SYNCHRONIZATION AND BIFURCATION IN LIMIT CYCLE OSCILLATORS WITH DELAYED COUPLINGS

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In this paper, a system of three globally coupled limit cycle oscillators with a linear time-delayed coupling are investigated. Considering the delay as a parameter, we also study the effect of time delay on the dynamics. Next, Hopf bifurcations induced by time delays using the normal form theory and center manifold reduction are obtained. Based on the symmetric Hopf bifurcation theorem, we investigate stable phase-locking and unstable waves. Then later, the directions of Hopf bifurcations are determined in some region, where stability switches may occur. The results show that the bifurcating periodic solutions are orbitally asymptotically stable. Numerical simulations are applied to verify the theoretical predictions.

Keywords: Coupled oscillators; synchronization; delay; bifurcation; stability; phase-locking.

1. Introduction

The rich dynamics of a system of coupled nonlinear oscillators has made great progress, it arises from the interaction of simple units, and provides a useful paradigm for the study of collective behavior of large complex systems — full of interesting mathematical challenges and novel applications: physics, chemistry, biology, economics and so on. Their applications include a wide variety of physical and biological problems such as interactions of arrays of Josephson junctions [Hadley et al., 1988; Wiesenfeld et al., 1996], semiconductor lasers [Varangis et al., 1997; Hohl et al., 1997], charge density waves, phase-locking of relativistic magnetrons [Benford et al., 1989], Belousov–Zhabotinski reactions in coupled Brusselator models [Dolnik & Epstein, 1996]. One of the most common occurrences of cooperative phenomena, which was highlighted by Winfree [1980, 1987] in a simple model of weakly coupled limit cycle oscillators, is that of frequency entrainment or synchronization of the diverse frequencies of the oscillator assembly to a single common frequency [Kuramoto & Nishikawa, 1987; Daido, 1990].
Time delay is ubiquitous in most physical and biological systems like optical bistable devices, electromechanical systems, predator-prey models, physiological systems, and so on. The presence of delay makes the system difficult to predict the change occurred, it is necessary to do a further study for the delay system. More and more theories and experiments show the interesting new directions of research in this field and suggest exciting future areas for exploration and applications [Sen et al., 2005]. Time-delay effects on the collective states of the two-coupled oscillator model can be studied by investigating the following set of equations

$$\begin{cases} 
    z_1(t) = (1 + i\omega_1 - |z_1(t)|^2)z_1(t) \\
    + k|z_2(t - \tau_1) - z_1(t)| \\
    z_2(t) = (1 + i\omega_2 - |z_2(t)|^2)z_2(t) \\
    + k|z_1(t - \tau_1) - z_2(t)|,
\end{cases}$$

where $k, \tau_1, \omega_1, \omega_2$ are positive constants and $z_1, z_2$ are complex numbers. The time-delay parameter has been introduced in the argument of the coupling oscillator to physically account for the fact that its phase and amplitude information is received by the other oscillator only after a finite time $\tau$.

When the number of oscillators $N$ is greater than 2, they can be mutually coupled in a variety of ways. The most common models are those that adopt a global coupling (where each oscillator is coupled to every other one in the system) or some form of a local coupling, e.g. nearest-neighbor coupling. The effect of time delay on both these models have been carried out by Ramana Reddy et al. [1999] and Dodla et al. [2004b] respectively. For example, a system of three globally coupled limit cycle oscillators with a linear time-delayed coupling can be described by the following set of model equations:

$$\begin{cases} 
    \dot{z}_1(t) = (1 + i\omega_1 - |z_1(t)|^2)z_1(t) \\
    + k|z_2(t - \tau_1) - z_1(t)| \\
    + k|z_3(t - \tau_1) - z_1(t)| \\
    \dot{z}_2(t) = (1 + i\omega_2 - |z_2(t)|^2)z_2(t) \\
    + k|z_1(t - \tau_1) - z_2(t)| \\
    + k|z_3(t - \tau_1) - z_2(t)|, \\
    \dot{z}_3(t) = (1 + i\omega_3 - |z_3(t)|^2)z_3(t) \\
    + k|z_1(t - \tau_1) - z_3(t)| \\
    + k|z_2(t - \tau_1) - z_3(t)|,
\end{cases}$$

where $k > 0$ is the coupled strength, $\tau_{1,2,3}$ is a discrete and constant delay time, $\omega_{1,2,3}$ is the intrinsic frequency of the oscillator, $z_{1,2,3}$ are complex numbers. The architecture of the coupled system (2) is illustrated in Fig. 1. In this paper, we consider $\tau_{1,2,3} = \tau$ and $\omega_{1,2,3} = \omega$, that is, discuss the case of identical oscillators.

Synchronization is one of the most widely studied phenomena in coupled oscillator systems. As demonstrated here with simple models, significant modifications in the characteristics of synchronized states can take place in the presence of time-delayed coupling. This sensitivity to time delay can in fact be exploited to achieve a desired synchronized state often in a very scientific manner. For example, recent studies (for example, see [Dhamala et al., 2004a; Peng & Guo, 2010]) on a network of neuronal oscillators with time-delayed coupling reveal an enhancement of neural synchrony by time delay. In other words, if coupling is time-delayed, stable synchronized states can be achieved with lower coupling strengths. Such enhanced neural synchrony by delay may have important implications, e.g. in understanding synchronization of distant neurons and information processing in the brain. The synchronization and phase-locking in coupled oscillators system with delay were studied by Popovych et al. [2006, 2007].

The remaining part of this paper is organized as follows. In the following section, we have devoted to the discussion the associated characteristic equation and obtain criteria ensuring the linear stability of the trivial solution. In Secs. 3 and 4, we establish the existence and stability of Hopf bifurcation synchronous and phase-locked periodic solution. And some numerical simulations are performed to illustrate the analysis results in Sec. 5. Our results are
summarized and some remarks are included in the concluding Sec. 6.

