Abstract

In this paper, for any prime \( p \geq 11 \), we consider \( C_p \)-decompositions of \( K_m \times K_n \) and \( K_m \times \overline{K}_n \) and also \( C_p \)-factorizations of \( K_m \times K_n \), where \( \times \) and \( * \) denote the tensor product and wreath product of graphs, respectively, \((K_m \times \overline{K}_n)\) is isomorphic to the complete \( m \)-partite graph in which each partite set has exactly \( n \) vertices. It has been proved that for \( m, n \geq 3 \), \( C_p \)-decomposes \( K_m \times K_n \) if and only if (1) either \( m \) or \( n \) is odd and (2) \( p \mid mn(m - 1)(n - 1) \). Further, it is shown that for \( m \geq 3 \), \( C_p \)-decomposes \( K_m \times \overline{K}_n \) if and only if (1) \( (m - 1)n \) is even and (2) \( p \mid mn(m - 1)n^2 \). Except possibly for some valid pairs of integers \( m \) and \( n \), the necessary conditions for the existence of \( C_p \)-factorization of \( K_m \times K_n \) are proved to be sufficient.

\( C_p \)-decompositions of some regular graphs

R.S. Manikandan, P. Paulraja

Department of Mathematics, Annamalai University, Annamalainagar 608 002, India

Received 6 February 2004; received in revised form 30 June 2005; accepted 29 August 2005

Available online 21 February 2006

1. Introduction

All graphs considered here are simple and finite. Let \( C_n \) denote a cycle of length \( n \). If the edge set of \( G \) can be partitioned into cycles \( C_{n_1}, C_{n_2}, \ldots, C_{n_r} \), then we say that \( C_{n_1}, C_{n_2}, \ldots, C_{n_r} \) decompose \( G \). If \( n_1 = n_2 = \cdots = n_r = k \), then we say that \( G \) has a \( C_k \)-decomposition and in this case we write \( C_k \mid G \). If \( G \) has a 2-factorization and each 2-factor of it has only cycles of length \( k \), then we say that \( G \) has a \( C_k \)-factorization, with notation \( C_k \parallel G \). We write \( G = H_1 \otimes H_2 \oplus \cdots \oplus H_k \) if \( H_1, H_2, \ldots, H_k \) are edge-disjoint subgraphs of \( G \) and \( E(G) = E(H_1) \cup E(H_2) \cup \cdots \cup E(H_k) \). The complete graph on \( m \) vertices is denoted by \( K_m \) and its complement is denoted by \( \overline{K}_m \). For some positive integer \( k \), the graph \( kH \) denotes \( k \) disjoint copies of \( H \). For a graph \( G \), \( G(\lambda) \) denotes the graph obtained from \( G \) by replacing each of its edges by \( \lambda \) edges. A cycle of length \( k \) is called a \( k \)-cycle. \( P_k \) denotes the path on \( k \) vertices.

For two graphs \( G \) and \( H \) their wreath product \( G \ast H \) has vertex set \( V(G) \times V(H) \) in which \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent whenever \( g_1 g_2 \in E(G) \) or \( g_1 = g_2 \) and \( h_1 h_2 \in E(H) \). Similarly, \( G \times H \), the tensor product of the graphs \( G \) and \( H \) has vertex set \( V(G) \times V(H) \) in which two vertices \((g_1, h_1)\) and \((g_2, h_2)\) are adjacent whenever \( g_1 g_2 \in E(G) \) and \( h_1 h_2 \in E(H) \). Clearly the tensor product is commutative and distributive over edge-disjoint union of graphs, that is, if \( G = H_1 \oplus H_2 \oplus \cdots \oplus H_k \), then \( G \times H = (H_1 \times H) \oplus (H_2 \times H) \oplus \cdots \oplus (H_k \times H) \). For \( h \in V(H), V(G) \times h = \{(v, h) \mid v \in V(G)\} \) is called the column of vertices of \( G \times H \) corresponding to \( h \). Further, for \( x \in V(G), x \times V(H) = \{(x, v) \mid v \in V(H)\} \) is called the layer of vertices of \( G \times H \) corresponding to \( x \). Similarly we

E-mail address: pprajaau@sify.com (P. Paulraja).

0012-365X/S - see front matter © 2005 Published by Elsevier B.V.
doi:10.1016/j.disc.2005.08.006
can define column and layer for wreath product of graphs also. We can easily observe that \( K_m \ast \overline{K}_n \) is isomorphic to the complete \( m \)-partite graph in which each partite set has exactly \( n \) vertices.

A Latin square of order \( n \) is an \( n \times n \) array, each cell of which contains exactly one of the symbols in \( \{1, 2, \ldots, n\} \), such that each row and each column of the array contains each of the symbols in \( \{1, 2, \ldots, n\} \) exactly once. A Latin square is said to be idempotent if the cell \((i, i)\) contains the symbol \( i \), \( 1 \leq i \leq n \).

Let the vertices of \( K_n \) be \( \{1, 2, \ldots, n\} \); then the edge \((i, j)\) of \( K_n \) is said to be of distance \( \min\{i - j \pmod{n}, \ j - i \pmod{n}\} \). Hence there are exactly two edges of distance \( k \), \( 1 \leq k \leq (n - 1)/2 \), incident with each of its vertices. If \( G \) is a bipartite graph with bipartition \((X, Y)\), where \( X = \{x_1, x_2, \ldots, x_n\} \), \( Y = \{y_1, y_2, \ldots, y_n\} \) and if \( G \) contains the set of edges \( F_1(X, Y) = \{x_jy_{i+j} \mid 1 \leq j \leq n\} \), where addition in the subscript is taken modulo \( n \) with residues 1, 2, \ldots, \( n \), \( 0 \leq i \leq n - 1 \), then we say that \( G \) has the 1-factor of distance \( i \) from \( X \) to \( Y \). Clearly, if \( G = K_{n,n} \), then \( E(G) = \bigcup_{i=0}^{n-1} F_i(X, Y) \). Note that \( F_1(Y, X) = E_{n-i}(X, Y) \), \( 0 \leq i \leq n - 1 \). In a bipartite graph with bipartition \((X, Y)\) with \( |X| = |Y| \), if \( x_iy_j \) is an edge, then \( x_iy_j \) is called an edge of distance \( j - i \) if \( i \leq j \), or \( n - (i - j) \), if \( i > j \), from \( X \) to \( Y \). (The same edge is said to be of distance \( i - j \) if \( i \geq j \) or \( n - (j - i) \), if \( i < j \), from \( Y \) to \( X \).

Recently, it has been proved that if \( n \) is odd and \( m \mid \binom{n}{2} \) or \( n \) is even and \( m \mid \binom{n}{2} - \frac{n}{2} \), then \( C_m \mid K_n \) or \( C_m \mid K_n - I \), where \( I \) is a 1-factor of \( K_n \). A similar problem can also be considered for regular complete multipartite graphs; Cavenagh and Billington [7] and Mahmoudian and Mirzakhan [11] have considered \( C_5 \)-decompositions of complete multipartite graphs. Moreover, Billington [3] has studied the decompositions of complete multipartite graphs into cycles of length 3 and 4. Further, Cavenagh and Billington [6] have studied the decompositions of complete multipartite graphs into cycles of length 4, 6 and 8. Recently, the present authors have proved [12,13] that the necessary conditions for the existence of a \( C_5 \)- and \( C_7 \)-decompositions of \( K_m \ast \overline{K}_n \) are sufficient; a similar problem has also been considered by them for \( K_m \ast K_n \). Billington et al. [4] have solved the problem of decomposing \((K_m \ast \overline{K}_n)(\lambda)\) into \( 5 \)-cycles. A detailed account of cycle decompositions of complete graphs can be seen in [8].

In this paper, we prove that the obvious necessary conditions for \( K_m \ast K_n \), \( m, n \geq 3 \), to have a \( C_p \)-decomposition are proved to be sufficient, where \( p \geq 11 \) is a prime. The proof technique employed here can be extended to prove existence of a \( C_p \)-decomposition of the complete multipartite graph \( K_m \ast \overline{K}_n \), \( m \geq 3 \). In the later part of the paper, we prove that the necessary conditions for a \( C_p \)-factorization of \( K_m \ast K_n \), \( m, n \geq 3 \), are sufficient for many values of \( m \) and \( n \). We list below some of the important results obtained here.

1. For a prime \( p \geq 11 \) and \( m \geq 3 \), \( C_p \mid K_m \ast \overline{K}_n \) if and only if (1) \( n(m - 1) \) is even and (2) \( p \mid m(m - 1)n^2 \).
2. For a prime \( p \geq 11 \) and \( m, n \geq 3 \), \( C_p \mid K_m \ast K_n \) if and only if (1) \( p \mid nm(n - 1) \) and (2) either \( m \) or \( n \) is odd.
3. For any prime \( p \geq 11 \), \( m, n \geq 3 \), \( C_p \parallel K_m \ast K_n \) if and only if (1) either \( m \) or \( n \) is odd and (2) \( p \mid mn \), except possibly for the following cases,
   (a) \( m = 7 \) or \( 11 \) and \( n \equiv 0 \pmod{2p} \) or \( n = 7 \) or \( 11 \) and \( m \equiv 0 \pmod{2p} \),
   (b) \( m \not\equiv \{7, 11\} \) and \( n = 2p \) or \( 6p \) or \( n \not\equiv \{7, 11\} \) and \( m = 2p \) or \( 6p \).

In [10] the necessary conditions for the existence of a \( C_k \)-factorization of \( K_m \ast \overline{K}_n \) are proved to be sufficient. For our future reference we list below some known results.

**Theorem A** (Alspach et al. [2]). Let \( t \) be an odd integer and \( p \) be a prime so that \( 3 \leq t \leq p \). Then \( C_t \ast \overline{K}_p \) has a 2-factorization so that each 2-factor is composed of \( t \) cycles of length \( p \).

