Multicarrier demodulation for frequency selective channels subject to narrow band interferences

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Abstract

Nowadays, multicarrier modulation (MCM) systems based on the discrete Fourier transform are commonly used to transmit over frequency selective channels subject to aggressive noise disturbances. However, these transceivers suffer from poor subchannel spectral containment, that is, the level of inter-channel interference is not negligible. It can be shown that the system performance decreases when it is subject to a disturbance with most of its energy concentrated on a narrow frequency band. To improve the quality of service under these conditions, we present general MCM systems based on filter bank transformations. Lately, we add an equalizer at the channel output to implement these techniques over frequency selective channels, and we introduce a set of design strategies for the equalizer. Finally, we evaluate the performance of the overall system by simulating a frequency selective channel subject to narrow band interference.

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1. Introduction

Multicarrier modulation (MCM) systems have become widely popular in the last decades in order to reach high data rates over frequency selective channels subject to strong disturbances. As such, MCM has been recognized as an effective technique for mitigating inter-symbol interference (ISI) over frequency selective channels. In that sense, MCM is an alternative to traditional equalized communication systems. Examples of MCM systems used for broadband data transmission include discrete multi-tone (DMT), adopted for broadband transmission over phone lines such as ADSL and VDSL services, orthogonal frequency division multiplexing (OFDM) considered for data transmission over wireless local area networks such as in the IEEE standard 802.11a, and discrete wavelet multi-tone (DWMT), used on some implementations of power line communications such as HD-PLC.

The working principle for MCM is to transmit the information through very narrow subchannels so that ISI within each subchannel is almost eliminated. To avoid inter-channel interference (ICI), zero spectral overlap among neigh-

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boring subchannels is required. Interference among subchannels becomes critical when the system is perturbed by narrow band disturbances. In this case, disturbances that would affect just a single subchannel when no ICI is present, they will perturb noticeably other subchannels when ICI is present. The result is a larger probability of error than the one obtained on the ideal system. A typical example of a communication system affected by narrow band noise is a DSL system subject to radio frequency interference (RFI). For example, licensed AM radios broadcast in the frequency range between 540 kHz and 1.7 MHz, which interacts with the ADSL and VDSL transmissions. Another major RFI for wired services are ham radio communications. In those cases, the interference signal may have a large energy resulting in a serious disturbance for the transmission of data.

In this paper, we propose a particular structure for a MCM system to achieve good performance in the presence of narrow band disturbances. For that matter, we follow through a three-step thinking process: first, we design the MCM system for an ideal channel with no noise; on a second step, we include an equalizer at the input of the receiver to consider the effects of a frequency selective channel; finally, on the third step, we evaluate the performance of the overall structure using numerical experiments. The main objective of the paper is not to present a narrow band interference suppression system for MCM, but rather to introduce a particular structure for MCM systems that has naturally more resistance against narrow band interferences than the structure commonly used in the present. Moreover, the results presented here will benefit from strategies for disturbance suppression such as those presented in [1,2].

In general, MCM systems may be framed as filter-bank modulation techniques [3,4]. In this case, transmitter and receiver filter banks are derived from a based band filter known as the prototype filter [5]. The spectral characteristics of the prototype filter determine the properties of the modulation structure. For example, both DMT and OFDM are based on the discrete Fourier transform (DFT). Then the corresponding prototype filter is the ideal sinc(·) function that has strong side lobes crossing adjacent subchannels. Therefore, at the output of each subchannel, the level of ICI can be considerable. To solve this problem, we analyze two different designs for the prototype filter. Each of these designs defines subchannels with better subchannel spectral containment than DMT, i.e., subchannels having smaller spectral overlap among neighboring subchannels than the overlap seen in DMT. The first design implements the extended lapped transform (ELT). In this case, we obtain a transceiver satisfying the perfect reconstruction (PR) condition that guarantees a filter bank with zero reconstruction error for an ideal channel with no-noise [5]. For the second design, we relax the PR condition and we set a near perfect reconstruction (NPR) condition to improve the spectral characteristics of the modulation scheme.

The next step on our thinking process is to deal with nonideal transmission channels. For that we add an equalizer at the output of the transmission channel and we analyze a set of equalization techniques. In particular, we show that the modulation schemes obtained in the previous step require phase equalization only, that is, the equalized channel must have just linear phase to recover the results obtained for the ideal channel. Given an FIR channel, this condition is fulfilled if the impulse response of the equalized channel has central symmetry. This observation has already been made by the authors of [6] who propose a design algorithm that obtains an approximately symmetric equalized channel. The approximation is made within a window of fixed length δ. In this paper, we explore this notion further and we determine necessary conditions to obtain a unique solution by appropriate selection of the window length δ. In addition, we propose a complementary approach that may result in a design problem with fewer numbers of parameters.