2. Stability of Equilibria

To explore the possible (spatial) symmetry of the system (2), we need to introduce three compact Lie groups. One is the cycle group $S^1$, another is $Z_3$, the cyclic group of order 3 (the order of a finite group is the number of elements it contains), the third is the dihedral group $D_3$ of order 6, which is generated by $Z_3$ together with the flip $\kappa$ of order 2. Clearly, we have the following lemma.

Lemma 1. Denoted by $\rho$ the generator of the cyclic subgroup $Z_3$ and $\kappa$ the flip, define the action of $D_3$ on $\mathbb{R}^3$ by

$$
\rho z_j = z_{j+1}, \quad \forall j \ (\bmod 3),
$$

$$
\rho z_1, (\kappa z_2 = z_3, (\kappa z_3) = z_2, \quad \forall \ z \in \mathbb{R}^3.
$$

Then it follows from [Guo & Huang, 2003] that system (2) is $D_3$-equivariant.

Next, we present the linear stability analysis at the trivial solution of system (2). Without loss of generality, we consider that in the Banach space $C([-\tau, 0], \mathbb{R}^3)$, if $\sigma \in \mathbb{R}, A \geq 0$ and $u : [\sigma - \tau, \sigma + A] \rightarrow \mathbb{R}^3$ is a continuous mapping, then $u(t) = u(t + \theta)$ for $-\tau \leq \theta < 0$. Clearly, the origin $z_0 = 0$ for $j = 1, 2, 3$ is always an equilibrium of system (2). To investigate the effect of time delays on the stability of the origin, we need to study the corresponding characteristic equation. Linearizing system (2) at the origin leads to the following linear system,

$$
\begin{align*}
\dot{z}_1(t) &= (1 - 2k + i\omega)z_1(t) + kz_2(t - \tau) + kz_3(t - \tau) \\
\dot{z}_2(t) &= (1 - 2k + i\omega)z_2(t) + kz_1(t - \tau) + kz_3(t - \tau) \\
\dot{z}_3(t) &= (1 - 2k + i\omega)z_3(t) + kz_2(t - \tau) + kz_1(t - \tau).
\end{align*}
$$

(3)

Note that $z_{1,2,3}$ are complex numbers. Thus, the characteristic matrix resulting from (3) is

$$
\Delta(\tau, \lambda) = \begin{pmatrix}
\lambda - 1 + 2k - i\omega & 1 & 0 \\
1 & \lambda - 1 + 2k & 0 \\
0 & 0 & \lambda - 1 + 2k + i\omega
\end{pmatrix} - B e^{-\lambda \tau},
$$

(4)

where $\text{Id}$ is the identity matrix of size 3, and

$$
B = \begin{bmatrix}
0 & k & k \\
k & 0 & k \\
k & k & 0
\end{bmatrix}.
$$

Thus, the characteristic equation is $\det \Delta(\tau, \lambda) = 0$, that is,

$$
\det \Delta(\tau, \lambda) = (\lambda - 1 + 2k)^2 e^{-2k\tau} - 3(\lambda - 1 + 2k) e^{-k\tau} - 2 k e^{-3k\tau} = 0.
$$

(5)

Obviously, $\det \Delta(\tau, \lambda)$ can be decomposed as

$$
\det \Delta(\tau, \lambda) = [(\lambda - 1 + 2k + i\omega) - 2k e^{-\lambda \tau}] \\
\times [(\lambda - 1 + 2k + i\omega) + 2k e^{-\lambda \tau}]^2.
$$

(6)

The sign $(\pm)$ in (5) arises from considering the complex conjugate of (3). This consideration ensures that we have the complete set of eigenvalues. It is well known that the zero equilibrium of (2) is asymptotically stable if and only if all the roots of (5) have negative real parts. Thus, it is necessary to study the distribution of roots of the transcendental equation (5).

It is easy to verify that $\lambda$ is a root of (5) if and only if it is a root of one of the following equations:

$$
\lambda - 1 + 2k - i\omega = 2k e^{-\lambda \tau},
$$

(6)

$$
\lambda - 1 + 2k - i\omega = -ke^{-\lambda \tau},
$$

(7)

$$
\lambda - 1 + 2k + i\omega = 2k e^{-\lambda \tau},
$$

(8)

$$
\lambda - 1 + 2k + i\omega = -ke^{-\lambda \tau}.
$$

(9)

It is not difficult to verify that $\alpha + i\beta$ is a root of (6) if and only if $\alpha - i\beta$ is a root of (8), and $\alpha + i\beta$ is a root of (7) if and only if $\alpha - i\beta$ is a root of (9). Therefore, it is sufficient to investigate only (6) and (7). We next want to study the movement of the roots in the parametric plane of $(k, \omega)$ and find the conditions which determine that all roots of (6)-(9) satisfy $\text{Re}(\lambda) < 0$.

Firstly, we consider (6). Clearly, $\lambda = 0$ is not a root of (6). Let $\beta (\beta \neq 0)$ be a root of (6), then $\beta$ satisfies

$$
\begin{align*}
2k - 1 &= 2k \cos(\beta \tau), \\
\beta - \omega &= -2k \sin(\beta \tau).
\end{align*}
$$

(10)

It follows that

$$
(\beta - \omega)^2 = 4k - 1.
$$

(11)

Clearly, (11) has no real root if $0 < k < 1/4$ and has two real roots $\beta_0$ if $k \geq 1/4$, where

$$
\beta_0 = \omega \pm \sqrt{4k - 1}.
$$

(12)

Consequently, by (10), combine (6) and (8) as follows:

$$
\lambda - 1 + 2k \pm i\omega = 2k e^{-\lambda \tau},
$$

(13)
Let $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$ be the root of (13) near $\hat{\tau}_j^+$ satisfying

$$\alpha(\hat{\tau}_j^+) = 0, \quad \beta(\hat{\tau}_j^+) = \beta_\pm.$$ 

By a direct computation, we can obtain that if $k > 1/4$, then

$$\alpha'(\hat{\tau}_j^+) > 0,$$ 

and

$$\alpha'(\hat{\tau}_j^+) = \begin{cases} < 0, & \text{for } \frac{1}{4} < k < \frac{1 + \omega^2}{4} \vspace{10pt} \\ > 0, & \text{for } k > \frac{1 + \omega^2}{4} \end{cases}$$ 

for $j = 0, 1, 2, \ldots$. \quad (15)

In addition, it is easy to see that (13) has a pair of roots with positive real parts when $\tau = 0$. Consequently, we can show the lemma below.