**Theorem B** (Alspach et al. [2]). For odd \( n \), \( K_n \) has a 2-factorization \( \mathcal{F} = \{F_1, F_2, \ldots, F_{(n-1)/2}\} \) such that each \( F_i \), \( 1 \leq i \leq (n - 1)/2 \), consists of cycles of length 3 or 5 if and only if \( n \neq 7 \) or 11.

**Theorem C** (Alspach et al. [2]). For any odd integer \( t \geq 3 \), if \( n \equiv t \pmod{2t} \), then \( C_t \parallel K_n \).

**Theorem D** (Alspach and Gavlas [1]). For any odd integer \( t \geq 3 \), if \( n \equiv 1 \) or \( t \pmod{2t} \), then \( C_t \mid K_n \).

The following theorem can be found in [9].
Theorem E. Let \( m \) be an odd integer, \( m \geq 3 \).

1. If \( m \equiv 1 \) or \( 3 \) (mod 6), then \( C_3 \mid K_m \).
2. If \( m \equiv 5 \) (mod 6), then \( K_m \) can be decomposed into \((m(m - 1) - 20)/6 \) 3-cycles and two 5-cycles.

Theorem F (Šajna [16]). Let \( n \) be an even (resp. odd) integer and \( m \) be an even (resp. even) integer with \( 3 \leq m \leq n \) (resp. \( 4 \leq m \leq n \)). Then the graph \( K_m - I \) (resp. \( K_n \)) can be decomposed into cycles of length \( m \) whenever \( m \) divides the number of edges in \( K_m - I \) (resp. \( K_n \)), where \( I \) is a 1-factor of \( K_n \).

Theorem G (Liu [10]). For \( t \geq 3 \) and \( m \geq 2 \), \( K_m \ast \overline{K}_n \) has a \( C_t \)-factorization if and only if \( t \) divides \( mn \) and \((m - 1)n\) is even, \( t \) is even if \( m = 2 \), and \((m, n, t) \neq (3, 2, 3), (3, 6, 3), (6, 2, 3), (2, 6, 6) \).

Theorem H (Piotrowski [15]). Let \( k \geq 1 \), \( m \geq 3 \). If \( a_1, a_2, \ldots, a_k \) are positive integers which are divisible by \( m \) and \( mn = \sum a_i \), then \( F \mid C_m \ast \overline{K}_n \), where \( F \) consists of 1 cycles, namely, \( C_{a_1}, C_{a_2}, \ldots, C_{a_k} \), except in the cases: (1) \( n = 2 \) and \( m \) odd, (2) \( n = 6 \), \( m = 3 \) and \((a_1, a_2, \ldots, a_k) = (3, 3, 3, 3, 3) \), in which case the contrary is true.

2. \( C_p \)-decompositions of \( C_3 \times K_m \) and \( C_3 \ast \overline{K}_m \)

Lemma 2.1. For any odd integer \( t \geq 3 \), \( C_t \mid C_3 \times K_t \).

Proof. Clearly, \( C_3 \times K_t \equiv K_t \times C_3 \equiv (C_t \times C_3) \oplus (C_t \times C_3) \oplus \cdots \oplus (C_t \times C_3) \), since \( K_t \) admits a Hamilton cycle decomposition and tensor product is distributive over edge-disjoint subgraphs. Let \( V(C_t) = \{x_1, x_2, \ldots, x_t\} \) and \( V(C_3) = \{y_1, y_2, y_3\} \). Then \( V(C_t \times C_3) = \bigcup_{i=1}^{t} \{x_i \times C_3\} = \bigcup_{i=1}^{t} \{C_3 \times y_j\} \mid 1 \leq j \leq 3 \). Let \( X_i = \{(x_i, y_j) \mid 1 \leq j \leq 3, 1 \leq i \leq t \} \). Then \( E(C_t \times C_3) = \bigcup_{i=1}^{t} \{F_i(X_i, X_{i+1}) \cup F_2(X_i, X_{i+1})\} \), where additions in the subscripts are taken modulo \( t \) with residues \( 1, 2, \ldots, t \). The graph \( C_t \times C_3 \) can be factorized into two \( C_t \)-factors, say, \( F_1' \) and \( F_2' \), as follows:

\[
F_1' = \left( \bigcup_{i=1}^{\frac{t-3}{2}} \{F_i(X_{2i-1}, X_{2i}) \cup F_2(X_{2i}, X_{2i+1})\} \right) \cup F_1(X_{t-2}, X_{t-1}) \cup F_1(X_{t-1}, X_t)
\]

and

\[
F_2' = \left( \bigcup_{i=1}^{\frac{t-3}{2}} \{F_2(X_{2i-1}, X_{2i}) \cup F_1(X_{2i}, X_{2i+1})\} \right) \cup F_2(X_{t-2}, X_{t-1}) \cup F_2(X_{t-1}, X_t) \cup F_2(X_t, X_1).
\]

This completes the proof. \( \square \)

Lemma 2.2. For any prime \( p \geq 7 \), \( C_p \mid C_3 \times K_{2p} \).

Proof. Let the partite sets (three layers) of the tripartite graph \( G = C_3 \times K_{2p} \) be \( U = \{u_1, u_2, \ldots, u_{2p}\} \), \( V = \{v_1, v_2, \ldots, v_{2p}\} \) and \( W = \{w_1, w_2, \ldots, w_{2p}\} \). Let us assume that the vertices having the same subscript are the corresponding vertices of the partite sets, that is, vertices in a column of \( G \). Let \( Z_i = \{u_{2i-1}, v_{2i}, w_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}\}, 1 \leq i \leq p \), be the set of vertices of column \( 2i - 1 \) and \( 2i \) of \( G \); \( Z_i \) induces a 6-cycle in \( G \), say, \( C_i = (u_{2i-1}, v_{2i}, w_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}) \). We associate with \( G \) the complete graph \( K_p \) as follows: to each subset \( Z_i \) of vertices of \( G \) introduce a vertex \( z_i \) and join any pair of distinct vertices \( z_i \) and \( z_j \) by an edge. The set \( \{z_1, z_2, \ldots, z_p\} \) of vertices thus induces the graph \( K_p \). We associate with each edge \( z_i z_j \) of \( K_p \) the set of edges between \( Z_i \) and \( Z_j \) in \( G \), see Fig. 1. Throughout the proof of this lemma the order of occurrence of the vertices of \( Z_i \) is assumed to be \( u_{2i-1}, v_{2i}, w_{2i-1}, u_{2i}, v_{2i-1}, w_{2i} \) in all the figures, see Fig. 1.

In this paragraph, we give the idea behind the proof of the lemma. The proof goes as follows: we decompose the \( K_p \) into paths \( \mathcal{P} = \{P^1, P^2, \ldots, P^p\} \) such that each \( P^i \) is of length \((p - 1)/2 \) and contains the edges having distances \( 1, 2, \ldots, (p - 1)/2 \), consecutively. Then we associate with each \( P^i \) a subgraph, say, \( H_i \), of \( G \) so that \( H_i \) and \( H_j \), \( i \neq j \),...
are edge-disjoint and \( \bigcup_{i=1}^{p} H_i = G \). Then, in each \( H_i \), we choose a set of six edge-disjoint cycles of length \( p \). Let \( G' \) be the subgraph of \( G \) obtained by deleting the 6\( p \) \( p \)-cycles that are chosen in the \( H_i \)'s in \( G \). Then we decompose \( G' \) into \( p \)-cycles. This is achieved by again finding a Hamilton cycle decomposition \( \mathcal{C} \) of \( K_p \) such that each cycle in the decomposition contains only edges of the same distance (this is possible since \( p \) is a prime) and associating with each Hamilton cycle in \( \mathcal{C} \) a subgraph of \( G' \) and decomposing it into cycles of length \( p \).

Now we shall give the proof. As we mentioned earlier, first we decompose \( K_p \) into \( p \) paths of length \((p - 1)/2\) each, so that the distances of any two edges of each of these paths are distinct. Let \( t_j = 0 + 1 + 2 + \cdots + j \), \( 0 < j < (p - 1)/2 \).

For each \( i \), \( 1 \leq i \leq p \), we define the path \( P_i = z_i + t_0 z_i + t_1 z_i + t_2 \cdots z_i + t_{(p-1)/2} \), where additions in the subscripts are taken modulo \( p \) with residues \( 1, 2, \ldots, p \), of length \((p - 1)/2\) in \( K_p \). (Note that \( P_i \) is indeed a path; for some \( i \), \( t_a \neq t_b \) (mod \( p \)) if and only if \( a \neq b \) (mod \( p \)). Then if \( 0 < a < b < (p - 1)/2 \), since \( p \) is prime and \( p > 3 \), \( a - 1 \neq b \) (mod \( p \)).)

Note that the edge \( z_i + t_{(p-1)/2} \) is a required path decomposition of \( K_p \) and hence the path \( P_i \), \( 1 \leq i \leq p \), consists of edges of distances \( 1, 2, \ldots, (p - 1)/2 \). Thus \( \mathcal{P} = \{ P_i \} \) is a required path decomposition of \( K_p \).

