An extended proportional-integral control algorithm for distributed average tracking and its applications in Euler-Lagrange systems

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Abstract—Under an extended proportional-integral (PI) control scheme, distributed average tracking (DAT) control algorithms are derived for networked Euler-Lagrange systems for two different kinds of reference signals: reference signals with steady states and reference signals with bounded derivatives.

Index Terms—Multi-agent systems, cooperative control, distributed average tracking, Euler-Lagrange systems.

I. INTRODUCTION

The focus of this paper is the following distributed average tracking (DAT) problem: given a group of agents, each having a reference signal, design a control law for each agent based on local sensing or communication information such that all the agents will finally track the average of the reference signals. While the DAT problem can be viewed as a generalization of the average consensus problem and the coordinated tracking problem, it has the unique challenge that the average of the reference signals to be tracked might not be available to any agent and is usually time-varying. The problem has received increasing attention in recent years from the control community due to its broad applications in control related problems. In particular, theoretical results on DAT have been successfully applied in distributed sensor fusion [1], [2] and multi-agent coordination [3].

In the early stage of the DAT development, researchers seek to find solutions to the problem via designing linear control algorithms [4]–[6]. Besides, some robustness issues, e.g., robustness to the initialization errors, are also addressed [6]. These early works help build up a general framework for further study on this topic. Later, some efforts are made to construct nonlinear DAT algorithms capable of tracking general time-varying signals. Nonsmooth algorithms are proposed in [7], [8] to track arbitrary time-varying reference signals with bounded derivatives or accelerations. When a tracking error is allowed, a class of nonlinear algorithms is reported in [9] for reference signals having a bounded deviation.

On the other hand, cooperative control of networked Euler-Lagrange systems has received intense attention recently, due to the fact that a large class of mechanical systems can be modelled by Euler-Lagrange equations. Consensus results of Euler-Lagrange systems are established for both undirected network topology [10] and directed network topology [11], [12]. For coordinated tracking of Euler-Lagrange systems, centralized controller design is first presented [11], [13]–[15], by assuming that all the agents have the information of the reference signal or so-called the leader. The results are then extended to the distributed control case where the reference signal is only available to a subset of the agents [16], [17].

Although some useful DAT algorithms have been reported in the literature, there are few results to guide the design of DAT algorithms systematically for various reference signals. It would be desirable if some control design conditions can be provided such that DAT algorithms can be constructed accordingly to suit for different control purposes. Meanwhile, agents' nonlinear dynamics have not been fully taken into account in most DAT studies. In fact, it is still challenging to be addressed with potential applications such as the region following formation control [18] or the coordinated path planning [19]. This motivates this study.

The contributions of this paper are two-fold. First, under an extended PI control framework, DAT algorithms are proposed for two different kinds of reference signals: reference signals with steady states and reference signals with bounded derivatives. Second, the DAT problem for networked Euler-Lagrange systems is solved with the aid of the above algorithms, which might lead to a broader spectrum of applications for DAT.

II. PROBLEM DESCRIPTION

Suppose that there are $N \in \mathbb{Z}^+$ agents, labeled from 1 to $N$. The dynamics of the agents are described by the following nonidentical Euler-Lagrange equations

$$M_i[x_i(t)]\ddot{x}_i(t) + C_i[x_i(t), \dot{x}_i(t)]\dot{x}_i(t) + g_i[x_i(t)] = u_i(t)$$

where $x_i(t) \in \mathbb{R}^n$ is the vector of generalized coordinates, $M_i[x_i(t)] \in \mathbb{R}^{n \times n}$ the positive definite inertial matrix, $C_i[x_i(t), \dot{x}_i(t)]\dot{x}_i(t) \in \mathbb{R}^n$ the vector of Coriolis and centrifugal forces, $g_i[x_i(t)] \in \mathbb{R}^n$ the vector of gravitational force, $u_i(t) \in \mathbb{R}^n$ its control input to be designed, and $n \in \mathbb{Z}^+$. The Euler-Lagrange equations (1) satisfy the following assumptions.