**Lemma 2** [Song et al., 2007]. Assume that $\beta_\pm$ are defined by (12) and $\hat{\tau}_j^+$ are shown as following

$$\hat{\tau}_j^+ = \begin{cases} 2\pi - \arccos \left( \frac{2k - 1}{2k} \right) + 2j\pi, & \text{for } 2k > 1, \\
\pi + \arccos \left( \frac{1 - 2k}{2k} \right) + 2j\pi, & \text{for } 2k \leq 1, \\
\beta_+ & \text{if } 1 < k \leq 1 + \omega^2/4, \\
\beta_- & \text{if } 0 < k < 1/4 \end{cases}$$ \quad (16)

$$\hat{\tau}_j^- = \begin{cases} \arccos \left( \frac{2k - 1}{2k} \right) + 2j\pi, & \text{for } 2k > 1, \\
\pi - \arccos \left( \frac{1 - 2k}{2k} \right) + 2j\pi, & \text{for } 2k \leq 1, \\
\beta_- & \text{if } 1 < k \leq 1 + \omega^2/4, \\
\beta_+ & \text{if } 0 < k < 1/4 \end{cases}$$ \quad (17)

(1) If $0 < k < 1/4$, then (13) has two roots with positive real parts for all $\tau \geq 0$.

(2) If $1/4 < k < (1 + \omega^2)/4$, and $\hat{\tau}_j^- < \hat{\tau}_0^+$, then (13) has at least two roots with positive real parts for all $\tau \geq 0$.

(3) If $1/4 < k < (1 + \omega^2)/4$, and $\hat{\tau}_j^+ > \hat{\tau}_0^+$, then there exists an integer $n$ such that $0 < \hat{\tau}_0^+ < \hat{\tau}_1^+ < \cdots < \hat{\tau}_n^+ < \hat{\tau}_{n+1}^+$. In this case, all roots of (13) have negative real parts when $\tau \in \bigcup_{j=0}^n (\hat{\tau}_j^+, \hat{\tau}_j^-)$. (13) has a pair of roots with positive real parts when $\tau \in \bigcup_{j=0}^n (\hat{\tau}_j^+, \hat{\tau}_j^-)$, where $\hat{\tau}_0^+ = 0$, and (13) has at least two roots with positive real parts when $\tau > \hat{\tau}_n^+$. In addition, when $\tau = \hat{\tau}_j^+$ and $\tau = \hat{\tau}_j^-$ (j = 0, 1, ..., n), all roots of (13) have negative real parts except the purely imaginary roots $\pm i\beta_\pm$, and $\pm i\beta_\pm$, respectively.

(4) If $k > (1 + \omega^2)/4$, then (13) has at least two roots with positive real parts for all $\tau \geq 0$.

Lemma 2, together with (14) and (15), allow us to state the following results on the stability of the zero equilibrium of the coupled system (2) and Hopf bifurcations.

**Theorem 1.** Assume that $\beta_\pm$ are defined by (12) and $\hat{\tau}_j^\pm$ are defined by (16) and (17), respectively.

(1) If $0 < k < 1/4$, then the zero equilibrium of system (2) is unstable for all $\tau \geq 0$.

(2) If $1/4 < k < (1 + \omega^2)/4$, then the zero equilibrium of system (2) is stable for $\tau \in \bigcup_{j=0}^n (\hat{\tau}_j^+, \hat{\tau}_j^-)$, and unstable for $\tau \in \bigcup_{j=0}^n (\hat{\tau}_j^+, \hat{\tau}_j^-) \cup (\tau_n^+, \infty)$, where $\tau_n^+ = 0$. In this case, system (2) undergoes a Hopf bifurcation at its zero equilibrium when $\tau = \tau_n^+, j = 0, 1, 2, \ldots$.

(3) If $k > (1 + \omega^2)/4$, then the zero equilibrium of system (2) is unstable for all $\tau \geq 0$, and system (2) undergoes a Hopf bifurcation at its zero equilibrium when $\tau = \tau_n^+, j = 0, 1, 2, \ldots$.

In a similar way, for Eq. (7), let $\bar{\tau}_k(k \neq 0)$ be a root of (7), it follows that

$$-\beta - \omega^2 = -3k^2 + 4k - 1.$$ \quad (18)

Clearly, (18) has no real root if $k < 1/3$ or $k > 1$ and has two real roots $\beta_\pm$ if $1/3 < k \leq 1$, where

$$\bar{\tau}_k = \omega \pm \sqrt{-3k^2 + 4k - 1}.$$ \quad (19)

Now, we consider the equation

$$\lambda - 2k \pm i\omega = -ke^{-\lambda\tau},$$ \quad (20)

which is the combination of (7) and (9), we can state the following results.

**Lemma 3**

(1) If $k < 1/3$ or $k > 1$, then (20) has no purely imaginary root for all $\tau \geq 0$.

(2) Suppose that $1/3 < k < 1$, then we have the following:
Therefore, from Eq. (21), we obtain

\[ \tau_j^+ = \begin{cases} 
\pi - \arccos \left( \frac{2k - 1}{k} \right) + 2j\pi & \text{for } 2k > 1, \\
\frac{\arccos \left( 1 - \frac{2k}{k} \right) + 2j\pi}{\beta_+} & \text{for } 2k < 1.
\end{cases} \]  

(21)

\[ \tau_j^- = \begin{cases} 
\pi + \arccos \left( \frac{2k - 1}{k} \right) + 2j\pi & \text{for } 2k > 1, \\
\frac{\arccos \left( 1 - \frac{2k}{k} \right) + 2j\pi}{\beta_-} & \text{for } 2k < 1.
\end{cases} \]  

(22)

where \( j = 0, 1, 2, \ldots \). Then \( \pm i\beta_k \) is a pair of double purely imaginary roots of (20) with \( \tau = \tau_j^\pm \), respectively.