Let \( H_i \) be the union of the subgraph of \( G \) associated with the edges of the path \( P_i \) of \( K_p \) and \( C' \), the 6-cycle induced by \( Z_i(=Z_i + t_0) \), see Fig. 2. Note that \( \bigcup_{i=1}^{p} H_i = G \), since each vertex \( z_i \) of \( K_p \) appears as the initial vertex of exactly one of the paths, namely, \( P_i \) of \( \mathcal{P} \). First we decompose the graph \( H_i \) into six \( p \)-cycles and \((p - 1)\) edges. (Those edges which are not on the six \( p \)-cycles will be used later.) To obtain the six \( p \)-cycles of \( H_i \), \( 1 \leq i \leq p \), first we consider two “base \( p \)-cycles”, say, \( C' \) and \( C'' \) in \( H_i \) as shown in Fig. 3. Let \( p = (Z_i + t_0)(Z_i + t_1)(Z_i + t_2) \cdots (Z_i + t_{(p-1)/2}) \) be the permutation, where \( (Z_k) \) stands for the permutation \( (u_{2k - 1} u_{2k} v_{2k - 1} v_{2k} w_{2k - 1} w_{2k}) \). Now the required six \( p \)-cycles are \( C', C'', \rho^2(C'), \rho^3(C''), \rho^4(C') \) and \( \rho^4(C'') \).

Let \( G' \) be the graph obtained from \( G \) by deleting the 6\( p \) \( p \)-cycles obtained above from the \( H_i \)'s. Since \( C' \), \( 1 \leq j \leq p \), is covered by these 6\( p \) \( p \)-cycles, the subgraph induced by \( Z_i \) in \( G' \) is the empty graph. Again, we associate with each edge \( z_j z_i \) of \( K_p \) the subgraph of \( G' \) induced by \( Z_i \cup Z_j \). Let \( H_i^j \), \( 1 \leq j \leq (p - 1)/2 \), \( 1 \leq i \leq p \), be the subgraph of \( G' \) induced by \( Z_{i+j-1} \cup Z_{i+j} \), that is, \( H_i^j \) contains those edges of \( G' \) corresponding to the edge of distance \( j \) in \( P_i \) of \( K_p \) (\( P_i \) is defined above). Clearly, for each \( i \), the graph \( H_i^j \) is isomorphic to \( H', H'' \) or \( H''' \) according as \( j \leq (p - 5)/2 \), \( j = (p - 3)/2 \) or \( (p - 1)/2 \), respectively, see Fig. 4. One can observe that \( H''' \) is nothing but a redrawing of \( H'' \). An isomorphism between \( H'' \) and \( H''' \) is obtained as follows: let the vertices of \( H'' \) (resp. \( H''' \)) in one part be \( a_1, a_2, \ldots, a_6 \) (resp. \( b'_1, b'_2, b'_3, b'_4, b'_5, b'_6 \)) and the other part be \( b_1, b_2, \ldots, b_6 \) (resp. \( a'_1, a'_2, a'_3, a'_4, a'_5, a'_6 \)), in order. Then the required isomorphism is \( a_i \rightarrow a'_i \) and \( b_i \rightarrow b'_i \). Consequently, we shall use \( H''' \) in the place of \( H'' \) for our future purpose.

Let \( G_j = \bigcup_{i=1}^{p} H_i^j \), \( 1 \leq j \leq (p - 1)/2 \). Observe that \( G_j \) is nothing but the subgraph of \( G' \) which corresponds to all the edges of distance \( j \) in \( K_p \); also note that the set of edges of distance \( j \) in \( K_p \) induce a Hamilton cycle of \( K_p \) (since \( p \) is a prime). Hence \( G_j \), \( 1 \leq j \leq (p - 5)/2 \), is isomorphic to the graph obtained by identifying the first and last layers of the graph \( G_j \) of Fig. 5(a) and for \( j = (p - 3)/2 \) or \( (p - 1)/2 \), \( G_j \) is obtained by identifying the last and first layers of the graph \( G_j' \) of Fig. 5(b) (since \( H'' \) and \( H''' \) are isomorphic) wherein the additions in the subscripts are taken modulo \( p \) with residues \( 1, 2, \ldots, p \).
To complete the proof, it is enough to prove that $G_j$ and $G''_j$ each have decompositions into paths of length $p$, and each path has its end vertices on the same column (and so, on superimposing the last and first layers, these yield the required cycle decomposition of $G_j$). Now $G'_j$ can be factorized into two $P_{p+1}$-factors, say, $F'_1$ and $F'_2$ as follows:

$$F'_1 = \left( \bigcup_{k=1}^{(p-3)/2} \{ F_2(Z_{1+(2k-2)j}, Z_{1+(2k-1)j}) \cup F_4(Z_{1+(2k-1)j}, Z_{1+2kj}) \} \right)$$

$$\cup \left( \bigcup_{k=p-3}^{p-1} F_2(Z_{1+kj}, Z_{1+(k+1)j}) \right)$$

and

$$F'_2 = \left( \bigcup_{k=1}^{(p-3)/2} \{ F_4(Z_{1+(2k-2)j}, Z_{1+(2k-1)j}) \cup F_2(Z_{1+(2k-1)j}, Z_{1+2kj}) \} \right)$$

$$\cup \left( \bigcup_{k=p-3}^{p-1} F_4(Z_{1+kj}, Z_{1+(k+1)j}) \right),$$

In $G''_j$, to find the required decomposition, first we construct four “base paths” of length $p$ each, in $G''_j$. Let $\rho = (Z_1)(Z_{1+j})(Z_{1+2j}) \cdots (Z_{1+pj})$ be the permutation, where $(Z_k)$ stands for the permutation $(u_{2k-1}v_{2k}w_{2k-1}u_{2k})$ and the addition in the subscripts is taken modulo $p$ with residues $1, 2, \ldots, p$. The required paths are the four base paths and the paths obtained by letting the permutations $\rho^2$ and $\rho^4$ act on these four “base paths”. The four base paths of $G''_j$ are obtained by attaching copies of $G''_1$, $G''_2$, and $G''_3$, shown in Fig. 6, one over the other.
For example if \( p = 7 \), then consider the four paths in \( G''_1 \) alone. If \( p = 11 \), put \( G''_3 \) over \( G''_1 \), that is the bottom layer of \( G''_3 \) is superimposed with the first layer of \( G''_1 \). If \( p = 13 \), then putting \( G''_1 \) over \( G''_2 \) would do. Similarly, if \( p = 17 \), we use \( G''_1 \), \( G''_2 \) and \( G''_3 \) by keeping one over the other, successively, and so on (because for any prime \( p \), \( p \geq 7 \), \( p + 1 \) can be obtained by adding multiples of 8, 6 and 4). This completes the proof. \( \square \)

**Lemma 2.3.** For any odd integer \( t \geq 11 \), \( C_t | C_3 \times K_{t+1} \).

**Proof.** Let the partite sets (three layers) of the tripartite graph \( G = C_3 \times K_{t+1} \) be \( U = \{u_1, u_2, \ldots, u_{t+1}\} \), \( V = \{v_1, v_2, \ldots, v_{t+1}\} \) and \( W = \{w_1, w_2, \ldots, w_{t+1}\} \). Let us assume that the vertices having the same subscript are the corresponding vertices of the partite sets, that is, vertices in a column of \( G \). We prove this lemma in two cases.

**Case 1.** \( t \equiv 3 \pmod{4} \).

Let \( t = 4k + 3 \). Consider the \( t \)-cycle \( C = (u_{2k+1}, v_1, u_{2k}, v_2, u_{2k-1}, v_3, u_{2k-2}, \ldots, v_{k-1}, u_{k+2}, v_k, u_{k-2}, v_{k+1}, u_{k-3}, v_{k+2}, u_{k-4}, \ldots, v_{2k}, u_{4k+2}, v_{2k+1}, w_{2k}) \), where addition in the subscripts is taken modulo \( t + 1 \) with residues

Solid edges denote \( C' \) and the broken edges denote \( C'' \)

**Fig. 3.**

---

**Fig. 4.**
1, 2, ..., \( t + 1 \), in \( G \). Note that the edges in the cycle \( C \) are of distances 1, 2, ..., \( t \), where distances are taken in the order \((U, V), (V, W)\) and \((W, U)\). Letting the permutation \( \sigma = (u_1 u_2 \cdots u_{t+1})(v_1 v_2 \cdots v_{t+1})(w_1 w_2 \cdots w_{t+1}) \) and its powers act on \( C \) give \((t+1)\) \( t \)-cycles, say, \( C_1^t, C_2^t, \ldots, C_{t+1}^t \). Again, letting the permutation \( (u_1 v_1 w_1)(u_2 v_2 w_2) \cdots (u_{t+1} v_{t+1} w_{t+1}) \) and its powers act on the \( t \)-cycles \( C_1^t, C_2^t, \ldots, C_{t+1}^t \) give us a required \( t \)-cycle decomposition of \( G \).

**Case 2.** \( t \equiv 1 \pmod{4} \).

Let \( t = 4k + 1 \). As in the proof of Lemma 2.2, let \( Z_i = \{u_{2i-1}, v_{2i-1}, w_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}\} \), \( 1 \leq i \leq 2k + 1 \). The subset \( Z_i \), \( 1 \leq i \leq 2k + 1 \), induces a 6-cycle, say, \( C_i^t = (u_{2i-1}, v_{2i-1}, w_{2i-1}, u_{2i}, v_{2i}, w_{2i}) \) in \( G \). To each subset \( Z_i \) of vertices of \( G \), introduce a vertex \( z_i \) and join any pair of distinct vertices \( z_i \) and \( z_j \) by an edge. The set \( \{z_1, z_2, \ldots, z_{2k+1}\} \) of vertices thus induces the graph \( K_{2k+1} \). We associate with each edge \( z_i z_j \) of \( K_{2k+1} \) the set of edges between \( Z_i \) and \( Z_j \) in \( G \), see Fig. 1. A word of caution! Throughout this lemma, in all the figures, the order of occurrence
of the vertices of $Z_i$ is assumed to be $u_{2i−1}, v_{2i}, w_{2i−1}, u_{2i}, v_{2i−1}, w_{2i}$ or a cyclic permutation of it (for example $w_{2i−1}, u_{2i}, v_{2i−1}, w_{2i}, u_{2i−1}, v_{2i}$).