Assumption 1 ([20]):
Define some constant \( u \) describe sensing or communications among the agents. The reference signals are governed by:

\[
\dot{Y}_i[x_i(t), \dot{x}_i(t), \chi, \eta, \theta_i] = Y_i[x_i(t), \dot{x}_i(t), \chi, \eta, \theta_i] \quad \text{for} \quad i = 1, \ldots, N \text{ and } x_i(t), \dot{x}_i(t), \chi, \eta \in \mathbb{R}^n, \text{ where the regressor } Y_i[x_i(t), \dot{x}_i(t), \chi, \eta] \in \mathbb{R}^{n \times m} \text{ is a known continuous function and } \theta_i \in \mathbb{R}^m \text{ is an unknown constant parameter vector of agent } i.
\]

Note that a continuous function maps compact sets into compact sets. Thus, if:

\[
\|Y_i[x_i^T(t), \dot{x}_i^T(t), \chi^T, \eta^T]\| \leq \alpha 
\]

for some constant \( \alpha \in \mathbb{R}^+ \), there exists \( \bar{Y}(\alpha) \in \mathbb{R}^+ \) such that:

\[
\|Y_i[x_i^T(t), \dot{x}_i^T(t), \chi^T, \eta^T]\| \leq \bar{Y}(\alpha). \quad (2)
\]

Define:

\[
x(t) \triangleq \begin{bmatrix} x_1^T(t), \ldots, x_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{nN},
\]

\[
M[x(t)] \triangleq \begin{bmatrix} M_1[x_1(t)], \ldots, M_N[x_N(t)] \end{bmatrix} \in \mathbb{R}^{nN \times nN},
\]

\[
\begin{bmatrix} C \{ x_1(t), \dot{x}_1(t), \ldots, x_N(t), \dot{x}_N(t) \} \end{bmatrix} \in \mathbb{R}^{nN \times nN},
\]

\[
g[x(t)] \triangleq \begin{bmatrix} g_1^T[x_1(t)], \ldots, g_N^T[x_N(t)] \end{bmatrix} \in \mathbb{R}^{nN},
\]

\[
u(t) \triangleq \begin{bmatrix} u_1^T(t), \ldots, u_N^T(t) \end{bmatrix} \in \mathbb{R}^{nN}. \quad (3)
\]

An undirected graph \( G \triangleq \{V, E\} \) is used to describe sensing or communications among the agents, where \( V \triangleq \{1, \ldots, N\} \) is the node set with each node representing an agent and \( E \triangleq \{(i, j) \mid \text{agent } i \text{ can sense or communicate with agent } j, \ i,j = 1, \ldots, N, i \neq j\} \) is the edge set.

**Assumption 2**: Graph \( G \) is connected.

Suppose that each agent has a reference signal \( r_i(t) \in \mathbb{R}^n, i = 1, \ldots, N \). Let \( r(t) \triangleq \begin{bmatrix} r_1^T(t), \ldots, r_N^T(t) \end{bmatrix}^T \in \mathbb{R}^{nN} \) be the stack vector. The reference signals are governed by:

\[
\dot{r}(t) = f_1[t, r(t)] \in \mathbb{R}^{nN} \text{ is continuously differentiable}. \quad (4)
\]

The main objective of DAT is to design controllers for the agents based on local information, such that for all \( i = 1, \ldots, N \),

\[
\left\| x_i(t) - \frac{1}{N} \sum_{j=1}^{N} r_j(t) \right\| \to 0, \quad \text{as } t \to \infty.
\]

**III. NOVEL DISTRIBUTED AVERAGE TRACKING ALGORITHMS**

In this subsection, several new DAT algorithms are derived. The control algorithms take the following form:

\[
\dot{x}(t) = (A_1 \otimes I_n)x(t) + (B_1 \otimes I_n)w(t) + (C_1 \otimes I_n)r(t) + (D_1 \otimes I_n)\dot{r}(t),
\]

\[
w(t) = (A_2 \otimes I_n)x(t) + (B_2 \otimes I_n)w(t) + (C_2 \otimes I_n)r(t) + (D_2 \otimes I_n)\dot{r}(t) + (E_2 \otimes I_n)\dot{r}(t),
\]