(2) If \((2 + \sqrt{1 - 3k^2})/3 < k < 1\), then (20) has a pair of double purely imaginary roots \( \pm i\beta_k \) when \( \tau = \tau_j^\pm \), respectively, where \( \tau_j^\pm \) is defined by (21) and

\[ \tau_j^+ = \begin{cases} 
\pi - \arccos \left( \frac{2k - 1}{k} \right) + 2j\pi & \text{for } 2k > 1, \\
\frac{\arccos \left( 1 - \frac{2k}{k} \right) + 2j\pi}{\beta_-} & \text{for } 2k < 1.
\end{cases} \]  

(23)

According to Theorems 1 and 2, the bifurcation curves in the \((k, \omega)\) parameter plane for the stability of the coupled system (2) are shown in Fig. 2. In region \( D_1 \cup D_2 \cup D_3 \), the origin is unstable, but in region \( D_4 \), there are stability switches for delay \( \tau \). And region \( D_5 \) is the stability region of the phase-locked states.

Under the previous discussion, we determined all points in \( \tau \)-parameter plane where \( \det \Delta(\tau, \lambda) \) has zero points \( \lambda \) with zero real parts, i.e. where the infinitesimal generator \( A_p \) has eigenvalues with zero real parts. For the sake of convenience,
we will use $\mu = \tau$ as the bifurcation parameter. Varying $\tau$ in system (2) so as to pass through such a point may cause a bifurcation, that is, changing the qualitative behavior of the asymptotic solutions of (2).

In view of (5), there are two codimension one bifurcations, i.e. two Hopf bifurcations, which can occur when $A_\mu$ has a pair of simple or double purely imaginary eigenvalues. The following assumptions will be useful in the forthcoming discussion of this section.

(H1) $\mu = \tau - \frac{p}{3}$
(H2) $\mu = \tau + \frac{p}{3}$

In fact, it follows from Lemmas 2 and 3 that $A_\mu$ has only a pair of simple (resp. double) complex conjugate eigenvalues on the imaginary axis if and only if (H1) (resp. (H2)) holds.

The first goal is to completely analyze all the codimension one bifurcations of system (2) and classify them. An equilibrium $(z_1; z_2; z_3)$ of system (2) is called a synchronous equilibrium if $z_1 = z_2 = z_3$: a mirror reflecting equilibrium if and only if two of the components are equal. It turns out that we may encounter the following bifurcations.

(i) Steady state bifurcations of equilibria;
(ii) Hopf bifurcations of mirror-reflecting waves of the form $z_j(t) = z_{j-1}(t)$, $t \in \mathbb{R}$, $j \equiv 1, 2$ (mod 3);
(iii) Hopf bifurcations of standing waves of the form $z_j(t) = z_{j+1}(t - (p/3))$, $t \in \mathbb{R}$, $j \equiv 0$ (mod 3), where $p > 0$ is a period of $z$;
(iv) Hopf bifurcations of discrete waves of the form $z_j(t) = z_{j+1}(t \pm (np/3))$, $t \in \mathbb{R}$, $j \equiv 0$ (mod 3), where $p > 0$ is a period of $z$.

Especially, the discrete waves are also called synchronous oscillations (if $\kappa = 0$ (mod 3)) or phase-locked oscillations (if $\kappa \neq 0$ (mod 3)) as each neuron oscillates just like others except not necessarily in phase with each other.

3. Hopf Bifurcated Synchronous Periodic Solutions

This section is devoted to the discussion of Hopf bifurcation under assumption (H1), while the next section is devoted to Hopf bifurcation under assumption (H2). We assume that system (2) undergoes a Hopf bifurcation at zero equilibrium when $\tau = \tau^*$, i.e. a family of periodic solutions bifurcate from zero equilibrium. In the sequel, using the normal form theory and center manifold reduction due to [Hassard et al., 1981], we are able to know more detailed information of Hopf bifurcations. Specifically, we can make clear whether the bifurcating branch of periodic solution exists locally for $\tau > \tau^*$ or $\tau < \tau^*$, and determine the properties of these bifurcating periodic solutions such as stability on the center manifold and period. For this purpose, we compute the reduced system on the center manifold associated with the pair of conjugate complex, purely imaginary solutions of the characteristic equation (5). By this reduction we can make clear whether the bifurcating branch of periodic solution exists locally for supercritical bifurcation or subcritical bifurcation.

For convenience, we let $\mu = \tau - \tau^*$, then $\mu = 0$ is the Hopf bifurcation value for system (2). Thus, we can rewrite system (2) as:

$$
\dot{u}(t) = L_\mu u + G(u, \mu) \quad (24)
$$

with $L_\mu \varphi = -(1 - 2k + i\omega)\varphi(0) + B_\varphi(-\tau)$ and $G(\varphi, \mu)$ is the higher-order terms in $\varphi$. By the Riesz representation theorem, there exists a matrix whose components are bounded variation functions $E(\theta, \mu)$ in $\theta \in (-\tau, 0)$, such that $L_\mu \varphi = \int_{-\tau}^0 E(\theta, \mu) \varphi(\theta) d\theta$ for $\varphi \in C([-\tau, 0], \mathbb{R}^3)$. Define $A : C([-\tau, 0], \mathbb{R}^3) \rightarrow C([-\tau, 0], \mathbb{R}^3)$ as $(A_\mu \varphi)(\theta) = \varphi'(\theta)$ when $\theta \in (-\tau, 0)$ and $(A_\mu \varphi)(0) = L_\mu \varphi$ for $\varphi \in C^1([-\tau, 0], \mathbb{R}^3)$.
In order to construct coordinates to describe the center manifold \( \mathcal{C}_n \) near the origin, we define a bilinear form as

\[
\langle \psi, \varphi \rangle = \overline{\psi}(0)\varphi(0)
- \int_{\theta=\tau}^{0} \int_{\zeta=0}^{\theta} \overline{\psi}(\xi - \theta)d\xi d\theta e^{i\beta} \varphi(\zeta)d\zeta
\]