First we show that $K_{2k+1}$ can be decomposed into paths of length $k$ such that each path has edges of distances 1, 2, ..., $k$. Let $t_j = 1 − 2 + 3 − 4 + \cdots + (-1)^{j+1}j$, $1 \leq j \leq k$. Let $P^i = z_i z_{i+t_1} z_{i+t_2} \cdots z_{i+t_k}$, $1 \leq i \leq 2k + 1$, where addition in the subscripts is taken modulo $2k + 1$ with residues 1, 2, ..., $2k + 1$. Note that, for each $i$, the edges in $P^i$ are of distances 1, 2, ..., $k$. Thus $\mathcal{P} = \{P^i \mid 1 \leq i \leq 2k + 1\}$ is a required path decomposition of $K_{2k+1}$.

Let $H_i$, $1 \leq i \leq 2k + 1$, be the union of the subgraph of $G$ corresponding to the path $P^i$ of the $K_{2k+1}$ and $C_i$, the 6-cycle induced by $Z_i$ (similar to $H_i$ of the proof of Lemma 2.2), see Fig. 7. Note that $H_i$ contains exactly one of the 6-cycles induced by the $Z_i$’s. As each vertex of $K_{2k+1}$ happens to be the origin of exactly one path $P^i$ of $\mathcal{P}$, $\bigcup_{i=1}^{2k+1} H_i = G$ and hence, it is enough to prove that the graph $H_i$, $1 \leq i \leq 2k+1$, has a $t$-cycle decomposition.

The rest of the proof goes as follows: first we construct two base $t$-cycles $C'$ and $C''$ in $H_i$. Then we fix a suitable permutation $\rho$, so that $C'$, $C''$, $\rho^2(C')$, $\rho^3(C'')$, $\rho^4(C')$ and $\rho^5(C'')$ are edge-disjoint $t$-cycles of $H_i$.

Next we describe the constructions of $C'$ and $C''$ in $H_i$. First we consider the case $k \geq 4$. Initially, we construct two paths $P'_1$ and $P'_2$ (in $H_i$) which will be used to construct the cycles $C'$ and $C''$. The sections of the paths $P'_1$ and $P'_2$ in the last three layers of $H_i$, namely, $Z_{i+t_k−2}$, $Z_{i+t_k−1}$ and $Z_{i+t_k}$ are shown in Fig. 8(a). These sections are extended further as follows: observe (from Fig. 8(a)) that these sections of the paths have their end vertices in $Z_{i+t_k−2}$. We shall build up these sections so that the resulting sections of $P'_1$ and $P'_2$ have their end vertices in $Z_{i+t_k−3}$. The end vertices of $P'_1$ (resp. $P'_2$) in Fig. 8(a) are in the first (resp. second) and third (resp. fourth) columns of $H_i$. Add to each of these paths edges having distances 1, 2, 4 and 5 (from $Z_{i+t_k−3}$ to $Z_{i+t_k−2}$) as shown in Fig. 8(b), so that the end vertices of the resulting sections of $P'_1$ (resp. $P'_2$) are in $Z_{i+t_k−3}$ and in columns three (resp. four) and five (resp. six) of $H_i$.

By “cyclically permuting” the columns of vertices of the graph $H_i$, the end vertices of the sections of the paths $P'_1$ (resp. $P'_2$) in $Z_{i+t_k−3}$ can be brought to columns one (resp. two) and three (resp. four) of the resulting graph (see Fig. 8(c)); here the graph is obtained by the “cyclic rotation” of the columns (column $i \rightarrow$ column $i + 4$, where addition is taken modulo 6 with residues 1, 2, ..., 6). Again attach to each of these resulting paths (in Fig. 8(c)), the edges of distances 1, 2, 4 and 5 (from $Z_{i+t_k−4}$ to $Z_{i+t_k−3}$) so that the sections of the paths $P'_1$ and $P'_2$ have their end vertices in $Z_{i+t_k−4}$. Again make cyclic rotation of the columns of $H_i$ so that the ends of the resulting section of $P'_1$ (resp. $P'_2$) are in columns one (resp. two) and three (resp. four). Extend these paths, as described above, up to the layer $Z_{i+t_2}$. Call
Theorem 2.5. For any prime \( p \) with each entry of the resulting paths two) and three (resp. four). Then complete these paths into cycles \( C' \) and \( C'' \) as shown in Fig. 8(d).

Cycles \( C' \) and \( C'' \) have the following properties: both the cycles \( C' \) and \( C'' \) together contain exactly two edges in each distance, namely, 1, 2, 4 and 5 between any two consecutive layers. The cycle \( C' \) (resp. \( C'' \)) uses two (resp. six) edges between \( Z_i \) and \( Z_{i+t_1} \) and six (resp. two) edges between \( Z_{i+t_1} \) and \( Z_{i+t_2} \); but in any two other consecutive layers each of them uses exactly four edges of distinct distances, namely, 1, 2, 4 and 5. The edges between \( Z_i \) and \( Z_{i+t_1} \) and \( Z_{i+t_1} \) and \( Z_{i+t_2} \) are suitably chosen, to include in \( C' \) and \( C'' \), so that \( C' \), \( C'' \), \( p^2(C') \), \( p^2(C'') \), \( p^4(C') \) and \( p^4(C'') \) are edge-disjoint \( t \)-cycles, where \( \rho \) is the permutation \( \rho = (Z_i)(Z_{i+t_1})(Z_{i+t_2}) \cdots (Z_{i+t_k}) \) and \( (Z_k) \) stands for the permutation \( (w_{2k-1}v_{2k-1}w_{2k}v_{2k-1}w_{2k}) \). For the case \( k = 3 \), the cycles \( C' \) and \( C'' \) are shown in Fig. 9. This completes the proof. □

Remark 2.4. Let the partite sets (layers) of the complete tripartite graph \( C_3 \times \overline{K}_m, m \geq 1 \), be \( \{u_1, u_2, \ldots, u_m\} \), \( \{v_1, v_2, \ldots, v_m\} \) and \( \{w_1, w_2, \ldots, w_m\} \). Consider a latin square \( \mathcal{L} \) of order \( m \). We associate a triangle of \( C_3 \times \overline{K}_m \) with each entry of \( \mathcal{L} \) as follows: if \( k \) is the \((i,j)\)th entry of \( \mathcal{L} \), then the triangle of \( C_3 \times \overline{K}_m \) corresponding to \( k \) is \( (u_i, v_j, w_k) \). Clearly the triangles corresponding to the entries of \( \mathcal{L} \) decompose \( C_3 \times \overline{K}_m \); see e.g. [3].

Theorem 2.5. For any prime \( p \geq 11 \), \( C_p \mid C_3 \times K_m \) if and only if \( m \equiv 0 \) or 1 \((\mod p)\).

Proof. The necessity is obvious. We prove the sufficiency in two cases.

Case 1. \( m \equiv 1 \) \((\mod p)\).

Let \( m = pk + 1 \).

Subcase 1.1. \( k \neq 2 \).

Let the partite sets of the tripartite graph \( C_3 \times K_m \) be \( U = \{u_0\} \cup \bigcup_{i=1}^k \{u_{i1}, u_{i2}, \ldots, u_{ip}\} \), \( V = \{v_0\} \cup \bigcup_{i=1}^k \{v_{i1}, v_{i2}, \ldots, v_{ip}\} \), \( W = \{w_0\} \cup \bigcup_{i=1}^k \{w_{i1}, w_{i2}, \ldots, w_{ip}\} \); we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. By the definition of the tensor product, \( \{u_0, v_0, w_0\} \) and \( \{u_{i1}, v_{i2}, w_{i3}\} \), \( 1 \leq j \leq p \), are independent sets and the subgraph induced by each of the sets \( U \cup V, V \cup W \) and \( W \cup U \) is isomorphic to \( K_{m,m} - F_0 \), where \( F_0 \) is the 1-factor of distance zero in \( K_{m,m} \).

We obtain a new graph from \( H = (C_3 \times K_m) - \{u_0, v_0, w_0\} \cong C_3 \times K_{pk} \) as follows: for each \( i, 1 \leq i \leq k \), identify the sets of vertices \( \{u_{i1}, u_{i2}, \ldots, u_{ip}\} \), \( \{v_{i1}, v_{i2}, \ldots, v_{ip}\} \) and \( \{w_{i1}, w_{i2}, \ldots, w_{ip}\} \) with new vertices \( u', v', w' \), respectively; two new vertices are adjacent if and only if the corresponding sets of vertices in \( H \) induce a complete bipartite subgraph \( K_{p,p} \) or \( K_{p,p} - F \), where \( F \) is a 1-factor of \( K_{p,p} \). This defines the graph isomorphic to \( C_3 \times \overline{K}_k \) with partite sets \( \{u^1, u^2, \ldots, u^k\}, \{v^1, v^2, \ldots, v^k\} \) and \( \{w^1, w^2, \ldots, w^k\} \). Consider an idempotent latin square \( \mathcal{L} \) of order \( k, k \neq 2 \) (which exists, see [9]). To complete the proof of this subcase, we associate with entries of \( \mathcal{L} \) edge-disjoint subgraphs of...
Theorem 2.6 \cite{Cavenagh}. For $k \geq 3$, $C_3 \ast \overline{K}_m$ can be decomposed into cycles of length $k$ if and only if $k \mid 3m^2$ and $k \leq 3m$. \hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[scale=0.5]{fig9.png}
\caption{Solid edges denote $C^\prime$ and the broken edges denote $C^\prime\prime$}
\end{figure}

$C_3 \ast \overline{K}_m$ which are decomposable by $C_p$. The $i$th diagonal entry of $\mathcal{L}$ corresponds to the triangle $(u^i, v^i, w^i)$, $1 \leq i \leq k$, of $C_3 \ast \overline{K}_k$, see Remark 2.4. The subgraph of $H$ corresponding to the triangle of $C_3 \ast \overline{K}_k$ is isomorphic to $C_3 \times K_p$. For each triangle $(u^i, v^i, w^i)$, $1 \leq i \leq k$, of $C_3 \ast \overline{K}_k$ corresponding to the $i$th diagonal entry of $\mathcal{L}$, associate the subgraph of $C_3 \times K_m$ induced by vertices $\{u_0, u_1, u_2, \ldots, u_p\} \cup \{v_0, v_1, v_2, \ldots, v_p\} \cup \{w_0, w_1, w_2, \ldots, w_p\}$; since this subgraph is isomorphic to $C_3 \times K_{p+1}$, it can be decomposed into cycles of length $p$, by Lemma 2.3. Again, if we consider the subgraph of $H$ corresponding to the triangle of $C_3 \ast \overline{K}_k$, which corresponds to a non-diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 \ast \overline{K}_p$. By Theorem A, $C_3 \ast \overline{K}_p$ can be decomposed into cycles of length $p$. Thus we have decomposed $C_3 \times K_m$ into cycles of length $p$ when $k \neq 2$.