Finally, the PR system and the NPR system, together with the regular DMT system, are evaluated with respect to their ability to mitigate additive narrow band interference at the output of the transmission channel. For frequency selective transmission channels, a phase equalizer is included at the input of the receiver. The overall communication system for each MCM structure is numerically tested for a DSL channel.

It has been observed already in [7] that general MCM systems outperform DMT systems. Here, we will show that MCM systems based on NPR filter banks have not only a better performance than Fourier-based DMT systems, but also they perform better than MCM systems based on PR filter banks as well.

The paper is organized as follows: Section 2 is an introduction to MCM and their use in communication; then, Section 3 illustrates the use of filter banks to construct a general MCM system for an ideal channel; the real case of a dispersive channel is considered in Section 4 where two equalization techniques are proposed; finally, the experimental results are presented in Section 5 that are followed by the conclusions.

The notation used in this paper is standard. Vectors and matrices are noted with boldface; the superscript $t$ is the transpose of a the corresponding vector and/or matrix; given a discrete time signal $x(n)$, $X(z)$ is its $z$-transform and $F(x(n)) = X(\omega)$ is the corresponding discrete time Fourier transform. The words carrier, subchannel or tone are used interchangeably to denote an individual component of the transmitted signal.
2. Multicarrier modulation

Multicarrier systems are efficient communication systems used to obtain high data rates over noisy channels. Suppose that the communication system contains $M$ carriers and that $T$ is the signaling period of the transmission. Then every $T$ seconds, the data stream is divided into $M$ sub-streams, one per carrier. Each carrier has a finite alphabet associated to it. The dimension of each alphabet is chosen according to the signal-to-noise ratio at the receiver at the particular carrier. Intuitively, large alphabets that are associated with a large number of bits are transmitted through subchannels with large signal-to-noise ratios.

The set of carriers or subchannels is defined by a set of orthonormal signals $\{\phi_k(n), k = 0, \ldots, M - 1\} \subset l_2$, where $l_2$ is the set of discrete time signals with finite energy. Modulated signals belong to the subspace spanned by the set of carriers, i.e., a modulated signal $x(n)$ is the linear combination

$$x(n) = \sum_{k=0}^{M-1} \alpha_k \phi_k(n). \quad (1)$$

On this equation, $\alpha_k$ is selected from the $k$th alphabet and it represents the message transmitted along the $k$th carrier.

At the receiver, the noise corrupted received signal is projected onto the set $\{\phi_k(n), k = 0, \ldots, M - 1\}$ to recover the information transmitted. Judicious choices of the family $\{\phi_k(n), k = 0, \ldots, M - 1\}$ lead to different communication systems able to accommodate very high bit rates over noisy channels.

A well-known example of a MCM system is discrete multi-tone (DMT) that is based on the discrete Fourier transform (DFT). In this case, the carriers are defined as follows:

$$\phi_k(n) = \frac{1}{\sqrt{M}} e^{j\frac{2\pi}{M} kn}, \quad n = 0, \ldots, M - 1, \quad k = 0, \ldots, M - 1. \quad (2)$$

Notice that the set $\{\phi_k(n), k = 0, \ldots, M - 1\}$ is composed by $M$ complex exponential functions multiplied by an $M$-point rectangular window in the time domain. Although carriers are considered independent, there is some level of ICI. For instance, let $\Phi_k(\omega)$ be the discrete time Fourier transform of $\phi_k(n)$. Then, the first side lobe of the subchannel $\Phi_k(\omega)$ crosses the main lobe of the adjacent subchannel at $-3$ dB, and the peak of the side lobe is at just $-13$ dB, as it can be seen in Fig. 1. In this case, a basis with better subchannel spectral containment is required to improve noise rejection.

3. Filter bank based MCM

A general framework to deal with MCM systems has been proposed in [4]. A simplified version is shown in Fig. 2. Under this scheme, the filter bank $\{F_i(z), i = 0, \ldots, M - 1\}$ is used to transmit the information and the bank $\{H_i(z), i = 0, \ldots, M - 1\}$ is implemented at the receiver. Using the notation introduced in this figure, the block of data transmitted at time $nT$ is $(x_0(n), \ldots, x_{M-1}(n))$, where $x_i(n)$ is the data sent through the $i$th subchannel at time $nT$. The transmission channel is assumed to be linear time invariant and it is represented by a finite impulse response (FIR) filter with transfer function $C(z)$. Also, the signal $r(n)$ represents the additive Gaussian noise that is not necessarily white.