where \( w(t) \in \mathbb{R}^{nN} \) is an auxiliary variable and the parameters \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, E_2 \in \mathbb{R}^{nN \times nN} \) are polynomials of the Laplacian matrix \( L \) and the identity matrix \( I \). Thus, the matrices \( A_1, B_1, C_1, D_1, A_2, B_2, C_2, D_2, E_2 \) can be simultaneously diagonalized. Specifically, let \( \xi_1, \xi_2, \ldots, \xi_N \in \mathbb{R}^n \) be the right eigenvectors of \( L \) corresponding to the eigenvalues \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \) respectively such that \( \xi_i^T \xi_j = 0 \) for \( i \neq j \) and \( \xi_i^T \xi_i = 1 \).

It follows that:

\[
[\xi_1, \ldots, \xi_N]^T A_i[\xi_1, \ldots, \xi_N] = \begin{bmatrix} a_{11}, \ldots, a_{1N} \end{bmatrix}, \quad i = 1, 2,
\]

\[
[\xi_1, \ldots, \xi_N]^T B_i[\xi_1, \ldots, \xi_N] = \begin{bmatrix} b_{11}, \ldots, b_{1N} \end{bmatrix}, \quad i = 1, 2,
\]

\[
[\xi_1, \ldots, \xi_N]^T C_i[\xi_1, \ldots, \xi_N] = \begin{bmatrix} c_{11}, \ldots, c_{1N} \end{bmatrix}, \quad i = 1, 2,
\]

\[
[\xi_1, \ldots, \xi_N]^T D_i[\xi_1, \ldots, \xi_N] = \begin{bmatrix} d_{11}, \ldots, d_{1N} \end{bmatrix}, \quad i = 1, 2,
\]

\[
[\xi_1, \ldots, \xi_N]^T E_2[\xi_1, \ldots, \xi_N] = \begin{bmatrix} e_{21}, \ldots, e_{2N} \end{bmatrix}. \quad (8)
\]

**Theorem 1**: 1) If the following conditions hold:

\[
b_{11} = 0, \quad \text{and} \quad (a_{11} + c_{11})(\xi_1^T \otimes I_n)r(t) = 0,
\]

\[
(d_{11} - 1)(\xi_1^T \otimes I_n)\dot{r}(t) = 0, \quad (11)
\]

\[
[(-\text{diag}[b_{22}, \ldots, b_{2N}]) \text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \times \text{diag}[c_{21}, \ldots, c_{2N}] + \text{diag}[e_{22}, \ldots, e_{2N}] \otimes I_n]
\]

\[
\left((\xi_2, \ldots, \xi_N)^T \otimes I_n\right) \dot{r}(t) = 0, \quad (12)
\]

\[
\left\{(-\text{diag}[b_{22}, \ldots, b_{2N}]) \text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \times \text{diag}[d_{12}, \ldots, d_{1N}] + \text{diag}[d_{22}, \ldots, d_{2N}] + \text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \text{diag}[c_{12}, \ldots, c_{1N}] \otimes I_n\right\}
\]

\[
\left((\xi_2, \ldots, \xi_N)^T \otimes I_n\right) \dot{r}(t) = 0, \quad (13)
\]

\[
\left\{\text{diag}[e_{22}, \ldots, e_{2N}] + \text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \times \text{diag}[d_{12}, \ldots, d_{1N}] \otimes I_n\right\}
\]

\[
\left((\xi_2, \ldots, \xi_N)^T \otimes I_n\right) \dot{r}(t) = 0, \quad (14)
\]
and

\[
A \triangleq \begin{bmatrix}
\text{diag}[a_{11}, \ldots, a_{1N}] & 0^T \\
0 & \text{diag}[b_{22}, \ldots, b_{2N}]
\end{bmatrix}
\begin{bmatrix}
\text{diag}[b_{12}, \ldots, b_{1N}] \\
0
\end{bmatrix}
\]

∈ \mathbb{R}^{(2N-1) \times (2N-1)} is Hurwitz, (15)

then \( e_{\text{DAT}}(t) \to 0 \) as \( t \to \infty \).