Subcase 1.2. $k = 2$.

By Theorem D, $C_p \mid K_{2p+1}$ and hence we write $C_3 \times K_{2p+1} \cong K_{2p+1} \times C_3 = (C_p \times C_3) \oplus (C_p \times C_3) \oplus \cdots \oplus (C_p \times C_3)$. Now $C_p \times C_3$ can be decomposed into cycles of length $p$; see the proof of Lemma 2.1. This proves that $C_p \mid C_3 \times K_{2p+1}$.

Case 2. $m \equiv 0 \pmod{p}$.

Let $m = pk$. If $k = 2$, then the result follows from Lemma 2.2. Hence we may assume that $k \neq 2$. Let the partite sets of the tripartite graph $C_3 \times K_m$ be $U = \bigcup_{i=1}^{k} \{u_1^i, u_2^i, \ldots, u_p^i\}$, $V = \bigcup_{i=1}^{k} \{v_1^i, v_2^i, \ldots, v_p^i\}$ and $W = \bigcup_{i=1}^{k} \{w_1^i, w_2^i, \ldots, w_p^i\}$. We assume that the vertices of the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, from $C_3 \times K_m = C_3 \times K_{pk}$ we obtain the graph $C_3 \ast \overline{K}_p$ with partite sets $\{u^1, u^2, \ldots, u^k\}, \{v^1, v^2, \ldots, v^k\}$ and $\{w^1, w^2, \ldots, w^k\}$.

Consider an idempotent latin square $\mathcal{L}$ of order $k$, $k \neq 2$. The diagonal entries of $\mathcal{L}$ correspond to the triangles $(u^i, v^i, w^i)$, $1 \leq i \leq k$, of $C_3 \ast \overline{K}_k$. If we consider the subgraph of $C_3 \times K_m$ corresponding to a triangle of $C_3 \ast \overline{K}_k$, which corresponds to a diagonal entry of $\mathcal{L}$, then it is isomorphic to $C_3 \times K_p$. By Lemma 2.1, $C_p \mid C_3 \times K_p$. Again, as in the previous case, the triangle of $C_3 \ast \overline{K}_k$ corresponding to a non-diagonal entry of $\mathcal{L}$, corresponds to a subgraph of $C_3 \times K_m$ isomorphic to $C_3 \ast \overline{K}_p$; by Theorem A, $C_p \mid C_3 \ast \overline{K}_p$. \hfill $\square$
3. $C_p$-decompositions of $C_5 \times K_m$ and $C_5 \ast K_m$

Lemma 3.1. For any odd integer $t \geq 5$, $C_t || C_5 \times K_t$.

Proof. Clearly, $C_t \times K_t \cong K_t \times C_5 \cong (C_t \times C_5) \oplus (C_t \times C_5) \oplus \cdots \oplus (C_t \times C_5)$. We shall show that $C_t \times C_5$ can be factorized into $t$-cycles. Let $V(C_t) = \{x_1, x_2, \ldots, x_t\}$ and $V(C_5) = \{y_1, y_2, y_3, y_4, y_5\}$. Let $V(C_t \times C_5) = V(C_t) \times V(C_5)$. Let $X_i = \{(x_i, y_j) \mid 1 \leq j \leq 5\}, 1 \leq i \leq t$. Then $E(C_t \times C_5) = \bigcup_{i=1}^{t} \{F_1(X_i, X_{i+1}) \cup F_4(X_i, X_{i+1})\}$, where addition in the subscripts is taken modulo $t$ with residues $1, 2, \ldots, t$. The graph $C_t \times C_5$ can be factorized into two $C_t$-factors, say $F_1'$ and $F_2'$, as follows:

$$F_1' = \left( \bigcup_{i=1}^{(t-5)/2} \{F_1(X_{2i-1}, X_{2i}) \cup F_4(X_{2i}, X_{2i+1})\} \right) \cup F_1(X_{t-4}, X_{t-3}) \cup F_1(X_{t-3}, X_{t-2}) \cup F_1(X_{t-2}, X_{t-1}) \cup F_1(X_{t-1}, X_t) \cup F_1(X_t, X_1)$$

and

$$F_2' = \left( \bigcup_{i=1}^{(t-5)/2} \{F_4(X_{2i-1}, X_{2i}) \cup F_1(X_{2i}, X_{2i+1})\} \right) \cup F_4(X_{t-4}, X_{t-3}) \cup F_4(X_{t-3}, X_{t-2}) \cup F_4(X_{t-2}, X_{t-1}) \cup F_4(X_{t-1}, X_t) \cup F_4(X_t, X_1).$$

This completes the proof. □

Lemma 3.2. For any prime $p \geq 11$, $C_p | C_5 \times K_{2p}$.

Proof. Let the partite sets (five layers) of the 5-partite graph $G = C_5 \times K_{2p}$ be $U = \{u_1, u_2, \ldots, u_{2p}\}$, $V = \{v_1, v_2, \ldots, v_{2p}\}$, $W = \{w_1, w_2, \ldots, w_{2p}\}$, $X = \{x_1, x_2, \ldots, x_{2p}\}$ and $Y = \{y_1, y_2, \ldots, y_{2p}\}$. Let us assume that the vertices having the same subscript are the corresponding vertices of the partite sets, that is, vertices in a column of $G$. Let $Z_i = \{u_{2i-1}, v_{2i-1}, w_{2i-1}, x_{2i-1}, y_{2i-1}\}, 1 \leq i \leq p$, be the vertices of columns $2i - 1$ and $2i$ of $G$ and it induces a 10-cycle in $G$, say, $C_i = (u_{2i-1}, v_{2i-1}, w_{2i-1}, x_{2i-1}, y_{2i-1}, u_{2i-1}, v_{2i-1}, w_{2i-1}, x_{2i-1}, y_{2i-1})$. We associate with $G$ the complete graph $K_p$ as follows: to each subset $Z_i$ of vertices of $G$ introduce a vertex $z_i$ and join any pair of distinct vertices $z_i$ and $z_j$ by an edge. The set $\{z_1, z_2, \ldots, z_p\}$ of vertices thus induces the graph $K_p$. We associate with each edge $z_iz_j$ of $K_p$ the set of edges between $Z_i$ and $Z_j$ in $G$ (see Fig. 10). Throughout the proof of this lemma the order of occurrence of the vertices of $Z_i$ is assumed to be $u_{2i-1}, v_{2i-1}, w_{2i-1}, x_{2i-1}, y_{2i-1}$ in all the figures, see Fig. 10.

We divide the proof into two cases.

Case 1. $p \neq 13$.

In this paragraph we give the idea behind the proof of the theorem. The proof goes as follows: we decompose the $K_p$ into paths $\mathcal{P} = \{P^1, P^2, \ldots, P^p\}$ such that each $P^i$ is of length $(p - 1)/2$ and contains the edges having distances
and, distance of any two edges of each of these paths are distinct. Let (Those edges which are not on the 10
exactly one of the paths, namely, $P_i$)
be the subgraph of $G$ obtained by deleting the 10 $p$-cycles that are chosen in the $H_i$’s in $G$. Then we decompose $G'$ into $p$-cycles. This is achieved by again finding a Hamilton cycle decomposition ‘$\mathcal{P}$’ of $K_p$ such that each cycle in the decomposition contains only edges of same distance (this is possible since $p$ is a prime) and associating with each Hamilton cycle ‘$\mathcal{P}$’ a subgraph of $G'$ and decomposing it into cycles of length $p$.

Now we shall give the proof. As we mentioned earlier, first we decompose $K_p$ into $p$ paths of length $(p - 1)/2$ each and, distance of any two edges of each of these paths are distinct. Let $t_j = 0 + 1 + 2 + \cdots + j$, $0 \leq j \leq (p - 1)/2$. For each $i$, $1 \leq i \leq p$, we define the path $P^i = z_{i+0}z_{i+1}z_{i+2} \cdots z_{i+(p-1)/2}$, where addition in the subscripts is taken modulo $p$ with residues 1, 2, \ldots, $p$, of length $(p - 1)/2$ in $K_p$. Note that the edge $z_{j+1}z_{i+j}$ is of distance $j$ in $K_p$, and hence the path $P^i$, $1 \leq i \leq p$, consists of edges of distances 1, 2, \ldots, $(p - 1)/2$. Thus $\mathcal{P} = \{P^i \mid 1 \leq i \leq p\}$ is a required path decomposition of $K_p$.