A first analysis is performed for an ideal channel and zero-noise conditions, that is, $C(z) = 1$ and $r(n) = 0$. In this case, the problem of designing the transmitter/receiver pair is equivalent to the problem of designing analysis/synthesis filter banks. The filter bank at the transmitter, $\{F_i(z), i = 0, \ldots, M - 1\}$, receives $M$ sub-streams of data and synthesizes the signal to be sent through the channel. At the other end of the channel, the receiver filter bank, $\{H_i(z), i = 0, \ldots, M - 1\}$, analyzes the signal to recover the information sent through each subchannel. The problem of designing analysis/synthesis filter banks has been fruitfully studied by several researchers in the field (see, for example, [5,8] and references therein). The idea is to use the results in the literature for designing transceivers tolerant to narrow band disturbances.

Define the transmitted vector as $X(z) := [X_0(z), \ldots, X_{M-1}(z)]^T$, where $X_i(z)$ are the $z$-transforms of the sequences $x_i(n)$, $i = 0, \ldots, M - 1$. Similarly, define the received vector as $\hat{X}(z) := [\hat{X}_0(z), \ldots, \hat{X}_{M-1}(z)]^T$. Then, using basic operations in multirate filter banks, we obtain the following relationship:

$$\hat{X}(z) = \frac{1}{M} H(z^{1/M})^{T} F(z^{1/M}) X(z), \quad (3)$$
Fig. 1. Spectral characteristic of a single DMT tone. The peak of the first side lobe is at 13 dB below the peak of the main lobe.

Fig. 2. Description of a MCM system with $M$ carriers. The filter bank $\{F_i(z)\}$ is at the transmitter, and $\{H_i(z)\}$ is the filter bank at the receiver end. The LTI channel is $C(z)$.

where $H(z^{1/M})$ is an $M \times M$ complex matrix whose $(k, l)$-element is defined as $(H(z^{1/M}))_{kl} = H_l(z_e^{-j2\pi k/M})$. The matrix $F(z^{1/M})$ is also an $M \times M$ complex matrix defined accordingly using the filters $F_i(z)$ instead of $H_i(z)$.

3.1. Perfect reconstruction filter banks

A zero-forcing criterion on the communication system would impose that the matrix $H(z^{1/M})'F(z^{1/M})$ be diagonal. It was shown in [9] that this condition is satisfied by the collection of pairs of filters $\{(H_i(z), F_i(z)), i = 0, \ldots, M - 1\}$ satisfying the following property:

$$A_l(z) := \sum_{k=0}^{M-1} H_k(z e^{-j2\pi l/M})F_k(z) = 0, \quad 1 \leq l \leq M - 1,$$

$$T(z) := \frac{1}{M} \sum_{k=0}^{M-1} H_k(z)F_k(z) = c z^{-n_0},$$

where $c$ and $n_0$ are two constant values.

When the filter banks $\{H_i(z), i = 0, \ldots, M - 1\}$ and $\{F_i(z), i = 0, \ldots, M - 1\}$ are used for sub-band coding, Eqs. (4) and (5) are known as the perfect reconstruction (PR) conditions [5]. The terms $A_l(z)$ are known as the aliasing terms and $T(z)$ is the direct gain. Under conditions (4) and (5), the reconstructed signal after the synthesis
filter bank is a scaled and time-shifted version of the original signal at the input of the analysis filter bank. All the machinery developed in the field of filter bank for sub-band coding can be used here for designing MCM systems. In particular, the filter banks that we will obtain are very simple to implement through the polyphase decomposition as shown in [8].

Let \( f_k(n) \) and \( h_k(n) \) be the inverse \( z \)-transforms of \( F_k(z) \) and \( H_k(z) \) respectively. Then, the PR condition is verified when \( f_k(n) \) and \( h_k(n) \) are \( N \)-tap FIR filters such that [5]

\[
f_k(n) = h_k(N-1-n), \quad k = 0, \ldots, M-1, \quad n = 0, \ldots, N-1.
\]

In this case, the resulting \( T(z) \) in (5) is an \( N \)-tap linear filter with linear phase. Moreover, filter banks obtained by modulation of a prototype filter satisfy condition (6) as it is shown in [10]. Notice that the usual DMT filter bank creates the filters \( h_k(n) \) by exponential modulation of a prototype filter \( p(n) \), i.e., \( h_k(n) = p(n)e^{i2\pi kn/M}, \quad k = 0, \ldots, M-1 \).