2) If Assumption (R1) holds, \( E_2 = 0 \), and (9), (10), and (15) hold, then \( e_{\text{DAT}}(t) \to 0 \) as \( t \to \infty \).

3) If Assumption (R2) holds, \( E_2 = 0 \), and (9), (10), (12), and (15) hold, then the tracking error \( \|e_{\text{DAT}}(t)\| \) is ultimately bounded.

**Proof:**
1) Define the DAT error

\[
e_{\text{DAT}}(t) \triangleq x(t) - \left( \frac{11^T}{N} \otimes I_n \right) r(t),
\]

It follows from (16) and (7) that

\[
\dot{e}_{\text{DAT}}(t) = (A_1 \otimes I_n) e_{\text{DAT}}(t) + (A_1 \otimes I_n) \left( \frac{11^T}{N} \otimes I_n \right) r(t)
\]

\[
+ (B_1 \otimes I_n) w(t) + (C_1 \otimes I_n) r(t)
\]

\[
+ \left( D_1 - \frac{11^T}{N} \otimes I_n \right) \dot{r}(t).
\]

Define \( e_{\text{DAT}}^\xi(t) \triangleq (\xi_1, \ldots, \xi_N)^T \otimes I_n \) \( e_{\text{DAT}}(t) \in \mathbb{R}^{N^2} \) and

\[
w_{\text{DAT}}^\xi(t) \triangleq (\xi_1, \ldots, \xi_N)^T \otimes I_n) w(t) \in \mathbb{R}^{(N-1)n}. \]

It follows from (8) and (9) that

\[
\dot{e}_{\text{DAT}}^\xi(t) = (\text{diag}[a_{11}, \ldots, a_{1N}] \otimes I_n) e_{\text{DAT}}^\xi(t)
\]

\[
+ \left( \text{diag}[b_{12}, \ldots, b_{1N}] \otimes I_n \right) w_{\text{DAT}}^\xi(t)
\]

\[
+ \left( \text{diag}[a_{11}, \ldots, a_{1N}] \otimes I_n \right) \left( \frac{11^T}{N} \otimes I_n \right) r(t)
\]

\[
+ \left( \text{diag}[c_{11}, \ldots, c_{1N}] \otimes I_n \right) \left( \frac{11^T}{N} \otimes I_n \right) \dot{r}(t)
\]

\[
+ \left( \text{diag}[d_{11}, \ldots, d_{1N}] \otimes I_n \right) \left( \frac{11^T}{N} \otimes I_n \right) \dot{r}(t).
\]

Let \( e_{\text{DAT}}^\xi(t) = 0 \) and \( w_{\text{DAT}}^\xi(t) \) be an equilibrium point of (17).

It follows from (10) and (11) that

\[
\dot{w}_{\text{DAT}}^\xi(t) = -(\text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \text{diag}[c_{12}, \ldots, c_{1N}])
\]

\[
\times (\xi_2, \ldots, \xi_N)^T \otimes I_n) r(t)
\]

\[
- (\text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \text{diag}[d_{12}, \ldots, d_{1N}])
\]

\[
\times (\xi_2, \ldots, \xi_N)^T \otimes I_n) \dot{r}(t),
\]

which yields that

\[
\dot{e}_{\text{DAT}}^\xi(t) = (\text{diag}[a_{11}, \ldots, a_{1N}] \otimes I_n) e_{\text{DAT}}^\xi(t)
\]

\[
+ \left( \text{diag}[b_{12}, \ldots, b_{1N}] \otimes I_n \right) \dot{w}_{\text{DAT}}^\xi(t),
\]

where \( w_{\text{DAT}}^\xi(t) = w_{\text{DAT}}^\xi(t) - w_{\text{DAT}}^\xi(t) \) are governed by

\[
\dot{w}_{\text{DAT}}^\xi(t) = (\xi_2, \ldots, \xi_N)^T \otimes I_n) (A_2 \otimes I_n) x(t)
\]

\[
+ (\xi_2, \ldots, \xi_N)^T \otimes I_n) (B_2 \otimes I_n) w(t)
\]