Let $H_i$ be the union of the subgraph of $G$ associated with the edges of the path $P^i$ of $K_p$ and $C'$, the 10-cycle induced by $Z_i (=Z_i+0)$, see Fig. 11. Note that $\bigcup_{i=1}^{p} H_i = G$, since each vertex $z_i$ of $K_p$ appears as the initial vertex of exactly one of the paths, namely, $P^i$ of $\mathcal{P}$. First we decompose the graph $H_i$ into $10$ $p$-cycles and $(10(p - 1)/2)$ edges. (Those edges which are not on the 10 $p$-cycles will be used later.) To obtain the 10 $p$-cycles of $H_i$, $1 \leq i \leq p$, first we construct two “base $p$-cycles”, say, $C'$ and $C''$ as shown in Fig. 12. Let $\rho = (Z_{i+0}Z_{i+1})Z_{i+2}Z_{i+3} \cdots Z_{i+(p-1)/2}$ be the permutation, where $(Z_k)$ stands for the permutation $(u_{2k-1} v_{2k-1} w_{2k-1} x_{2k-1} y_{2k-1} w_{2k} v_{2k-1} u_{2k} x_{2k-1} y_{2k})$. Now the required 10 $p$-cycles are $C', C'', \rho^2(C'), \rho^2(C''), \rho^4(C'), \rho^4(C''), \rho^8(C'), \rho^8(C'')$. $\rho^5(C')$ and $\rho^5(C'')$.

Let $G'$ be the graph obtained from $G$ by deleting the 10 $p$-cycles obtained above from the $H_i$’s. Since $C'_i$, $1 \leq i \leq p$, is covered by these 10 $p$-cycles, the subgraph induced by $Z_i$ in $G'$ is the empty graph. Again, we associate with each edge $z_iz_j$ of $K_p$ the subgraph of $G'$, induced by $Z_i \cup Z_j$. Let $H^j_i$, $1 \leq j \leq (p - 1)/2$, $1 \leq i \leq p$, be the subgraph of $G'$ induced by $Z_i+1 \cup Z_j$, that is, $H^j_i$ contains those edges of $G'$ corresponding to the edge of distance $j$ in $P^i$ of $K_p$. ($P^i$ is defined above). Clearly, for each $i$, the graph $H^j_i$ is isomorphic to one of the graphs in \{ $H', H'', H'''$ \}, see Fig. 13. One can observe that $H'''$ is nothing but a redrawing of $H''$. An isomorphism between $H''$ and $H'''$ is obtained as follows: let the vertices of $H''$ (resp. $H'''$) in one part be $a_1, a_2, \ldots, a_{10}$ (resp. $a_1', a_2', \ldots, a_{10}'$) and the other part be $b_1, b_2, \ldots, b_{10}$ (resp. $b_1', b_2', \ldots, b_{10}'$), in order. Then the required isomorphism is $a_i \rightarrow a_i'$ and $b_i \rightarrow b_i'$. Consequently, we shall use $H''$ in the place of $H'''$ for our future purpose.
Let $G_j = \bigcup_{i=1}^{p} H_j^i$, $1 \leq j \leq (p - 1)/2$. Observe that $G_j$ is nothing but the subgraph of $G'$ which corresponds to all the edges of distance $j$ in $K_p$; also note that the set of edges of distance $j$ in $K_p$ induce a Hamilton cycle of $K_p$ (since $p$ is a prime). Hence $G_j$, $j \neq (p - 3)/2$, $(p - 7)/2$, is isomorphic to the graph obtained by identifying the first and last layers of the graph $G_j'$ of Fig. 14(a) and for $j = (p - 3)/2$ or $(p - 7)/2$, $G_j$ is obtained by identifying the last and first layers of the graph $G_j''$ of Fig. 14(b) (since $H''$ and $H'''$ are isomorphic) wherein the additions in the subscripts are taken modulo $p$ with residues $1, 2, \ldots, p$.

To complete the proof, it is enough to prove that $G_j'$ and $G_j''$ have decompositions into paths of length $p$ each and each path has its end vertices on the same column (and so, on superimposing the last layer with the first layer yield the required cycle decomposition of $G_j$). Now $G_j'$, $j \neq (p - 3)/2$, $(p - 7)/2$, can be factorized into two $P_{p+1}$-factors, say, $F_1'$ and $F_2'$ as follows:

\[
F_1' = \left( \bigcup_{k=1}^{(p-5)/2} \{ F_6(Z_1+(2k-2)j, Z_1+(2k-1)j) \cup F_4(Z_1+(2k-1)j, Z_1+2kj) \} \right)
\]
\[
\bigcup \left( \bigcup_{k=p-5}^{p-1} F_4(Z_1+kj, Z_1+(k+1)j) \right)
\]

and
In $G''_j$, to find the required decomposition first we construct four “base paths” of length $p$ each. Let $\rho = (Z_1)(Z_1+j) (Z_1+j+1) \cdots (Z_1+pj)$ be the permutation, where $(Z_k)$ stands for the permutation $(u_{2k-1}v_{2k}w_{2k-1}x_{2k}y_{2k-1} w_{2k}v_{2k-1}w_{2k-1}x_{2k-1}y_{2k})$ and the additions in the subscripts are taken modulo $p$ with residues $1, 2, \ldots, p$. The required path decomposition consists of four base paths and the paths obtained by letting the permutations $\rho^2$, $\rho^3$, $\rho^6$ and $\rho^8$ act on them. The four base paths of $G''_j$ are obtained by attaching copies of $G''_1$, $G''_2$ and $G''_3$, shown in Fig. 15, one over the other.
For example if \( p = 11 \), then consider the four paths in \( G''_{1} \) alone. If \( p = 17 \), put \( G''_{2} \) over \( G''_{1} \), that is, the bottom layer of \( G''_{2} \) is superimposed with the first layer of \( G''_{1} \). Similarly, if \( p = 19 \), we use \( G''_{1} \) and two copies of \( G''_{3} \) by keeping one over the other, successively, and so on (as for any prime \( p \geq 19 \), \( p + 1 \) can be obtained by adding multiples of 12, 6 and 4).

Case 2. \( p = 13 \).

Clearly, \( K_{13} = P_{4}^{1} \oplus P_{4}^{2} \oplus \cdots \oplus P_{4}^{13} \oplus C_{13}^{1} \oplus C_{13}^{2} \oplus C_{13}^{3} \), where \( P_{4}^{i} = z_{i}z_{i+4}z_{i-1}z_{i+5} \), \( 1 \leq i \leq 13 \), and addition in the subscripts is taken modulo 13 with residues 1, 2, ..., 13 and \( C_{13}^{i} \), \( 1 \leq i \leq 3 \), is the 13-cycle induced by the edges of distance \( i \) in \( K_{13} \). All the edges of distance 4, 5, 6 in \( K_{13} \) are covered by the paths \( P_{4}^{i} \), \( 1 \leq i \leq 13 \), and the rest of the edges of \( K_{13} \) are covered by the cycles \( C_{13}^{i} \), \( 1 \leq i \leq 3 \). We associate with each \( P_{4}^{i} \) of \( K_{13} \), the subgraph \( H_{i} \) of \( G \), isomorphic to the graph of Fig. 16(a).

Let \( \rho = (Z_{i})(Z_{i+4})(Z_{i-1})(Z_{i+5}) \) be a permutation, where \( (Z_{k}) \) stands for the permutation \((u_{2k-1}v_{2k}w_{2k-1}x_{2k}y_{2k-1}u_{2k}v_{2k-1}w_{2k}x_{2k-1}y_{2k})\). Let \( C' \) and \( C'' \) be two “base 13-cycles” of \( H_{i} \), shown in Fig. 16(b). Now \( C', C'', \rho^{2}(C'), \rho^{3}(C'), \rho^{4}(C'), \rho^{5}(C'), \rho^{6}(C'), \rho^{8}(C') \) and \( \rho^{8}(C'') \) are the 13-cycle decomposition of \( H_{i} \). Observe that all the edges of the cycles \( C_{i} \), \( 1 \leq i \leq 13 \), induced by \( Z_{i} \) are covered by these 130 13-cycles. Let \( G' \) be the subgraph of \( G \) obtained by deleting these 130 13-cycles. Let \( G'_{i} \) be the subgraph of \( G' \) corresponding to the cycle \( C_{13}^{i} \), \( 1 \leq i \leq 3 \), of \( K_{13} \); \( G'_{i} \) is isomorphic to the graph of Fig. 17 by superimposing the first layer vertices with the last layer vertices.
To complete the proof it is enough to decompose the graph $G_i^t$, $1 \leq i \leq 3$, into 13-cycles. In fact, $G_i^t$ can be factorized into four $C_{13}$-factors, say $F_1^t$, $F_2^t$, $F_3^t$ and $F_4^t$ as follows:

$$F_1^t = \left( \bigcup_{i=1}^{7} F_1(Z_i, Z_{i+1}) \right) \cup F_9(Z_8, Z_9) \cup F_4(Z_9, Z_{10}) \cup F_1(Z_{10}, Z_{11}) \cup F_1(Z_1, Z_{12})$$
$$\quad \cup F_9(Z_{12}, Z_{13}) \cup F_9(Z_{13}, Z_1),$$

$$F_2^t = \left( \bigcup_{i=1}^{7} F_0(Z_i, Z_{i+1}) \right) \cup F_1(Z_8, Z_9) \cup F_9(Z_9, Z_{10}) \cup F_6(Z_{10}, Z_{11}) \cup F_9(Z_{11}, Z_{12})$$
$$\quad \cup F_1(Z_{12}, Z_{13}) \cup F_1(Z_{13}, Z_1),$$

$$F_3^t = \left( \bigcup_{i=1}^{6} F_6(Z_i, Z_{i+1}) \right) \cup F_4(Z_7, Z_8) \cup F_4(Z_8, Z_9) \cup F_6(Z_9, Z_{10}) \cup F_4(Z_{10}, Z_{11})$$
$$\quad \cup F_6(Z_{11}, Z_{12}) \cup F_4(Z_{12}, Z_{13}) \cup F_6(Z_{13}, Z_1) \quad \text{and}$$

$$F_4^t = \left( \bigcup_{i=1}^{6} F_4(Z_i, Z_{i+1}) \right) \cup F_6(Z_7, Z_8) \cup F_6(Z_8, Z_9) \cup F_1(Z_9, Z_{10}) \cup F_9(Z_{10}, Z_{11})$$
$$\quad \cup F_4(Z_{11}, Z_{12}) \cup F_6(Z_{12}, Z_{13}) \cup F_4(Z_{13}, Z_1).$$