If the FIR filters \( h_k(n) \) are four times longer than the number of subchannels, i.e., \( N = 4M \), a closed formula for a PR filter bank can be obtained [11]. This filter bank is known as the extended lapped transform (ELT) and it uses the following prototype filter [8]:

\[
p(n) = -\frac{1}{4\sqrt{M}} + \frac{1}{2\sqrt{2M}} \cos \left( \frac{n + \frac{1}{2} \pi}{2M} \right). \tag{7}
\]

Then, the modulated filter defining each subchannel of the ELT system has the following expression:

\[
h_k(n) = p(n) \frac{2}{M} \cos \left( \frac{\pi}{M} \left( k + \frac{1}{2} \right) \left( n + \frac{M + 1}{2} \right) \right). \tag{8}
\]

Notice that \( f_k(n + lM) \) is orthogonal to \( h_k(n) \) when \( l \) is a positive integer number. That is, translations of \( f_k(n) \) by integer numbers of \( M \) are not detected at the receiver by \( h_k(n) \). Then, it is possible to overlap the signals sent through each subchannel without introducing undesired ISI along that subchannel. Therefore, the symbol rate on each subchannel may still be 1 symbol every \( M \) samples, even though the filter lengths are four times longer than \( M \).

We have plotted the spectral characteristic of a single ELT subchannel in Fig. 3. Comparing Figs. 1 and 3, we notice that the ELT filter bank has side lobes that are more attenuated with respect to the main lobe than those of the DMT system.

### 3.2. Near perfect reconstruction filter banks

It is interesting to relax the zero-forcing condition imposed by the PR structure to improve the subchannel spectral containment of the overall modulation scheme. In particular, we obtain nonperfect reconstruction (NPR) conditions for a family of modulated filter banks.

Let \( P(\omega) \) be the Fourier transform of the prototype filter. Using a cosine modulated filter bank, the set of Fourier transforms \( \{H_k(\omega), \quad k = 0, \ldots, M-1\} \) is obtained as the sum of two exponentially modulated versions of the prototype filter [5], i.e.,

\[
H_k(\omega) = \alpha_k P \left( \omega - \frac{(2k+1)\pi}{2M} \right) + \alpha_k^* P \left( \omega - \left( 2\pi - \frac{(2k+1)\pi}{2M} \right) \right). \tag{9}
\]

On the other hand, \( f_k(n) \) is chosen to satisfy (6). It has been proved in [5] that in this case, the direct gain \( T(z) \) has a linear phase and the magnitude of \( T(z) \) only needs to be considered for the present analysis.

It can be shown from (4) that when \( P(\omega) \) is band limited to \( |\omega| < \pi/2M \), the aliasing terms \( A_l(z) \) are zero. Moreover, since \( H_k(\omega) \) is the sum of two disjoint band limited filters, the magnitude of the direct gain results:

\[
|T(\omega)| = \sum_{k=0}^{M-1} |H_k(\omega)|^2 = \sum_{k=0}^{2M-1} |P_k(\omega - k\pi/M)|^2. \tag{10}
\]

Consider \( G(\omega) = |P(\omega)|^2 \). Then,

\[
|T(\omega)| = \sum_{k=0}^{2M-1} |P(\omega - k\pi/M)|^2 = \sum_{k=0}^{2M-1} G(\omega - k\pi/M). \tag{11}
\]
The NPR condition is obtained by selecting $G(\omega)$ as a “near perfect” Nyquist filter, i.e.,

$$|T(\omega)| \simeq 1. \quad (12)$$

Let $g(n)$ be the inverse discrete time Fourier transform of $G(\omega)$, i.e., $g(n) = \mathcal{F}^{-1}(G(\omega))$. Then, using (11), an equivalent condition for (12) is obtained as [12]

$$g(2Mn) \simeq \frac{1}{2M}\delta(n). \quad (13)$$

Suppose that the prototype filter is a low-pass filter whose bandwidth is the parameter $w_c$. Then, a sensible design strategy is to find the solution to the following optimization problem:

$$\min_{w_c} \max_{n, n \neq 0} |g(2Mn)|. \quad (14)$$

Under certain conditions on the family of filters $P(\omega, \omega_c)$, (14) is a convex optimization problem. For instance, if $P(\omega)$ is obtained by multiplying an ideal filter with bandwidth $w_c$ by a Kaiser window, (14) is convex in the parameter $w_c$ [12].