\[
+ (\xi_2, \ldots, \xi_N)^T \otimes I_n) (C_2 \otimes I_n) r(t)
\]

\[
+ (\xi_2, \ldots, \xi_N)^T \otimes I_n) (D_2 \otimes I_n) \dot{r}(t)
\]

\[
+ (\xi_2, \ldots, \xi_N)^T \otimes I_n) (E_2 \otimes I_n) \ddot{r}(t)
\]

\[
+ (\text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \text{diag}[c_{12}, \ldots, c_{1N}])
\]

\[
\times (\xi_2, \ldots, \xi_N)^T \otimes I_n) \dot{r}(t)
\]

\[
+ (\text{diag}[b_{12}^{-1}, \ldots, b_{1N}^{-1}] \text{diag}[d_{12}, \ldots, d_{1N}])
\]

\[
\times (\xi_2, \ldots, \xi_N)^T \otimes I_n) \ddot{r}(t).
\]

Due to (12)–(14), Eqs. (18) and (22) can be rewritten as

\[
\begin{bmatrix}
\dot{e}_{\text{DAT}}^\xi(t) \\
\dot{w}_{\text{DAT}}^\xi(t)
\end{bmatrix}
= \begin{bmatrix}
A \otimes I_n \\
A \otimes I_n
\end{bmatrix}
\begin{bmatrix}
e_{\text{DAT}}^\xi(t) \\
w_{\text{DAT}}^\xi(t)
\end{bmatrix},
\]

where \( A \) is defined by (15). It follows from (15) that \( e_{\text{DAT}}^\xi(t) \to 0 \) as \( t \to \infty \), which indicates that \( e_{\text{DAT}}(t) \to 0 \) as \( t \to \infty \). The proof of the first part is thus completed.
2) Redefine the equilibrium

\[ \bar{w}_\xi^* \equiv - (\text{diag}[b_1^{-1}, \ldots, b_N^{-1}]) \text{diag}[c_1, \ldots, c_N] \times [\xi_2, \ldots, \xi_N]^T \otimes I_n \hat{r}(t). \]  

Due to (9), (10), and (24), Eq. (17) can be rewritten as

\[ \dot{\bar{w}}_\xi^* = \text{diag}[b_1^{-1}, \ldots, b_N^{-1}] \text{diag}[c_1, \ldots, c_N] \times [\xi_2, \ldots, \xi_N]^T \otimes I_n \hat{r}(t). \]

Using a similar proof to that of the first part and \( E_2 = 0 \), it can be shown that

\[ \dot{\bar{w}}_\xi^* = \text{diag}[d_1, \ldots, d_N] [\xi_2, \ldots, \xi_N]^T \otimes I_n \hat{r}(t). \]

Define \( A_w \triangleq \text{diag}[d_2, \ldots, d_N] [\xi_2, \ldots, \xi_N]^T + \text{diag}[b_1^{-1}, \ldots, b_N^{-1}] [\xi_2, \ldots, \xi_N]^T \) and \( A_e \triangleq \text{diag}[d_1, \ldots, d_N] [\xi_2, \ldots, \xi_N]^T - [\xi_1, \ldots, \xi_N]^T \frac{11}{N} \). It follows from (12) that

\[ \begin{bmatrix} \dot{\bar{w}}_\xi^* \\ \dot{\xi}^* \end{bmatrix} = \begin{bmatrix} A_w \\ A_e \end{bmatrix} \begin{bmatrix} \bar{w}_\xi^* \\ \xi^* \end{bmatrix} \]  

Due to (15), there exists a positive definite matrix \( Q \in \mathbb{R}^{(2N-1) \times (2N-1)} \) such that the Lyapunov equation

\[ A^T Q + QA = -2I \]  

holds. Define the function

\[ V(t) \triangleq \langle T \rangle \bar{w}_\xi^* (t), \dot{\xi}^* (t) \rangle (Q \otimes I_n) \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \| \]

which can be bounded by class \( \mathcal{K} \) functions as follows:

\[ (1/2) \lambda_{\min}(Q) \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \leq V(t) \]

\[ \leq (1/2) \lambda_{\max}(Q) \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2, \]

where \( \lambda_{\max}(Q) \geq \lambda_{\min}(Q) > 0 \). The Lie derivative of \( V(t) \) along the solutions of (25) is given by

\[ \dot{V}(t) = \langle T \rangle \bar{w}_\xi^* (t), \dot{\xi}^* (t) \rangle (Q \otimes I_n) \begin{bmatrix} A_e \\ A_w \end{bmatrix} \begin{bmatrix} \bar{w}_\xi^* \\ \xi^* \end{bmatrix} \hat{r}(t). \]

