarchical” distribution among indices,
  
  2. Build a labeling scheme that decomposes the graph along separators to assign nodes the right labels of M.
Matrix augmentation with labeling

Matrix augmentation with labeling:

1. Build an augmentation matrix with “hierarchical” distribution among indices,

2. Build a labeling scheme that decomposes the graph along separators to assign nodes the right labels of $M$.

$\Rightarrow$ Done through a path-decomposition.
**Theorem:**
There is a matrix $M$ and a labeling scheme $L$ s.t. in any graph $G$ augmented with $(M,L)$, greedy routing performs in:

$$O\left(\min\left(\log^2 n \times \text{pathshape}(G), \sqrt{n}\right)\right)$$

steps.

**Pathshape:** $\min(\text{pathwidth}, \text{pathlength})$

- $\text{distance} \leq \text{pathlength}$
- $\text{size} \leq \text{pathwidth}$
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- Improvement from $O(\sqrt{n})$ to $O(\log^2 n)$ for paths.
- New $O(\text{polylog } n)$-navigable graphs: interval, AT-free...
Conclusion & Perspectives

- Augmentation of arbitrary graphs: still a gap between $\tilde{O}(n^{1/3})$ and $\Omega(n^{1/\sqrt{\log n}})$. 
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• $\tilde{O}(n^{1/k})$ for any $k<\sqrt{\log n}$?
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Conclusion & Perspectives

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  - $\tilde{O}(n^{1/k})$ for any $k<\sqrt{\log n}$?
  - But raises graph decomposition $Q^\circ$.

- Matrix augmentation: can we get rid of $O(\sqrt{n})$ in the bound?