This completes the proof. \qed

**Lemma 3.3.** For any odd integer $t \geq 11$, $C_t \mid C_{5} \times K_{t+1}$.

**Proof.** Let the partite sets (five layers) of the 5-partite graph $G = C_{5} \times K_{t+1}$ be $U = \{u_1, u_2, \ldots, u_{t+1}\}$, $V = \{v_1, v_2, \ldots, v_{t+1}\}$, $W = \{w_1, w_2, \ldots, w_{t+1}\}$, $X = \{x_1, x_2, \ldots, x_{t+1}\}$ and $Y = \{y_1, y_2, \ldots, y_{t+1}\}$. Let us assume that the vertices having the same subscript are the corresponding vertices of the partite sets, that is, vertices in a column of $G$. We prove this lemma in two cases.

**Case 1.** $t \equiv 3 \pmod{4}$.

Let $t = 4k + 3$. Consider the $t$-cycle $C = (w_{2k+1}, w_1, u_1, u_2, w_2, u_2, u_{2k-1}, v_3, u_{2k-2}, \ldots, v_1, u_{k+3}, v_k, u_k, u_{k-2}, v_k, u_{k-2}, v_k, u_{k+2}, v_{k-4}, \ldots, v_{4k+3}, v_{2k}, w_{4k+2}, z_{2k+1}, y_{2k})$, where the additions in the subscripts are taken modulo $t + 1$ with residues $1, 2, \ldots, t + 1$, in $G$. Note that the edges in the cycle $C$ are of distances 1, 2, \ldots, $t$, where distances are taken in the order $(U, V)$, $(V, W)$, $(W, X)$, $(X, Y)$ and $(Y, U)$. Letting the permutation $\sigma = (u_1 u_2 u_3 \cdots u_{t+1})(v_1 v_2 \cdots v_{t+1})(w_1 w_2 \cdots w_{t+1})(x_1 x_2 \cdots x_{t+1})(y_1 y_2 \cdots y_{t+1})$ and its powers act on $C$ give $(t + 1)$ $t$-cycles, say, $C_1^t, C_2^t, \ldots, C_{t+1}^t$. Now let the permutation $(u_1 v_1 w_1 x_1 y_1)(u_2 v_2 w_2 x_2 y_2) \cdots (u_{t+1} v_{t+1} w_{t+1} x_{t+1} y_{t+1})$ and its powers act on the $t$-cycles $C_1^t, C_2^t, \ldots, C_{t+1}^t$ give us a required $t$-cycle decomposition of $G$.

**Case 2.** $t \equiv 1 \pmod{4}$.

Let $t = 4k + 1$. As in the proof Lemma 3.2, let $Z_i = \{u_{2i-1}, v_{2i}, w_{2i-1}, x_{2i}, y_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}, x_{2i-1}, y_{2i}\}$, $1 \leq i \leq 2k+1$. The subset $Z_i$, $1 \leq i \leq 2k+1$, induces a 10-cycle, say, $C_i = (u_{2i-1}, v_{2i}, w_{2i-1}, x_{2i}, y_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}, x_{2i-1}, y_{2i})$ in $G$. To each subset $Z_i$ of vertices of $G$, introduce a vertex $z_i$ and join any pair of distinct vertices $z_i$ and $z_j$ by an edge. The set of $z_1, z_2, \ldots, z_{2k+1}$ vertices induce the graph $K_{2k+1}$. We associate with each edge $z_i z_j$ of $K_{2k+1}$ the set of edges between $Z_i$ and $Z_j$ in $G$, see Fig. 10. A word of caution! Throughout this lemma, in all the figures, the order of occurrence of the vertices of $Z_i$ is assumed to be $u_{2i-1}, v_{2i}, w_{2i-1}, x_{2i}, y_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}, x_{2i-1}, y_{2i}$ or a cyclic permutation of it (for example $w_{2i-1}, x_{2i}, y_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}, x_{2i-1}, y_{2i}$ or a cyclic permutation of it (for example $u_{2i-1}, x_{2i}, y_{2i-1}, u_{2i}, v_{2i-1}, w_{2i}, x_{2i-1}, y_{2i}$)).

First we show that $K_{2k+1}$ can be decomposed into paths of length $k$ such that each path has edges of distances 1, 2, \ldots, $k$. Let $t_j = 1 - 2 + 3 - 4 + \cdots + (-1)^{i+1} j$, $1 \leq j \leq k$. We define a path $P_i = z_i z_{i+1} z_{i+2} \cdots z_{i+t_i}$, $1 \leq i \leq 2k+1$, where the additions in the subscripts are taken modulo $2k + 1$ with residues $1, 2, \ldots, 2k + 1$, of length $k$. Note that, for each $i$, the edges in $P_i$ are of distances 1, 2, \ldots, $k$. Thus $P = \{P_i \mid 1 \leq i \leq 2k + 1\}$ is a required path decomposition of $K_{2k+1}$.

Let $H_i$, $1 \leq i \leq 2k + 1$, be the union of the subgraph of $G$ corresponding to the path $P_i$ of $K_{2k+1}$ and $C_i$, the 10-cycle induced by $Z_i$, see Fig. 18. Note that $H_i$ contains exactly one of the 10-cycles induced by the $Z_i$'s. As each vertex of
The rest of the proof goes as follows: first we construct two “base $t$-cycles” $C’$ and $C”$ in $H_t$. Then we fix a suitable permutation $\rho$, so that $C’, C”, \rho^2(C’), \rho^2(C”), \rho^4(C’), \rho^4(C”), \rho^8(C’), \rho^8(C”)$, and $\rho^{8}(C’)$ and $\rho^{8}(C”)$ are edge-disjoint $t$-cycles of $H_t$

Next we describe the constructions of $C’$ and $C”$ in $H_t$. First we consider the case $k \geq 4$. Initially, we construct two paths $P_1$ and $P_2$ in $H_t$ which will be used to construct the cycles $C’$ and $C”$. The sections of the paths $P_1$ and $P_2$ in the last three layers of $H_t$, namely, $Z_{i+1}$, $Z_{i+2}$ and $Z_{i+3}$ are shown in Fig. 19(a). These sections are extended further as follows: observe (from Fig. 19(a)) that these sections of the paths have their end vertices in $Z_{i+1}$ and $Z_{i+2}$. We shall build up these sections so that the resulting sections of $P_1$ and $P_2$ have their end vertices in $Z_{i+1}$ and $Z_{i+2}$. The end vertices of $P_1'$ (resp. $P_2'$) in Fig. 19(a) are in the fifth (resp. sixth) and ninth (resp. tenth) columns of $H_t$. Add to each of these paths edges having distances 1, 4, 6 and 9 (from $Z_{i+1}$ to $Z_{i+2}$) as shown in Fig. 19(b), so that the end vertices of the resulting section of $P_1'$ (resp. $P_2'$) are in columns five (resp. six) and nine (resp. ten) of $H_t$. By “cyclically permuting” the columns of vertices of the graph $H_t$, the end vertices of the section of the path $P_1'$ (resp. $P_2'$) in columns five (resp. six) and nine (resp. ten) can be brought to columns five (resp. six) and nine (resp. ten) of the resulting graph (see Fig. 19(c)). The rest of the proof goes as follows: first we construct two “base $t$-cycles” $C’$ and $C”$ in $H_t$. This completes the proof.

$K_{2k+1}$ happens to be the origin of exactly one path $P_i$ of $\mathcal{P}$, $\bigcup_{i=1}^{2k+1} H_i = G$ and hence, it is enough to prove that the graph $H_i$, $1 \leq i \leq 2k + 1$, has a $t$-cycle decomposition.
Theorem 3.4 (Manikandan and Paulraja [12]). For $m \geq 3$, $k \geq 1$, $C_{2k+1} \mid C_{2k+1} \times K_m$.

Theorem 3.5 (Manikandan and Paulraja [12]). For $m$, $k \geq 1$, $C_{2k+1} \mid C_{2k+1} \ast \overline{K}_m$.

Remark 3.6. Let the vertex set of the 5-partite graph $C_5 \ast \overline{K}_m$, $m \neq 2$, be $\{u_1, u_2, \ldots, u_m\}$, $\{v_1, v_2, \ldots, v_m\}$, $\{w_1, w_2, \ldots, w_m\}$, $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2, \ldots, y_m\}$. From Theorem 3.4, $C_5 \ast \overline{K}_m$ has a 5-cycle decomposition containing the cycles of length 5 $\{(u_i, v_i, w_i, x_i, y_i)|1 \leq i \leq m\}$, since $C_5 \ast \overline{K}_m$ differs from $C_5 \times K_m$ only by $m$ vertex disjoint copies of $C_5$, a 2-factor of $C_5 \ast \overline{K}_m$.

Theorem 3.7. For any prime $p \geq 11$, $C_p \mid C_5 \times K_m$ if and only if $m \equiv 0$ or $1 \pmod{p}$.

Proof. The proof of the necessity is obvious. We prove the sufficiency in two cases.
Case 1. $m \equiv 1 \pmod{p}$.
Let $m = pk + 1$.