The spectral characteristic of a tone for the Kaiser-based system is shown in Fig. 4. By comparing Figs. 1, 3, and 4 we observe that the NPR system has the best subchannel spectral containment among the three studied here.

4. Equalization for MCM systems

When considering frequency selective channels, PR and NPR conditions alone do not guarantee good performance. For instance, frequency selective channels have gain and phase that depend on the frequency. Then, data blocks transmitted through the channel suffer not only from ICI but also from inter-block interference (IBI).

In DMT systems, IBI is avoided by inserting a guard period at the beginning of the transmitted signal. This guard period consists of a cyclic prefix longer than the duration of the channel impulse response. Notice that in this case,
transmitted signals do not overlap in time. This is not the case for the ELT system or the Kaiser-based system introduced above. In these two cases, successive symbols overlap in time. To avoid IBI, on each subchannel we need to preserve orthogonality among time shifted versions of $f_k(n)$. Orthogonality is not affected by a change in magnitude when the signals go through the channel. Therefore, equalization is reduced to phase treatment. It is clear that the overall magnitude change will introduce a bias that should be removed before the decision step. However, the bias may be removed on each subchannel separately by multiplying the filtered signal $\hat{x}_k$ on each subchannel by a constant computed so that the overall gain on the $k$th subchannel is 1. The process resembles the usual frequency equalizer (FEQ) used in DMT systems after the FFT operation.

Let $c_{eq}(n)$ be the convolution between the impulse responses of the channel $c(n)$ and the equalizer $w(n)$. Perfect linear phase on $c_{eq}(n)$ is obtained when the equalizer is matched to the impulse response of the channel, that is

$$w(n) = c(-n).$$

(15)

In this case, the frequency response of the equalized system, $C_{eq}(\omega)$, has a perfect linear phase. Unfortunately, $w(n)$ is a noncausal system and its length equals the channel length. Then, its implementation would increase the delay of the system.

Observe that we are working with FIR systems, and therefore $c_{eq}(n)$ has linear phase if the impulse response is symmetric with respect to its central point. In particular, we will exploit this property of general FIR systems to design short and causal equalizers for the MCM systems introduced in the previous section.

4.1. Perfect symmetry equalization

A first approach, is to obtain a causal equalizer that imposes even symmetry to the impulse response of the equalized channel within a window of length $2\delta + 1$ samples. Then, the design problem is to obtain the best causal equalizer $w(n)$ that guarantees even symmetry of the equalized channel inside the time window and minimizes the error energy outside this window. The symmetry axis will be at $n_s = \delta$ samples.

Let $N_c$ and $N_w$ be the lengths of the FIR channel and the equalizer respectively. Define the vector $w = [w(0), \ldots, w(N_w - 1)]^T$ and the Toeplitz matrix $C \in \mathbb{R}^{(N_c + N_w - 1) \times (N_w)}$ that characterizes the channel response as follows:
Symmetry equalizer design problem is formulated as the following optimization problem:

\[ \min_{\delta} \quad c(0) \quad 0 \quad \ldots \quad 0 \\
\begin{pmatrix}
c(1) & c(0) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
c(Nw - 1) & \vdots & & c(0) \\
\vdots & \vdots & \ddots & \vdots \\
c(Nc - 1) & \vdots & & 0 \\
0 & c(Nc - 1) & \vdots & \vdots \\
0 & 0 & \ldots & c(Nc - 1)
\end{pmatrix}. \tag{16} \]

Using the previous definitions, the impulse response of the equalized channel may be arranged in a \( Nc + Nw - 1 \) vector \( \mathbf{c}_{eq}(n) = \mathbf{C} \mathbf{w} \).

For convenience, define the vector \( \mathbf{c}_i^{eq} \) as the \( i \)th row of matrix \( \mathbf{C} \). Then, for each fixed \( \delta \), it is possible to define the following matrices:

\[
\mathbf{\Gamma} = \begin{pmatrix}
c_{2\delta+1} \\
\vdots \\
c_{Nc+Nw-2}
\end{pmatrix}, \quad \mathbf{\Psi} = \begin{pmatrix}
c_0 - c_{2\delta} \\
\vdots \\
c_{\delta-1} - c_{\delta+1}
\end{pmatrix}. \tag{17}
\]