From (26), it follows that

\[ \dot{V}(t) \leq \langle T \rangle \bar{w}_\xi^* (t), \dot{\xi}^* (t) \rangle (Q \otimes I_n) \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \hat{r}(t) \|, \]

which with (27) yields

\[ \dot{V}(t) \leq \frac{\sqrt{2}}{2} \lambda_{\max}(Q) \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \| \hat{r}(t) \|. \]

Integrating on both sides and using Assumption (R1) yields

\[ \sqrt{V(t)} \leq \sqrt{V(0)} + \frac{\sqrt{2}}{2} \lambda_{\max}(Q) \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \rho_1. \]

Using (27) again, it follows that

\[ \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \leq \frac{\sqrt{2V(0)}}{\lambda_{\min}(Q)} + \lambda_{\max}(Q) \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \rho_1. \]

Thus, the solutions of (25) are bounded. In addition, recall that Assumption (R1) implies that \( \| \hat{r}(t) \| \leq \rho_2 \) for all \( t \geq 0 \).

Therefore, it follows from (15) and Lemma 2 in [21] that (25) is globally asymptotically stable because (23) is globally asymptotically stable. The proof of the second part is thus completed.

3) It follows from (26) and (28) that

\[ \dot{V}(t) \leq -\| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \]

\[ + \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \| \| Q \| \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \| \hat{r}(t) \|. \]  

Choose \( \alpha \in (0, 1) \), then (29) can be rewritten as

\[ \dot{V}(t) \leq -\alpha \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \]

\[ - (1 - \alpha) \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \]

\[ + \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \| \lambda_{\max}(Q) \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \| \hat{r}(t) \|. \]

If

\[ \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \leq \lambda_{\max}(Q) \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \| \hat{r}(t) \| / \alpha, \]

then \( \dot{V}(t) \leq -(1 - \alpha) \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \). It thus follows from Assumption (R2) that \( \| \bar{w}_\xi^* (t), \dot{\xi}^* (t) \|^2 \) is ultimately upper bounded by \( \lambda_{\max}(Q) \| \begin{bmatrix} A_e \\ A_w \end{bmatrix} \| \rho_2 / (\alpha \sqrt{\lambda_{\min}(Q)}). \)

The proof of the third statement is thus completed.

In (14), if \( B_1 \neq 0 \) and \( D_1 \neq 0 \), then \( E_2 \neq 0 \) is needed to satisfy (14). That is why in (7), \( \bar{w}(t) \) employs a higher-order derivative of \( r(t) \) that \( \bar{x}(t) \). Theorem 1 can be used to design various DAT algorithms for different reference signals. The following are a few examples.

Theorem 2:

1) If Assumption (R1) holds and the control algorithm (7) takes the following form

\[ \begin{cases} A_1 = -(I + L), B_1 = L, C_1 = I, D_1 = 0, \\ A_2 = -L, B_2 = C_2 = D_2 = E_2 = 0, \end{cases} \]

then the conditions (9), (10), (12), and (15) are satisfied and \( \bar{w}_\xi^* (t) \to 0 \) as \( t \to \infty \).
2) If Assumption (R2) holds and the control algorithm (7) takes the following form
\[
\begin{align*}
A_1 &= -\gamma(I + L), \\
B_1 &= \gamma L, \\
C_1 &= \gamma I, \\
D_1 &= \cdots = Y[x(t), \dot{x}(t), \dot{\nu}(t), \nu(t)] \hat{\theta}(t) - [\Gamma + (1/4)I]s(t), \\
\dot{\hat{\theta}}(t) &= -YT[x(t), \dot{x}(t), \dot{\nu}(t), \nu(t)]s(t). \\
\end{align*}
\] (43) (44)

Assumption (R1). A set of auxiliary variables are designed, and it is shown that under the proposed control algorithm DAT can be achieved asymptotically.