(25)

for \( \psi \in \mathcal{C}([0, \tau], \mathbb{C}^m) \) and \( \varphi \in \mathcal{C}([-\tau, 0], \mathbb{C}^m) \), where \( \mathcal{C}^m \) is the space of three-dimensional complex row vectors. Then, as usual,

\[
\langle \psi, A \varphi \rangle = \langle A_0^* \psi, \varphi \rangle
\]

where \( A_0^* \) is the adjoint operator of \( A_0 \). Note that the eigenvalue \( i\beta \) of \( A_0 \) is simple, with the associated eigenvectors \( q \). Then, as usual, the associated eigenvectors \( q \) are the eigenvectors of \( A_0 \) with \( \langle q, q \rangle = 1 \) and \( \langle q, \overline{q} \rangle = 0 \).

For each \( u \in \text{Dom}(A_0) \), we may then associate the pair \( (z, w) \), where \( z = (q^*, u) \) and \( w = u - 2\text{Re}\{qz\} \). The center manifold \( \mathcal{C}_n \) we have \( w(\zeta, \theta) = w(z(t), \omega(t), \theta) \). Since the center manifold is a graph over the center eigenspace with values in the complement of the center eigenspace and this graph is tangent to the center eigenspace at the equilibrium so that \( w(0, 0, \theta) = 0 \), we can rewrite \( w \) as follows:

\[
w(z, \zeta, \theta) = w_{20}(\theta) z^2 + w_{11}(\theta) z \zeta + w_{02}(\theta) \zeta^2
+ w_{00}(\theta) z^2 + \ldots,
\]

where \( z \) and \( \zeta \) are local coordinates for center manifold \( \mathcal{C}_n \) in the direction of \( q^* \) and \( \overline{q}^* \). Note that \( w \) is real if \( u \) is real. We consider only real solutions.

For a solution \( u(t) \) of (24) at \( \mu = 0 \), we have

\[
\dot{z} = i\beta z + g(z, \zeta),
\]

(26)

where \( g(z, \zeta) = \langle \overline{q}^*(0) f_0(z, \zeta) \rangle \) and \( f_0(z, \zeta) = G_1 w(z, \zeta, \theta) + 2\text{Re}\{zq(\theta)\} \). Let

\[
g(z, \zeta) = \sum_{j+k \geq 1} g_{jk} z^j \zeta^k \frac{i^k}{j! k!}.
\]

Once the coefficients of \( g(z, \zeta) \) have been figured out, we can compute the following quantities [Hassard et al., 1981]:

\[
C_1(0) = \frac{i}{2\beta}\left(g_{20}g_{11} - \frac{1}{3}g_{02}^2\right) + \frac{1}{2}g_{21},
\]

(27)

\[
\mu_2 = -\frac{1}{\alpha}\left|g_{11}\right|,
\]

where \( \beta_2 = 2\text{Re}\{C_1(0)\} \), which determine the properties of bifurcating periodic solutions in the center manifold at the critical value \( \mu = 0 \), that is, \( \mu_2 \) determines the directions of the Hopf bifurcation: if \( \mu_2 > 0 \) (resp., \( \mu_2 < 0 \)), then the Hopf bifurcation is supercritical (resp., subcritical) and the bifurcating periodic solutions exist for \( \mu > 0 \) (resp., \( \mu < 0 \)); \( \delta \) determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are stable (resp., unstable) if \( \beta_2 < 0 \) (resp., \( \beta_2 > 0 \)). Therefore, in the following, we compute the coefficients of \( g(z, \zeta) \).

It follows from Theorem 1 that for each \( \tau = \tau^* \), near \( \tau = \tau^* \), the trivial solution of system (2) undergoes an Hopf bifurcation, giving rise to one branch of synchronous periodic solutions. In this case, by direct computation, we can choose

\[
q(\theta) = (1, d, d^T) e^{i\beta \theta}, \quad \theta \in [-\tau, 0]
\]

(28)

and

\[
q^* (\xi) = \overline{D}\theta(\xi, 1) e^{i\xi}, \quad \xi \in [0, \tau],
\]

(29)

where \( d = \{2k - 1 + i(\beta - \omega)/2k\} e^{i\beta \tau} \) and \( D \) is \( \{3d[1 + \tau^*(2k - 1 + i(\beta - \omega))]^{-1}\). It follows that \( g_{20} = g_{11} = g_{02} = 0 \) and

\[
g_{21} = -3Dd(1 + 2|d|^2).
\]

(30)

More specifically, if the Hopf bifurcation results from (13), i.e. \( \tau^* \) is taken as one of \( \tau^*_1 \), \( \tau^*_2 \), where \( \tau^*_1 \) and \( \tau^*_2 \) are as shown in Lemma 2, then it follows from (6) and (8) that \( d = 1 \) or \( d = 1 + i(\omega/k)e^{i\beta \tau} \). So we first consider \( d = 1 \), and combine with (27) and (30), we can obtain,

\[
C_1(0) = \frac{1}{1 + \tau^*(2k - 1 + i(\beta - \omega))}.
\]

(21)

It follows that

\[
\text{Re}\{C_1(0)\} = -\frac{1 + \tau^*(2k - 1)}{N},
\]

where \( N = [1 + \tau^*(2k - 1)]^2 + \tau^2(\beta - \omega)^2 \). On the other hand, i.e. \( d = 1 + i(\omega/k)e^{i\beta \tau} \), then we can
obtain
\[ C_1(0) = \frac{1 + 2|d|^2}{2[1 + r^*(2k - 1 + r(\beta - \omega))]}. \]

Consequently, we also obtain
\[ \text{Re}\{C_1(0)\} = \frac{1 + r^*(2k - 1)}{2N}(1 + 2|d|^2). \]

In any case, since the characteristic equation associated with system (2) has at least one root with positive real part when \((k, \omega) \in D_3 \cup D_4\), the non-trivial periodic solutions bifurcating from the zero equilibrium must be unstable in the phase space even though they are stable on the center manifold. So, we are interested especially in the case when \((k, \omega) \in D_3\), i.e. \(k \in (1/2, (1 + \omega^2)/4)\), where stability switches may occur.