Notice that the vector \( \mathbf{\Gamma} \mathbf{w} \) contains the samples of the equalized channel \( \mathbf{c}_{eq}(n) \) for \( n = 2\delta + 1, 2\delta + 2, \ldots, Nc + Nw - 2 \). Similarly, the vector \( \mathbf{\Psi} \mathbf{w} \) contains the differences \( \mathbf{c}_{eq}(n) - \mathbf{c}_{eq}(2\delta - n) \) for \( n = 0, \ldots, \delta - 1 \). Then, the perfect symmetry equalizer design problem is formulated as the following optimization problem:

\[
\min_{\mathbf{w} \in \mathbb{R}^{Nw}} \| \mathbf{\Gamma} \mathbf{w} \|_2^2 \quad \text{such that} \quad \begin{cases}
\mathbf{\Psi} \mathbf{w} = 0, \\
\mathbf{d}' \mathbf{w} = 1.
\end{cases} \tag{18}
\]

The vector \( \mathbf{d} \in \mathbb{R}^{Nw} \) and the condition \( \mathbf{d}' \mathbf{w} = 1 \) have been included into the design problem to avoid the trivial solution \( \mathbf{w} = \mathbf{0} \). Notice that when the channel \( c(n) \) is already symmetric, then \( \mathbf{\Psi} = \mathbf{0} \). When the channel is not symmetric, it is possible to establish necessary conditions for a unique solution to this problem by restricting the set of possible parameters \( \delta \). For instance, to avoid an empty solution space, the dimension of the null space of \( \mathbf{\Psi} \in \mathbb{R}^{\delta \times Nw} \) should be greater than 0, i.e.,

\[
\dim(\text{Ker}(\mathbf{\Psi})) \geq 1. \tag{19}
\]

By construction, the last \( Nw - 2\delta - 1 \) columns of matrix \( \mathbf{\Psi} \) are all zero. Moreover, the \( i \)th row of \( \mathbf{\Psi} \) is of the form \( [\ldots, -c(0), 0, -c_{2\delta-1}] \). Therefore, if \( \delta < Nw \) and \( c(0) \neq 0 \), \( \mathbf{\Psi} \) has \( \delta \) linearly independent rows, which implies that \( \dim(\text{Ker}(\mathbf{\Psi})) = Nw - \delta \).

On the other hand, notice that \( \mathbf{\Gamma} \in \mathbb{R}^{Nc+Nw-2\delta-2 \times Nw} \) and it is a tall matrix if \( Nc + Nw - 2\delta - 2 > Nw \), i.e., \( \delta < (Nc - 2)/2 \). Moreover, if \( c(Nc - 1) \neq 0 \), last \( Nw \) rows of \( \mathbf{\Gamma} \) are linearly independent and \( \text{rank}(\mathbf{\Gamma}) = Nw \). Then, the parameter \( \delta \) should satisfy

\[
\delta < \min(Nw, Nc/2 - 1). \tag{20}
\]

When \( \mathbf{\Psi} \) has full row rank and \( \mathbf{\Gamma} \) has full column rank, we use Lagrange multipliers to solve the design problem. In this case, the optimization of the Lagrangian function results

\[
\min_{\mathbf{w}, \lambda_1, \lambda_2} \mathbf{w}' \mathbf{\Gamma} \mathbf{w} + \lambda_1 \mathbf{\Psi} \mathbf{w} + \lambda_2 \mathbf{d}' \mathbf{w}, \tag{21}
\]

where \( \lambda_1 \in \mathbb{R}^3 \) and \( \lambda_2 \in \mathbb{R} \). To solve this unconstrained minimization problem we compute the gradient and set it to zero. Define the following auxiliary matrix:

\[
\mathbf{M} = (\mathbf{\Gamma}' \mathbf{\Gamma})^{-1} [\mathbf{\Gamma}' \mathbf{\Gamma} - \mathbf{\Psi}' (\mathbf{\Psi} (\mathbf{\Gamma}' \mathbf{\Gamma})^{-1} \mathbf{\Psi}')^{-1} (\mathbf{\Gamma}' \mathbf{\Gamma})^{-1}]. \tag{22}
\]

Then, the optimal value of \( \mathbf{w} \) is

\[
\mathbf{w} = \frac{\mathbf{M} \mathbf{d}}{\mathbf{d}' \mathbf{M} \mathbf{d}}. \tag{23}
\]
Using this solution, we evaluate the energy of the samples of $c_{eq}(n)$ outside the symmetry window
\[ \| \Gamma w \|_2^2 = (d' M d)^{-1}. \] (24)
Notice that $M$ is a positive definite matrix that depends on the channel coefficients only. It is now feasible to select the vector $d$ as the eigenvector associated with the larger eigenvalue of matrix $M$ to minimize the energy outside the symmetry window.