Define the following auxiliary variables
\[
s_i(t) \triangleq \dot{x}_i(t) + x_i(t) + \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)]
- \sum_{j=1}^{N} a_{ij}[w_i(t) - w_j(t)] - r_i(t),
\] (35)
\[
\dot{w}_i(t) \triangleq -\sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)], \quad i = 1, \ldots, N, 
\] (36)
where $a_{ij}$ is the $(i, j)$th element of the adjacency matrix of $G$. Define $s(t) \triangleq [s^T_1(t), \ldots, s^T_N(t)]^T \in \mathbb{R}^{Nn}$. Eqs. (35) and (36) can be rewritten into a vector form as
\[
s(t) = \dot{x}(t) + [(I + L) \otimes I_n]x(t) - (L \otimes I_n)w(t) - r(t),
\] (37)
\[
\dot{w}(t) = -(L \otimes I_n)x(t).
\] (38)

It follows from (25) that
\[
\begin{bmatrix}
\dot{e}_{\text{DAT}}(t) \\
\dot{\bar{w}}(t)
\end{bmatrix}
= (A \otimes I_n)\begin{bmatrix}
e_{\text{DAT}}(t) \\
\bar{w}(t)
\end{bmatrix}
+ \begin{bmatrix}
A_e \\
A_w
\end{bmatrix} \otimes I_n \hat{\theta}(t)
+ (B \otimes I_n)s(t),
\] (39)
where $A$ is defined in (32), $A_e \triangleq -[\xi_1, 0, \ldots, 0]^T$, $A_w \triangleq \text{diag}[\lambda_2^1, \ldots, \lambda_N^1][\xi_2, \ldots, \xi_N]^T$, and $B \triangleq \begin{bmatrix}
\xi_1, \ldots, 0
\end{bmatrix}^T$.

The control input $u_i(t)$ of (1) is designed as
\[
u_i(t) \triangleq Y_i[x_i(t), \dot{x}_i(t), \dot{\nu}_i(t), \nu_i(t)]\hat{\theta}_i(t)
- [\Gamma_i + (1/4)I_n]s_i(t),
\] (40)
\[
\nu_i(t) \triangleq -x_i(t) - \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)]
+ \sum_{j=1}^{N} a_{ij}[w_i(t) - w_j(t)] + r_i(t),
\] (41)
where $\hat{\theta}_i(t) \in \mathbb{R}^{n}$ is an estimate of $\theta_i$ defined in Assumption (EL3), $\Gamma_i \in \mathbb{R}^{n \times n}$ a symmetric positive definite matrix, and $Y_i[x_i(t), \dot{x}_i(t), \dot{\nu}_i(t), \nu_i(t)]$ the regressor defined in Assumption (EL3). The adaptation law of $\hat{\theta}_i(t)$ is designed as
\[
\dot{\hat{\theta}}_i(t) \triangleq -Y_i^T[x_i(t), \dot{x}_i(t), \dot{\nu}_i(t), \nu_i(t)]s_i(t).
\] (42)
Define $u(t) \triangleq \begin{bmatrix}
u_1^T(t), \ldots, \nu_N^T(t)
\end{bmatrix}^T$, $\nu(t) \triangleq \begin{bmatrix}
u_1^T(t), \ldots, \nu_N^T(t)
\end{bmatrix}^T$, $\hat{\theta}(t) \triangleq \begin{bmatrix}\hat{\theta}_1^T(t), \ldots, \hat{\theta}_N^T(t)
\end{bmatrix}^T$, $\Gamma \triangleq \text{diag}(\Gamma_1, \ldots, \Gamma_N)$, and let
\[
Y[x(t), \dot{x}(t), \dot{\nu}(t), \nu(t)] \triangleq \text{diag}[Y_1[x_1(t), \dot{x}_1(t), \dot{\nu}_1(t), \nu_1(t)],
\ldots, Y_N[x_N(t), \dot{x}_N(t), \dot{\nu}_N(t), \nu_N(t)]].
\]
Eqs. (40) and (42) can be rewritten into a vector form as
\[
u(t) = Y[x(t), \dot{x}(t), \dot{\nu}(t), \nu(t)]\hat{\theta}(t) - [\Gamma + (1/4)I]s(t),
\] (43)
\[
\dot{\hat{\theta}}(t) = -Y^T[x(t), \dot{x}(t), \dot{\nu}(t), \nu(t)]s(t).
\] (44)
It follows from (37) and (41) that
\[ \dot{x}(t) = s(t) + \nu(t). \]  
(45)
Substituting (45) into (3) yields
\[ M[x(t)]\dot{s}(t) + \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)] - \sum_{j=1}^{N} a_{ij}w_i(t) - w_j(t) - r_i(t) \]  
and
\[ \dot{w}_i(t) \equiv -\gamma \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)] + \sum_{j=1}^{N} a_{ij}[w_i(t) - w_j(t)] + r_i(t), \]  
where \( \gamma > 1 \) is a constant.
The control input \( u_i(t) \) of (1) is designed as
\[ u_i(t) \equiv Y_i[x_i(t), \dot{x}_i(t), \nu_i(t), \nu(t)]\dot{\theta}_i(t), \]  
(47)
\[ \nu_i(t) \equiv -\gamma x_i(t) - \gamma \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)] \]  
\[ + \gamma \sum_{j=1}^{N} a_{ij}w_i(t) - w_j(t) + \gamma r_i(t). \]  
(48)
The adaptation law of \( \dot{\theta}_i(t) \) is given by
\[ \dot{\theta}_i(t) \equiv -Y_i^T[x_i(t), \dot{x}_i(t), \nu_i(t), \nu(t)]s_i(t). \]  
(49)