**Theorem 3.** Suppose that \((k, \omega) \in D_3\) and \(\tau_j^\pm (j = 0, 1, 2, \ldots )\) are defined by
\[ \tau_j^+ = \frac{2\pi - \arccos\left(\frac{2k - 1}{2k}\right) + 2j\pi}{\beta_+}, \]
and
\[ \tau_j^- = \frac{\arccos\left(\frac{2k - 1}{2k}\right) + 2j\pi}{\beta_-}, \]
respectively. Then, for the coupled system (2), we have the following:

(i) The Hopf bifurcation occurs as \(\tau\) crosses \(\tau_j^+\) to the right, i.e. the Hopf bifurcation is supercritical, and the corresponding bifurcating periodic solutions are orbitally asymptotically stable.

(ii) The Hopf bifurcation occurs as \(\tau\) crosses \(\tau_j^-\) to the left, i.e. the Hopf bifurcation is subcritical, and the corresponding bifurcating periodic solutions are also orbitally asymptotically stable.

**Proof.** Noting the fact that
\[ \text{Re}\{C_1(0)\} = \frac{1 + r^*(2k - 1)}{N} < 0, \quad \text{for } 2k > 1, \]
and \(1 + 2|d|^2\) is always positive,
\[ \alpha'(\tau_j^+) > 0, \quad \alpha'(\tau_j^-) < 0 \]
the results follow immediately. 

### 4. Asynchronous: Stable Phase-Locking and Unstable Waves

Throughout this section, we always assume that (H2) holds, i.e. \(r^* = r^*_k\). In view of Lemma 3, we have under assumption (H2), \(A_{\mu}\) has a pair of double purely imaginary eigenvalues \(\pm i\zeta\). On the other hand, the considered system (2) is equivariant with respect to the \(\mathbb{Z}_2\)-action where the \(\mathbb{Z}_2\) subgroup acts by permutation and the flip acts by interchanging. This allows us to apply the symmetric Hopf bifurcation theorem for delay differential equations established by Wu [1998] (as an extension of the well-known Golubitsky–Stewart Theorem [Golubitsky et al., 1988] for symmetric ordinary differential equations) to obtain eight branches of asynchronous periodic solutions. More precisely, the generalized eigenspace \(U_{\pm i\zeta}(A_{\mu})\) consists of eigenvectors of \(A_{\mu}\) associated with eigenvalue \(\pm i\zeta\). Moreover, \(U_{\pm i\zeta}(A_{\mu}) = \text{span} \Phi\), where \(\Phi = (1, \zeta_1, \zeta_2, \zeta_3)\). \(\zeta_j\) are given by \(\zeta_j(\theta) = \nu_je^{\mu_0i\theta}\) for \(\theta \in [-\pi, \pi]\), and \(\nu_j = \cos(1, 2\sin(\theta), \sin(\theta/2), j = 1, 2\). Obviously, \(\lambda(\mu)\) is also a double eigenvalue of the adjoint operator \(A^\ast_{\mu}\) with adjoint eigenvector defined by
\[ A^\ast_{\mu}e^{\pm i\zeta_1} = \lambda(\mu)e^{\pm i\zeta_2}, \quad j = 1, 2. \]

We normalize the eigenvectors such that \(\zeta_j^\pm\), \(\zeta_j^{0\pm}\) for all \(\mu \in B(\mu_0, \delta)\), \(j, k = 1, 2\). Then
\[ \zeta_j^\pm = \zeta_j^0e^{\pm i\gamma}, \]
where \(D_1 = \{ 3\delta [1 + r^*(2k - 1 + i(\beta - \omega))] \}^{-1}\). The center space of \(A_{\mu_0}\) is \(U_{\pm i\zeta}(A_{\mu_0})\). Let \(\varphi_0\) and \(\varphi\) be the representations of \(D_3\) on \(\mathbb{C}^3 \cong \text{span} \Phi\) and \(\mathbb{R}^3\), respectively. In fact,
\[ \varphi_0(\gamma)(z_1, z_2) = (z_1e^{\gamma}, z_2e^{-\gamma}, \gamma) \quad \text{and} \quad \varphi_0(\kappa)(z_1, z_2) = (z_2, z_1) \]
for \(\gamma \in \mathbb{Z}_3\) and \(\kappa \in \mathbb{Z}_2\) defined in Lemma 1. There exist \(W \in C^0(\mathbb{C}^3 \times \mathbb{R}^2, Q)\), we choose to satisfy \(W(0, 0, 0) = 0\), \(DW(0, 0, 0) = 0\), and
\[ W(\varphi_0(\rho)z, \mu) = \varphi(\rho)W(z, \mu) \quad \text{for all } z \in \mathbb{C}^2 \text{ and } \rho \in D_3. \]

