Strong Equivalence of Non-Monotonic Temporal Theories

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Abstract
In this paper we solve the following open problem: we prove that equivalence in the logic of Temporal Here-and-There (THT) is not only a sufficient, but also a necessary condition for strong equivalence of two Temporal Equilibrium Logic (TEL) theories. This result has allowed constructing a tool, ABSTEM, that can be used to check different types of equivalence between two arbitrary temporal theories. More importantly, the tool provides a context theory that makes them behave differently, together with a Büchi automaton showing the temporal stable models that arise from that difference.

Introduction
With the consolidation of Answer Set Programming (ASP) (Brewka, Eiter, and Truszczynski 2011) as a successful paradigm for practical knowledge representation, many examples and benchmarks formalising dynamic scenarios became available. ASP usually treats time as an integer index restricting all reasoning tasks to finite narratives. As a piece of example, consider an extremely simple ASP program where a fluent \( p \) represents that a switch is on and \( q \) represents that it is off. Moreover, suppose we have freedom to add \( p \) arbitrarily at any moment and that either \( p \) or \( q \) holds initially. A typical ASP representation could be:

\[
\begin{align*}
 p(0) & \lor q(0) & (1) \\
 p(I+1) & \leftarrow p(I), \lnot q(I+1), \text{sit}(I) & (2) \\
 q(I+1) & \leftarrow q(I), \lnot p(I+1), \text{sit}(I) & (3) \\
 p(I) & \lor \lnot p(I) & (4)
\end{align*}
\]

where (1) is the initial state, (2) and (3) are the inertia rules and (4) is a choice rule for \( q \). This strat- would have a finite domain \( \{0, \ldots, n\} \) for some constant \( n \geq 0 \). Planning for a goal like \( p \land \lnot q \) implies including two constraints for the last situation. \( \bot \leftarrow \lnot p(n) \) and \( \bot \leftarrow q(n) \), and go increasing \( n \) until a solution is found. This strategy is not always the best choice for many temporal reasoning problems that involve dealing with infinite time such as proving the non-existence of a plan or checking the satisfaction of temporal properties of a given dynamic system. For instance, questions such as “is there a reachable state in which both \( p \) and \( q \) are false?” or “can we show that whenever \( p \) is true it will remain so forever?” can be answered by an analytical inspection of our simple program, but cannot be solved in an automated way.

To overcome these limitations, (Aguado et al. 2013) proposed a temporal extension of Equilibrium Logic (Pearce 1996), the best-known logical formalisation of ASP. This extension, which received the name of Temporal Equilibrium Logic (TEL), is defined as follows. First, it extends the monotonic basis of Equilibrium Logic, the intermediate logic of Here-and-There (HT) (Heyting 1930), by introducing the full syntax of the well-known Linear-time Temporal Logic (LTL) (Pnueli 1977). The result of this combination is called Temporal Here-and-There (THT). Then, a selection criterion on THT models is imposed, obtaining non-monotonicity in this way. As a result, TEL constitutes a full non-monotonic temporal logic that allows a proper definition of temporal stable models for any arbitrary theory in the syntax of LTL. For instance, the ASP program (1)-(4) would be represented in TEL as:

\[
\begin{align*}
 p \lor q & \quad (5) \\
 \Box (p \land \lnot q) & \rightarrow \Diamond p & (6) \\
 q & \rightarrow \Box q & (7) \\
 \Box (p \lor \lnot p) & & (8)
\end{align*}
\]

with ‘\( \Diamond \)’ meaning “eventually.” Similarly, to test whether \( p \) remains true after becoming true we would add:

\[
\Box (p \rightarrow \Box p) \rightarrow \bot & \quad (9)
\]

and check that, indeed, no temporal stable model exists.

In the past years, several interesting results about TEL were obtained – see survey (Aguado et al. 2013) – but a few important questions about TEL remained unsolved. One of them has to do with the property of strong equivalence in TEL. In any Non-Monotonic Reasoning approach, we say that \( \Gamma_1 \) and \( \Gamma_2 \) are strongly equivalent when, for any arbitrary theory \( \Gamma \), both \( \Gamma_1 \cup \Gamma \) and \( \Gamma_2 \cup \Gamma \) have the same
selected operators such as $\alpha$ is read as "always" and is the dual of "until." Derived operators such as $\square$ ("always") and $\diamond$ ("at some future time") are defined as $\square \varphi \triangleq \bot \varphi$ and $\diamond \varphi \triangleq \top \cup \varphi$.

A temporal formula $\varphi$ can be expressed following the grammar shown below:

$$\varphi ::= \perp | p | \alpha \land \beta | \alpha \lor \beta | \alpha \rightarrow \beta | \bigcirc \alpha | \alpha U \beta | \alpha R \beta$$

where $p$ is an atom of some finite signature $At$, and $\alpha$ and $\beta$ are temporal formulas in their turn.