The main results of this section are stated in the following theorem.

**Theorem 3:** For system (1) with the control inputs (40)–(42), if Assumptions 1, 2, and (R1) hold, then the control objective (6) is achieved asymptotically.

**V. DISTRIBUTED AVERAGE TRACKING FOR EULER-LAGRANGE SYSTEMS UNDER REFERENCE SIGNALS WITH BOUNDED DERIVATIVES**

Design the following auxiliary variables
\[ s_i(t) \equiv \dot{x}_i(t) + \gamma \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)] - \sum_{j=1}^{N} a_{ij}w_i(t) - w_j(t) - r_i(t), \]  
and
\[ \dot{w}_i(t) \equiv -\gamma \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)] + \sum_{j=1}^{N} a_{ij}[w_i(t) - w_j(t)] + r_i(t). \]  
The control input \( u_i(t) \) of (1) is designed as
\[ u_i(t) \equiv Y_i[x_i(t), \dot{x}_i(t), \nu_i(t), \nu(t)]\dot{\theta}_i(t), \]  
(47)
\[ \nu_i(t) \equiv -\gamma x_i(t) - \gamma \sum_{j=1}^{N} a_{ij}[x_i(t) - x_j(t)] \]  
\[ + \gamma \sum_{j=1}^{N} a_{ij}w_i(t) - w_j(t) + \gamma r_i(t). \]  
(48)
The adaptation law of \( \dot{\theta}_i(t) \) is given by
\[ \dot{\theta}_i(t) \equiv -Y_i^T[x_i(t), \dot{x}_i(t), \nu_i(t), \nu(t)]s_i(t). \]  
(49)

**Theorem 4:** For system (1) with the control inputs (47)–(49), if Assumptions 1, 2, and (R2) hold, then the DAT error is ultimately upper bounded by
\[ \sqrt{\max(1, m)} \| \dot{\theta}_i(t) \|_2 + \gamma \| \nu_i(t) \|_2 + \| w_i(t) \|_2 \]  
where \( \dot{\theta}_i(t) \) are finite constants.

**VI. CONCLUSION**

Under an extended PI control scheme, several novel DAT control algorithms have been developed for various kinds of tracking reference signals, which have been further applied in solving the DAT problem for networked Euler-Lagrange systems.