Moreover, on the center manifold \(C_0\) which is four-dimensional, we obtain the complex two-dimensional ordinary differential equation
\[ \dot{z} = g(z, \mu) \]
with \( z = \text{col}(z_1, z_2) \in \mathbb{C}^2 \) and \( g = \text{col}(g_1, g_2) \) is given by
\[
g^j(z, \mu) = z \mathbf{D}_j + 3 \mathbf{D}_j (\tau - \frac{\omega}{2})(z e^{-i \beta \tau} + z_{\pm \omega} e^{i \beta \tau}) + O(|z|^2)
\]
for \( j = 1, 2 \). It follows from Lemma 1 and the \( S^1 \)-invariance of eigenspace \( U_{\pm \omega}(\mathcal{A}_{\mu_0}) \) that the action of \( \mathcal{D}_3 \times S^1 \) on \( \mathbb{C}^2 \) is given by
\[
\theta \cdot (z_1, z_2) = (e^{-i \theta} z_1, e^{-i \theta} z_2), \quad \theta \in S^1.
\]
Assume that the normal form of (32) is
\[
z = \mathbf{G}z + \mathcal{G}(z, \mu)
\]
where \( \mathcal{G} = \text{col}(g^1, g^2) \). It follows from [Wu et al., 1999] that \( g(z, \mu) \) is \( \mathcal{D}_3 \times S^1 \)-equivariant according to the action defined. Therefore, \( \mathbf{G}_j \) (\( j = 1, 2 \)) takes the form
\[
g^j(z, \mu) = K_0 z_j + K_1 |z_j|^2 z_j + K_2 z_{\pm \omega} |z|^4 z_j + O(|\mu - \mu_0|^2 |z|^4) + |\mu - \mu_0| |z|^4 z_j.
\]
First, it is easy to see that \( K_0 = 3 \mathbf{D}_j (\tau - \frac{\omega}{2}) e^{-i \beta \tau} \). Next, we need to calculate \( K_1 \) and \( K_2 \). For this purpose, we focus on the case where \( \mu = \mu_0 \). Then (32) and (35) can be rewritten as
\[
z = \mathbf{G}z + g(\mu_0)
\]
and
\[
\dot{\nu} = i \beta \nu + g(\nu, \mu_0),
\]
respectively. In order to transform (37) to (38), we use the near identity transformation
\[
z = \nu + \psi(\nu, \tau), \quad \psi = O(|\nu|^2).
\]
According to [Wu et al., 1999; Guo & Yuan, 2009; Golubitsky et al., 1988], the bifurcations of small-amplitude periodic solutions of (36) are completely determined by the zeros of equation
\[
- \beta z e^{-i \beta \tau} = 0,
\]
where \( \beta \) is a new period-scaling parameter, and their orbital stability is determined by the signs of three eigenvalues of
\[
\mathbf{DG}(\nu, 0) - \beta \mathbf{M} \mathbf{I}
\]
that are not forced to zero by the group action. It is known that (40) can be written as
\[
A \text{col}(\nu_1, \nu_2) + B \text{col}(\nu_1^2, \nu_2^2, \nu_1 \nu_2) + \cdots = 0,
\]
with \( A = \mathcal{A}_0 + \mathcal{A}_1_1 |\nu_1|^2 + |\nu_2|^2, B = \mathcal{B}_0 \). For some complex numbers \( \mathcal{A}_0, \mathcal{A}_1_1 \) and \( \mathcal{B}_0 \) given by
\[
\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2, \quad \mathcal{B}_0 = \mathcal{B}_1 = \mathcal{B}_2.
\]
By the results of [Golubitsky et al., 1988], we know that the bifurcation of phase-locked oscillation is supercritical (resp., subcritical) depends on whether \( \text{Re}(\mathcal{A}_1 + \mathcal{B}_0) > 0 \) (resp., \( \text{Re}(\mathcal{A}_1 + \mathcal{B}_0) < 0 \)) and these are orbitally asymptotically stable if \( \text{Re}(\mathcal{A}_1 + \mathcal{B}_0) > 0 \) and \( \text{Re} \mathcal{B}_0 < 0 \). And the bifurcation of mirror-reflecting waves and standing waves are orbitally asymptotically stable provided that \( \text{Re}(2 \mathcal{A}_1 + \mathcal{B}_0) < 0 \) and \( \text{Re} \mathcal{B}_0 < 0 \). By direct computation, we can obtain \( \mathcal{A}_1 = (1/2) \mathcal{B}_0 \mathcal{A}_2 \) and \( \mathcal{B}_2 = \mathcal{A}_1 \mathcal{B}_0 \). Then if follows from (7) and (9), we obtain
\[
2 \text{Re}(\mathcal{K}_1) = \text{Re}(\mathcal{K}_2)
\]
\[
= \frac{1 + \tau^2(2k - 1)}{(1 + \tau^2(2k - 1))^2 + \tau^2(\mu - \omega)^2}
\]
or
\[
2 \text{Re}(\mathcal{K}_1) = \text{Re}(\mathcal{K}_2)
\]
\[
= \frac{1 + \tau^2(2k - 1)}{(1 + \tau^2(2k - 1))^2 + \tau^2(\mu - \omega)^2}
\times (1 + 2|\mu|^2).
\]
Therefore, we have the following results.

**Theorem 4.** Under assumption (H2), near \( \tau = \tau^* \), system (22) undergoes Hopf bifurcations. The direction of Hopf bifurcation and stability of bifurcating periodic solutions are completely determined by \( \text{Re}(\mathcal{K}_1) \) and \( \text{Re}(\mathcal{K}_2) \). Moreover, the bifurcations are supercritical, that is, each branch of the bifurcated phase-locked periodic solutions exists for \( \tau \) satisfying \( \tau < \tau^* \). Then all the bifurcated phase-locked oscillations have the same stability as the trivial solution before the bifurcation, while all the bifurcated mirror-reflecting waves and standing waves are unstable.

5. **Numerical Simulations**

By numerical simulations and simple analysis, Ramana Reddy et al. (1998, 1999) have shown that amplitude death exists in the model (1) when \( \tau_1 = \tau_2 = 0 \), and the stability switches when \( \tau_1 \neq \tau_2 \) has been shown by Song et al. [2007]. We obtain more explicit conditions for the amplitude death by analyzing the distribution of zeros of characteristic
Fig. 3. Synchronous phenomena of system (2) for \( \tau \geq 0 \). Initial conditions: \( x_1(0) = -0.8, x_2(0) = 0.8, x_3(0) = -0.6, x_4(0) = 0.6, x_5(0) = -0.2, x_6(0) = 0.2, \tau = 0.3 \).

equations, that is, only when \((k, \omega) \in D_3\) causing amplitude death to occur. Furthermore, using the normal form theory and center manifold reduction, we have rigorously demonstrated conditions that the bifurcating periodic solutions are asymptotically stable.