### 4.2. Near perfect symmetry equalization

A complementary approach is to request zero energy outside the symmetry window and to achieve near symmetry within the symmetry window. Using the matrices defined previously in (17), the new design problem results as follows:
\[
\min_{w \in \mathbb{R}^{Nw}} \| \Psi w \|_2^2 \quad \text{such that} \quad \begin{cases} \Gamma w = 0, \\ d' w = 1. \end{cases}
\] (25)

Since the equalized channel is the convolution of two FIR filters of duration $N_c$ and $N_w$, the impulse response $c_{eq}(n)$ has a finite duration equal to $N_c + N_w - 1$ samples. Therefore, considering that the largest symmetry point may not be beyond the mid-point of $c_{eq}(n)$, an upper bound on $\delta$ is determined, $\delta < (N_c + N_w - 1)/2$.

Now, an analysis similar to the one performed before would require that $\Gamma$ be full row rank and $\Psi$ be full column rank. Then, the feasible range for $\delta$ results
\[ \max(N_c/2 - 1, N_w) < \delta < \frac{N_c + N_w - 1}{2}. \] (26)

Notice that in this case, $2\delta + 1 > N_c - 1$. Then, the first $2\delta - N_c + 2$ columns of matrix $\Gamma$ are equal to zero. Hence, by exploiting this particular structure we can solve a problem with a smaller number of parameters and therefore with a lower complexity than the perfect symmetry approach presented above.

### 4.3. Selection of the width of the symmetry window

So far, we have obtained two equalization techniques so that the equivalent channel $c_{eq}(n) = c(n) * w(n)$ has almost linear phase. Let $\vartheta_{eq}(\omega)$ be the phase of $C_{eq}(\omega)$. To quantify the goodness of each design technique, we compute the mean square error between $\vartheta_{eq}(\omega)$ and its best linear approximation. It is possible to obtain a linear approximation to the phase of the equalized channel using a minimum mean square error approach. We define the linear phase approximation as the line that passes through the origin and minimizes the mean square difference between $\vartheta_{eq}(\omega)$ and its approximation. Let $\hat{\vartheta} = a \omega$ be the best linear phase approximation for $\vartheta_{eq}(\omega)$. Then, the mean square error along the set of carrier frequencies $\{\omega_i, i = 0, \ldots, M - 1\}$ is defined as follows:
\[ \epsilon(\delta) = \frac{1}{M} \sum_{i=0}^{M-1} (\vartheta_{eq}(\omega_i; \delta) - a \omega_i)^2. \] (27)

Here, we have emphasized the dependency of $\vartheta_{eq}$ on $\delta$, the length of the symmetry window. Notice that (27) may be used to select the width of the symmetry window. Then, for each fixed value of $\delta$ in the feasible set, the optimization problem in (18) (in (25)) is solved and the performance measure $\epsilon(\delta)$ computed. We select the window length that minimizes $\epsilon(\delta)$.

### 5. Experimental results

Numerical experiments are performed to analyze the modulation and equalization schemes separately. First, each equalization strategy is characterized by its ability to linearize the phase of the equalized channel $c_{eq}(n)$, regardless of the pair of filter banks used in the transmitter and the receiver.

On a second analysis, the equalizer that performed best on the initial test is used to compare different transceivers when the channel is perturbed by narrow band interference.
5.1. Experimental setup

The numerical experiments are performed for a DSL channel used for broadband transmission over telephone cables. The system is modeled as a linear time invariant system that has low-pass characteristics and a long impulse response. In particular, the model used approximates the impulse response described in [13] and it has the following $z$-transform:

$$C(z) = \frac{-0.080 - 0.054z^{-1} + 0.594z^{-2}}{1 - 1.212z^{-1} + 0.259z^{-2}}.$$  \hspace{1cm} (28)

The impulse response of this IIR system is truncated at 100 samples. The frequency response of the channel is shown in Fig. 5.

5.2. Analysis of the equalization strategies

For the particular channel introduced above, the optimization problem in (18) is solved for a 7-tap equalizer. The frequency response of the equalized channel is shown in Fig. 6. A second causal equalizer is obtained using the near perfect symmetry problem in (25). The frequency response of the equalized channel is shown in Fig. 7.