The formula $\alpha U \beta$ stands for "$\alpha$ until $\beta$" whereas $\alpha R \beta$ is read as "$\alpha$ release $\beta$" and is the dual of "until." Derived operators such as $\square$ ("always") and $\diamond$ ("at some future time") are defined as $\square \varphi \triangleq \bot \varphi$ and $\diamond \varphi \triangleq \top \cup \varphi$.

Other usual propositional operators are defined as follows: $\neg \varphi \triangleq \varphi \rightarrow \bot$, $\top \varphi \triangleq \top \land \varphi$ and $\varphi \leftrightarrow \psi \triangleq (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$.

Given a finite propositional signature $At$, an LTL-interpretation $T$ is an infinite sequence of sets of atoms, $T_0, T_1, \ldots$ with $T_i \subseteq At$ for all $i \geq 0$. Given two LTL-interpretations $H, T$ we define $H \leq T$ as: $H_i \subseteq T_i$ for all $i \geq 0$. A THT-interpretation $M$ for $At$ is a pair of LTL-interpretations $(H, T)$ satisfying $H \leq T$. A THT-interpretation is said to be total when $H = T$.

**Definition 1 (Satisfaction)** We define when an interpretation $M = (H, T)$ satisfies a formula $\varphi$ at a state $i \geq 0$, written $M, i \models \varphi$, recursively as follows:

1. $M, i \models p$ iff $p \in H_i$, with $p$ an atom.
2. $\land, \lor, \bot$ as usual.
3. $M, i \models \varphi \rightarrow \psi$ iff for all $x \in \{H, T\}$, $(x, T), i \not\models \varphi$ or $(x, T), i \models \psi$.
4. $M, i \models \bigcirc \varphi$ iff $M, i+1 \models \varphi$.
5. $M, i \models \varphi U \psi$ iff $\exists k \geq i$ such that $M, k \models \psi$ and $\forall j \in \{i, \ldots, k-1\}, M, j \not\models \varphi$.
6. $M, i \models \varphi R \psi$ iff $\forall k \geq i$ such that $M, k \not\models \psi$ then $\exists j \in \{i, \ldots, k-1\}, M, j \models \varphi$.

We say that $(H, T)$ is a model of a theory $\Gamma$, written $(H, T) \models \Gamma$, iff $(H, T), 0 \models \alpha$ for all formulas $\alpha \in \Gamma$.

It is easy to see that restricting the study to total interpretations, THT-satisfaction collapses to LTL-satisfaction, i.e.:

**Proposition 1 (from (Aguado et al. 2013))** $(T, T), i \models \varphi$ in THT iff $T, i \models \varphi$ in LTL.

An interpretation $M$ is a temporal equilibrium model of a theory $\Gamma$ if it is a total model of $\Gamma$, that is, $M = (T, T), i \models \Gamma$, and there is no $H < T$ such that $(H, T) \models \Gamma$. An LTL-interpretation $T$ is a temporal stable model (TS-model) of a theory $\Gamma$ iff $(T, T)$ is a temporal equilibrium model of $\Gamma$. By Proposition 1 it is easy to see that any TS-model of a temporal theory $\Gamma$ is also an LTL-model of $\Gamma$.

As happens in LTL, the set of TS-models of a theory $\Gamma$ can be captured by a Büchi automaton (Büchi 1962), a kind of $\omega$-automaton (that is, a finite automaton that accepts words of infinite length). In this case, the alphabet of the automaton would be the set of states (classical propositional interpretations) and the acceptance condition is that a word (a sequence of states) is accepted iff it corresponds to a run of the automaton that visits some acceptance state an infinite number of times. As an example, Figure 1 shows the TS-models for the theory (5)-(8) which coincide with sequences of states of the forms $\{q\}^* \{p\}^\omega$ or $\{q\}^\omega$.

**Computing Temporal Stable Models of Arbitrary Theories**

Based on the techniques introduced in (Cabalar and Demri 2011), the tool ABSTEM constitutes the first implementation capable of computing TS-models for arbitrary temporal formulas, without syntactic restrictions. The method obtains the TS-models of a formula $\varphi$ by performing several operations on a pair of automata derived from $\varphi$. The first automaton, denoted as $A_\varphi$, accepts the total THT-models...
(T, T) of \( \varphi \). By Proposition 1 this simply amounts to compute the LTL models T of \( \varphi \) using an automata construction method for LTL. The second automaton, denoted as \( A_{\varphi''} \), accepts the non-total THT-models \( \langle H, T \rangle \) of \( \varphi \). The final set of TS-models is obtained from the composition \( A_{\varphi} \cap h(A_{\varphi''}) \) where \( h(A_{\varphi''}) \) filters out the H component of non-total models, \( h(A_{\varphi''}) \) is the complementary of \( h(A_{\varphi''}) \) and finally \( \cap \) denotes the automata product.