It follows from Theorem 1, we know that, taking \( k = 0.8, c = 1 \), every solution of (26) with arbitrarily given time delay \( \tau \) is asymptotically synchronous, which can be shown in Fig. 3.

When \( k = 0 \) (without coupling), system (2) becomes the single limit-cycle oscillator which has simple dynamics, i.e. an unstable zero equilibrium and an attracting limit cycle \(|z| = 1\), as shown in [Song et al., 2007].

When \( k \neq 0 \), to illustrate the theoretical results obtained in the previous sections, we choose some particular parameters as \((k, \omega) = (0.8, 4) \in D_3\), and \( z_1(t) = x_1(t) + ix_2(t), z_2(t) = x_3(t) + ix_4(t), z_3(t) = x_5(t) + ix_6(t)\). By (16) and (17), we can obtain

\[
\tau_j^+ = \frac{2\pi(j + 1) - \arccos\left(\frac{3}{8}\right)}{4 + \sqrt{22}}, \quad j = 0, 1, 2, \ldots
\]

\[
\tau_j^- = \frac{2j\pi + \arccos\left(\frac{3}{8}\right)}{4 - \sqrt{22}}
\]

which lead to \( \tau_0^+ = 0.4714 < \tau_0^- = 0.9295 < \tau_1^- = 2.9679 < \cdots \). Theorem 2 shows that the zero equilibrium of system (2) is unstable for \( \tau \in [0, \tau_0^-) \cup (\tau_0^+, \tau_1^-) \cup (\tau_1^+, \infty) \) and asymptotically stable for \( \tau \in (\tau_0^-, \tau_0^+) \cup (\tau_1^-, \tau_1^+) \). This means that as the delay \( \tau \) varies, the zero equilibrium of system (2) switches from stability to instability, then to stability, as shown in Figs. 4-6, and finally becomes unstable. This phenomenon extends the previous results, showing the important effect of time delays on the dynamics of the delay-coupled system (2).

Fig. 4. When \( \tau \in [0, \tau_0^-) \), the zero equilibrium is unstable. Furthermore, when \( \tau < \tau_0^- \) near \( \tau_0^- \), a periodic solution bifurcates from the zero equilibrium and is orbitally asymptotically stable. (a) The corresponding waveform and (b) phase plot. Initial conditions: \( x_1(0) = x_3(0) = x_5(0) = 0.3, x_2(0) = x_4(0) = x_6(0) = 0.5, \tau = 0.4 \).
Fig. 5. When $\tau \in (\tau_j^+, \tau_j^-)$, the zero equilibrium is asymptotically stable. (a) The corresponding waveform and (b) phase plot. Initial conditions: $x_1(0) = x_3(0) = x_5(0) = 0.3$, $x_2(0) = x_4(0) = x_6(0) = 0.5$, $\tau = 0.65$.

Fig. 6. When $\tau \in (\tau_j^+, \tau_j^-)$, the zero equilibrium is unstable. Furthermore, when $\tau < \tau_j^+$ near $\tau_j^+$, a periodic solution bifurcates from the zero equilibrium and is orbitally asymptotically stable. (a) The corresponding waveform and (b) phase plot. Initial conditions: $x_1(0) = x_3(0) = x_5(0) = 0.3$, $x_2(0) = x_4(0) = x_6(0) = 0.5$, $\tau = 1.5$.

Theorem 3 further shows that the Hopf bifurcation occurs as $\tau$ crosses $\tau_j^+$ to the right but $\tau_j^-$ to the left, and the corresponding bifurcating periodic solutions are orbitally asymptotically stable.

The numerical simulation results in Figs. 4–6 show the oscillations of the delay-coupled system (2) are in good agreement. And in Figs. 4–6, (a) and (b) are the corresponding waveform and phase plot, respectively. Taking $(k, \omega) = (0.8, 0.25) \in D_5$, by (26) we have $\tau_j^+ = \pi(2j + 1) \mp \sqrt{2/3}$, $j = 0, 1, 2, \ldots$, so we choose $\tau^* = \tau_j^* = 3.1045$, then by Theorem 4,

Fig. 7. All bifurcated phase-locked oscillations have same stability as the trivial solution before the bifurcation. Initial conditions: $x_1(0) = x_3(0) = x_5(0) = 0.3$, $x_2(0) = x_4(0) = x_6(0) = 0.5$, (a) $\tau = 3.5$, (b) $\tau = 4.5$, (c) $\tau = 5.5$.
there exists stable phase-locked periodic solution for $\tau > \tau^*$, and the corresponding phase plot for $\tau = 3.5, 4.5$ and $\tau = 5.5$ are shown in Figs. 7(a)–7(c), respectively.

6. Conclusions and Remarks

In this paper, we have investigated limit-cycle oscillators with the effect of time delays on the dynamics with delayed-coupling. By analyzing the associated characteristic equation, the first quadrant of the $(k, \omega)$ plane can be divided into five regions, $D_1, D_2, D_3, D_4, D_5$, and $D_6$ in each of which we can determine the stability and Hopf bifurcations of the coupled system (2). In particular, taking the delay as a parameter, we investigate the existence of Hopf bifurcations. Using the center manifold reduction, normal form theory and symmetric Hopf bifurcation theorem, the stability and direction of the Hopf bifurcation are determined. Specifically, we have shown that if $(k, \omega) \in D_5$, then the Hopf bifurcation occurs as $\tau$ crosses $\tau_5^*$ to the right but $\tau_5^*$ to the left, and the corresponding bifurcating periodic solutions are orbitally asymptotically stable. And if $(k, \omega) \in D_6$, all the bifurcated phase-locked oscillations have the same stability as the trivial solution before the bifurcation.

This paper is only a first step toward limit cycle oscillators modeling with delayed coupling, which describes a more realistic complex oscillators system. There are also some limitations in our models. For example, in our model all the delays and intrinsic frequency are the same, and the coupling strengths are all constants. Future work regarding this topic includes, for example, oscillators modeling with different time-delay constants in different couplings and different intrinsic frequencies in oscillator, and time-varying coupling strengths, and the dynamics and control of such models.

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References


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