To compare the equalization strategies (18) and (25), we design the equalizers so that they both attain equal phase error, as defined in (27). Under this condition, the perfect symmetry strategy (18) requires a 7-tap equalizer, while the near perfect symmetry strategy (25) needs a 93-tap FIR equalizer. In this case, the perfect symmetry equalizer results in a shorter filter than the near perfect symmetry technique.

5.3. Analysis of the modulation schemes

The goal of this particular analysis is to compare the modulation techniques presented before: traditional DFT-based system, perfect reconstruction system based on the ELT transform, and near perfect reconstruction Kaiser-based system. In particular, these three communication systems will be compared when narrow band interference centered at $\omega_n$ is added to the channel defined in (28). The comparison will be on the probability of error for each communication system when the interference signal is present.
Following the analysis performed in subsection 5.2, a 7-tap near perfect equalizer is used. All three systems are defined with 128 subchannels. For the ELT transform and the Kaiser-based system the length of the basis functions is chosen as four times the number of channels ($N = 512$). For the DMT system, we use a cyclic prefix of length 5, and we design an appropriate time equalizer (TEQ). The signal-to-noise and interference ratio (SNIR) at the input of the equalizer is chosen at 0 dB on all systems.

The probability of error for each modulation scheme is estimated by sending $10^6$ bits through each subchannel. Narrow band noise is modeled as a cosine signal with frequency $\omega_n$. Notice that the performance of the communication
Fig. 8. Probability of error for varying narrow band interference center frequency $\omega_n$. The bottom plot shows the detail for $1.57 \text{ rad s}^{-1} \leq \omega_n \leq 1.76 \text{ rad s}^{-1}$.

Fig. 9. Histogram of the probability of error on each subchannel $P_e(k)$ for a fixed narrow band interference.

The system depends on the location of the center frequency $\omega_n$. To show this property, $P_e$ was computed for different values of $\omega_n$ as it is shown in Fig. 8.

When the narrow band disturbance is at the center of a carrier, the disturbance affects that carrier only. In that case, $P_e$ has its lowest value for all three systems. However, when the interference frequency is shifted away from the center of a carrier, the narrow band disturbance has a spilling effect over neighboring carriers, which increases $P_e$. 
Recall from Fig. 2 that the signal \( y(n) \) is multiplied by a square window at the input of the receiver to form the received block of data. The effective disturbance processed by the filter bank is then a sinc(·) function obtained after filtering the ideal narrow band noise with the square window. When we use DMT, the length of the window equals \( M \), the number of carriers; on the other two cases, the window length is \( N = 4M \). Therefore, the effective disturbance affecting the detection on each subchannel in the DMT system has a broader main lobe than in the other two cases. This additional effect is detrimental for the performance of the DMT system. On the other hand, since the Kaiser-based system achieves the best subchannel spectral containment, the probability of error is more independent from \( \omega_n \) than it is in the case of the ELT transform system.

Finally, for the case of a fixed interference concentrated on a single tone that does not coincide with the center frequency of any carrier, Fig. 9 shows the histograms of the probabilities of error on each subchannel, i.e., \( P_e(k), k = 1, \ldots, 128 \). We observe that DMT has a large number of subchannels with high probability of error, while the other two systems have most of their subchannels concentrated on the low-probability bin of the histograms. Since DMT has a poor subchannel spectral containment, spilling of the disturbance is more evident. On the other hand, it is the NPR system that has the largest number of subchannels in the low-probability bin resulting in the system with the smallest overall \( P_e \) and best performance.

6. Conclusions

In this paper, two multicarrier modulation systems are presented, the first one based on a perfect reconstruction set of filter banks and the second one based on a set of filter banks satisfying a near perfect reconstruction condition.

To transmit over a real channel, it is necessary to introduce an equalizer at the receiver input. A new equalizer design criteria is proposed by using a causal FIR filter to linearize approximately the phase of the equalized channel. To account for the inevitable constellation scaling at the output of the equalizer, we implement a scalar multiplication at the output of each subchannel so that the overall gain on each subchannel remains unity.

The key feature of the PR and NPR systems is that they have better subchannel spectral containment than traditional DMT scheme. Moreover, since both PR and NPR systems implement filter banks that use filter lengths four times longer than the number of carriers, the actual disturbance that reaches the detection system is narrower than the one observed in the DMT system. As a consequence, it is possible to achieve better narrow band noise immunity than with the usual DFT-based system. Finally, the NPR filter bank was proposed as a way to improve further on the spectral characteristics of the MCM structure.

References

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