The computation of \( A_{\varphi''} \) is done exploiting a translation of THT into LTL first presented in (Aguado et al. 2008) and directly extrapolating the translation of HT into classical logic in (Pierce, Tompits, and Woltran 2001). It uses an extended signature \{p' | p \in At\} so that \( p' \) represents the truth of \( p \) in H while \( p \) is used for T. The translation of \( \varphi \), written \( \varphi^* \), is recursively defined as follows:

- \( (p^*) \equiv p' \), for any atom \( p \in At \)
- \( (\alpha \rightarrow \beta)^* \equiv (\alpha \rightarrow \beta) \land (\alpha^* \rightarrow \beta^*) \)

being homomorphic for the rest of logical connectives. To impose the restriction \( H < T \) we further include the axiom:

\[
\left( \bigwedge_{p \in At} \Box(p' \rightarrow p) \right) \land \left( \bigvee_{p \in At} \Diamond(\neg p' \land p) \right) \quad (Ax1)
\]

Automaton \( A_{\varphi''} \) is built from formula \( \varphi'' \equiv \varphi^* \land (Ax1) \). The operation \( h(A_{\varphi''}) \) returns a new automaton that results from removing the atoms \( p' \) from transitions in \( A_{\varphi''} \). This captures the T-components of non-total models; in this way, its complementary automaton \( \bar{h}(A_{\varphi''}) \) accepts the T sequences that do not form a non-total model, but perhaps they are not models either. Thus, the final product \( A_{\varphi} \cap \bar{h}(A_{\varphi''}) \) captures those T such that \( (T, T) \) is a total model of \( \varphi \) and no non-total model \( (H, T) \) can be formed.

**Temporal Strong Equivalence**

For simplicity, we assume finite theories and we indistinctly represent them as the conjunction of their formulas.

**Theorem 1 (Sufficient condition (Aguado et al. 2008))**

If two temporal formulas \( \alpha \) and \( \beta \) are THT-equivalent then they are strongly equivalent in TEL.

To prove the other direction, namely, that THT-equivalence is also a necessary condition for strong equivalence, we begin defining \( \gamma_0 \) as the conjunction of all formulas \( \Box(p \lor \neg p) \) for all atoms \( p \in At \).

**Proposition 2** Let \( \langle H, T \rangle \) be a THT interpretation for signature \( At \). If \( \langle H, T \rangle \models \gamma_0 \) then \( H = T \).

**Corollary 1** For any formula \( \alpha \) for signature \( At \), the LTL-models of \( \alpha \land \gamma_0 \) coincide with its TS-models.

**Lemma 1** Let \( \alpha \) and \( \beta \) be two LTL-equivalent formulas and let \( \gamma = (\beta \rightarrow \gamma_0) \). Then, the following conditions are equivalent:

(i) There exists some \( H < T \) such that \( \langle H, T \rangle \not\models \alpha \rightarrow \beta \);

(ii) \( T \) is TS-model of \( \beta \land \gamma \) but not TS-model of \( \alpha \land \gamma \).

**Theorem 2 (Main theorem: necessary condition)** If two temporal formulas \( \alpha \) and \( \beta \) are strongly equivalent in TEL then they are THT-equivalent.

Proof We will prove that if \( \alpha \) and \( \beta \) are not THT-equivalent then there is some context formula \( \gamma \) for which \( \alpha \land \gamma \) and \( \beta \land \gamma \) have different TS-models. Assume first that \( \alpha \) and \( \beta \) have different total models, i.e., different LTL-models. Then, the LTL-models of \( \alpha \land \gamma_0 \) and \( \beta \land \gamma_0 \) also differ (since \( \gamma_0 \) is an LTL tautology). But by Corollary 1, LTL-models of these theories are exactly their TS-models, and so, they also differ.

Suppose now that \( \alpha \) and \( \beta \) are LTL-equivalent but not THT-equivalent. Then, there is some THT-countermodel \( \langle H, T \rangle \) of either \( (\alpha \rightarrow \beta) \) or \( (\beta \rightarrow \alpha) \), and given LTL-equivalence of \( \alpha \) and \( \beta \), the countermodel is non-total, \( H < T \). Without loss of generality, assume \( \langle H, T \rangle \not\models \alpha \rightarrow \beta \). By Lemma 1, taking the formula \( \gamma = (\beta \rightarrow \gamma_0) \), we get that \( T \) is TS-model of \( \beta \land \gamma \) but not TS-model of \( \alpha \land \gamma \).

**Algorithm 1 StrongEquivalenceTest(\( \alpha, \beta \))**

**Require:** Two propositional temporal formulas \( \alpha, \beta \).

**Ensure:** If \( \alpha \) and \( \beta \) are THT-equivalent, it returns true. Otherwise, it returns a triple \( \langle \gamma_0, A_1, A_2 \rangle \).

\[
A_1 := \text{ltl} \rightarrow \text{Buchi}(\alpha \land \neg \beta) \\
A_2 := \text{ltl} \rightarrow \text{Buchi}(\beta \land \neg \alpha) \\
\text{if } A_1 \neq \emptyset \text{ or } A_2 \neq \emptyset \text{ then} \\
\quad \text{return } \langle \gamma_0, A_1, A_2 \rangle \\
\text{end if} \\
A := \text{ltl} \rightarrow \text{Buchi}(\neg (\alpha \rightarrow \beta) \land (Ax1)) \\
\text{if } A \neq \emptyset \text{ then} \\
\quad A_2 := h(A) \\
\quad \text{return } ((\beta \rightarrow \gamma_0), \emptyset, A_2) \\
\text{end if} \\
A := \text{ltl} \rightarrow \text{Buchi}(\neg (\beta \rightarrow \alpha) \land (Ax1)) \\
\text{if } A \neq \emptyset \text{ then} \\
\quad A_1 := h(A) \\
\quad \text{return } ((\alpha \rightarrow \gamma_0), A_1, \emptyset) \\
\text{end if} \\
\text{return true}
\]

**Implementation and a practical example**

The procedure for checking strong equivalence in ABSTEM is shown in Algorithm 1. It takes two arbitrary propositional temporal formulas \( \alpha \) and \( \beta \) and returns either true, if they are strongly equivalent, or a triple with a formula \( \gamma \) and two
automata $A_1, A_2$ otherwise. The meaning of this information is that $A_1$ captures TS-models of $\alpha \land \gamma$ that are not TS-models of $\beta \land \gamma$, and analogously, $A_2$ captures TS-models of $\beta \land \gamma$ that are not TS-models of $\alpha \land \gamma$. The procedure uses an auxiliary routine ltl_to_Buechi($\varphi$) to obtain a Büchi automaton from an LTL-formula $\varphi$.

As an example of use, let $\beta_1$ be our “switch” domain (5)-(8) plus the rule $\Box (\neg p \rightarrow q)$ trying to capture the idea that, when no information on $p$ is available, $q$ becomes true. This new rule is actually a new default for $q$ that interacts with inertia rules (6),(7) destroying somehow their effect. Using ABSTEM to check the TS-models of $\beta_1$ we obtain the automaton in Figure 2(a) which corresponds to arbitrary sequences formed with states $\{p\}$ and $\{q\}$. This set of TS-models actually coincides with what one would expect from a formula of the form $\Box (p \lor q)$ since, as happens in ASP, truth minimality converts the disjunction into an exclusive or. So, we may conjecture that $\beta_1$ is equivalent to $\alpha_1 = \Box (p \lor q)$. Using ABSTEM we can check that, in fact, $\alpha_1$ and $\beta_1$ have the same TS-models and, furthermore, they are also LTL-equivalent. However, $\alpha_1$ and $\beta_1$ are not THT-equivalent and so, they are not strongly equivalent. The answer displayed this time by ABSTEM is negative and shows the context formula $\gamma_1 = \beta_1 \rightarrow \gamma_0$ plus a file containing the automaton in Figure 2(b). This automaton captures all the TS-models of $\beta_1 \land \gamma_1$ that are not TS-models of $\alpha_1 \land \gamma_1$.

Figure 2: Temporal stable models related to $\alpha_1$ and $\beta_1$.

Conclusions

In this paper we have adapted (Lifschitz, Pearce, and Valverde 2007) to the temporal case, to prove that equivalence in the logic of Temporal Here-and-There (THT) is not only a sufficient but also a necessary condition for strong equivalence in Temporal Equilibrium Logic (TEL). Using this proof, we have implemented a system, ABSTEM, for analysing TEL arbitrary theories in different ways. First, ABSTEM constitutes the first tool for computing temporal equilibrium models of any arbitrary temporal theory. Second, it also allows checking three types of equivalence between two arbitrary theories: LTL, weak and strong equivalence. When equivalence is not satisfied, ABSTEM shows counterexamples in the form of a Büchi automaton. Regarding efficiency and scalability, this prototype works satisfactorily for small theories like the ones presented in the paper. It must be noted, however, that checking THT-equivalence is a PSPACE-complete problem.

References


Chen, Y.; Lin, F.; and Li, L. 2005. SELP - a system for studying strong equivalence between logic programs. In LPNMR’05, volume 3662 of Lecture Notes in Computer Science, 442–446.


\footnote{For further details, see the extended version of this document at http://kr.irlab.org/sites/default/files/papers/kr2014-ex.pdf.}