Subcase 1.1. $k \neq 2$.

Let the partite sets (the layers) of the $5$-partite graph $C_5 \times K_m$ be $U = \{u_0\} \cup \bigcup_{i=1}^{k} \{u_i, u_{i+1}, \ldots, u_{p_i}\}$, $V = \{v_0\} \cup \bigcup_{i=1}^{k} \{v_i, v_{i+1}, \ldots, v_{p_i}\}$, $W = \{w_0\} \cup \bigcup_{i=1}^{k} \{w_i, w_{i+1}, \ldots, w_{p_i}\}$, $X = \{x_0\} \cup \bigcup_{i=1}^{k} \{x_i, x_{i+1}, \ldots, x_{p_i}\}$ and $Y = \{y_0\} \cup \bigcup_{i=1}^{k} \{y_i, y_{i+1}, \ldots, y_{p_i}\}$, where we assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. From the definition of the tensor product, $\{u_0, v_0, w_0, x_0, y_0\}$ and $\{u_1, v_1, w_1, x_1, y_1\}$, $1 \leq j \leq p$, $1 \leq i \leq k$, are independent sets of $C_5 \times K_m$, and the subgraph induced by each of the sets $U \cup V$, $V \cup W$, $W \cup X$, $X \cup Y$ and $Y \cup U$ is isomorphic to $K_{m, m} - F_0$, where $F_0$ is the 1-factor of distance zero.

We obtain a new graph from $H = (C_5 \times K_m) - \{u_0, v_0, w_0, x_0, y_0\} \cong C_5 \times K_{pk}$ as follows: for each $i$, $1 \leq i \leq k$, identify the subsets of vertices $\{u_1, u_2, \ldots, u_p\}$, $\{v_1, v_2, \ldots, v_p\}$, $\{w_1, w_2, \ldots, w_p\}$, $\{x_1, x_2, \ldots, x_p\}$ and $\{y_1, y_2, \ldots, y_p\}$ into new vertices $u^i$, $v^i$, $w^i$, $x^i$ and $y^i$, respectively, and two new vertices are adjacent if and only if the corresponding sets of vertices in $H$ induce a complete bipartite subgraph $K_{p, p}$ or $K_0, p - F$, where $F$ is a 1-factor of $K_{p, p}$. The new graph thus obtained is isomorphic to the graph $C_5 \star \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$, $\{w^1, w^2, \ldots, w^k\}$, $\{x^1, x^2, \ldots, x^k\}$ and $\{y^1, y^2, \ldots, y^k\}$. The graph $C_5 \star \overline{K}_k$ has a $C_5$-decomposition containing the 5-cycles $(\{u^i, v^i, w^i, x^i, y^i\} | 1 \leq i \leq k)$, by Remark 3.6.

The subgraph of $H$ corresponding to these $k$ 5-cycles of the graph $C_5 \star \overline{K}_k$ is isomorphic to $k$ vertex disjoint copies of $C_5 \star K_p$. To each of these $k$ 5-cycles $(u^i, v^i, w^i, x^i, y^i)$, $1 \leq i \leq k$, associate the 5-partite subgraph of $C_5 \times K_m$ induced by $\{u_0, u_1, u_2, \ldots, u_p\} \cup \{v_0, v_1, v_2, \ldots, v_p\} \cup \{w_0, w_1, w_2, \ldots, w_p\} \cup \{x_0, x_1, x_2, \ldots, x_p\} \cup \{y_0, y_1, y_2, \ldots, y_p\}$, since this induced subgraph is isomorphic to $C_5 \times K_{p+1}$, it can be decomposed into $p$-cycles, by Lemma 3.3. Again, the subgraphs of $C_5 \times K_m$ corresponding to the other 5-cycles in the decomposition of $C_5 \star \overline{K}_k$ are isomorphic to $C_5 \star K_{p^2}$, and they can be decomposed into $p$-cycles, by Theorem A. Thus we have decomposed $C_5 \times K_m$ into $p$-cycles when $k \neq 2$.

Subcase 1.2. $k = 2$.

By Theorem D, $C_p \mid K_{2p+1}$ and hence $C_5 \times K_{2p+1} \cong K_{2p+1} \times C_5 \cong (C_p \times C_5) \oplus (C_p \times C_5) \oplus \cdots \oplus (C_p \times C_5)$. Further, $C_p \times C_5$ can be decomposed into $p$-cycles, see the proof of Lemma 3.1.

Case 2. $m \equiv 0 \pmod{p}$.

Let $m = pk$. If $k = 2$, then the result follows from Lemma 3.2. Hence we may assume that $k \neq 2$. Let the partite sets (the layers) of the 5-partite graph $C_5 \times K_m$ be $U = \bigcup_{i=1}^{k} \{u_i, u_{i+1}, \ldots, u_{p_i}\}$, $V = \bigcup_{i=1}^{k} \{v_i, v_{i+1}, \ldots, v_{p_i}\}$, $W = \bigcup_{i=1}^{k} \{w_i, w_{i+1}, \ldots, w_{p_i}\}$, $X = \bigcup_{i=1}^{k} \{x_i, x_{i+1}, \ldots, x_{p_i}\}$ and $Y = \bigcup_{i=1}^{k} \{y_i, y_{i+1}, \ldots, y_{p_i}\}$. We assume that the vertices having the same subscript and superscript are the corresponding vertices of the partite sets. As in the proof of Subcase 1.1, we obtain the graph $C_5 \star \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$, $\{w^1, w^2, \ldots, w^k\}$, $\{x^1, x^2, \ldots, x^k\}$ and $\{y^1, y^2, \ldots, y^k\}$, by suitable identification of vertices of $C_5 \times K_m$. By Remark 3.6, the graph $C_5 \star \overline{K}_k$ has a $C_5$-decomposition containing the 5-cycles $(u^i, v^i, w^i, x^i, y^i)$, $1 \leq i \leq k$. Consequently, each of these $k$ 5-cycles, associated with the corresponding 5-partite subgraph of $C_5 \times K_m$ induced by $\{u_1, u_2, \ldots, u_p\} \cup \{v_1, v_2, \ldots, v_p\} \cup \{w_1, w_2, \ldots, w_p\} \cup \{x_1, x_2, \ldots, x_p\} \cup \{y_1, y_2, \ldots, y_p\}$, can be decomposed into $p$-cycles, by Lemma 3.3. Corresponding to each of these $k$ 5-cycles, we associate the corresponding subgraph of $C_5 \times K_m$, namely, a subgraph isomorphic to $C_5 \star \overline{K}_p$. Now $C_5 \star \overline{K}_p$ can be decomposed into $p$-cycles, by Theorem A. Thus we have decomposed $C_5 \times K_m$ into $p$-cycles. \(\square\)

Theorem 3.8. For any prime $p \geq 11$, $C_p \mid C_5 \star \overline{K}_m$ if and only if $m \equiv 0 \pmod{p}$.

Proof. The proof of the necessity is obvious. We prove the sufficiency. Let $m = pk$. Let the partite sets of the 5-partite graph $C_5 \star \overline{K}_m$ be $\bigcup_{i=1}^{k} \{u_i, u_{i+1}, \ldots, u_{p_i}\}$, $\bigcup_{i=1}^{k} \{v_i, v_{i+1}, \ldots, v_{p_i}\}$, $\bigcup_{i=1}^{k} \{w_i, w_{i+1}, \ldots, w_{p_i}\}$, $\bigcup_{i=1}^{k} \{x_i, x_{i+1}, \ldots, x_{p_i}\}$ and $\bigcup_{i=1}^{k} \{y_i, y_{i+1}, \ldots, y_{p_i}\}$. As in the proof of Subcase 1.1 of Theorem 3.7, obtain the graph $C_5 \star \overline{K}_k$ with partite sets $\{u^1, u^2, \ldots, u^k\}$, $\{v^1, v^2, \ldots, v^k\}$, $\{w^1, w^2, \ldots, w^k\}$, $\{x^1, x^2, \ldots, x^k\}$ and $\{y^1, y^2, \ldots, y^k\}$, by suitable identification of vertices of $C_5 \star \overline{K}_m$. Now $C_5 \star \overline{K}_p$ can be decomposed into 5-cycles, by Theorem 3.5. Each 5-cycle of the 5-cycle decomposition of $C_5 \star \overline{K}_k$ corresponds to a subgraph of $C_5 \star \overline{K}_m$ isomorphic to $C_5 \star \overline{K}_p$. By Theorem A, $C_p \mid C_5 \star \overline{K}_p$. This completes the proof. \(\square\)
4. $C_p$-factorization of $K_m \times K_n$

**Theorem 4.1.** (Manikandan and Paulraja [12]). If $m$ is a positive integer, $m \neq 2, 6$, then the graph $C_3 \ast \overline{K}_m$ with partite sets $U = \{u_1, u_2, \ldots, u_m\}$, $V = \{v_1, v_2, \ldots, v_m\}$ and $W = \{w_1, w_2, \ldots, w_m\}$ has a $C_3$-factorization in which $\{(u_i, v_i, w_i) | 1 \leq i \leq m\}$ is a $C_3$-factor.

**Corollary 4.2** (Manikandan and Paulraja [12]). If $m \neq 2, 6$, then $C_3 \parallel C_3 \times K_m$.

**Theorem 4.3.** For prime $p \geq 11$, $C_p \parallel C_3 \times K_m$ if and only if $m \equiv 0 \pmod{p}$ except possibly $m = 2p, 6p$.

**Proof.** The proof of the necessity is obvious. We prove the sufficiency. Let $m = pk$. Let the partite sets of the tripartite graph $C_3 \times K_m$ be $U = \bigcup_{i=1}^{k} \{u_{1i}, u_{2i}, \ldots, u_{pi}\}$, $V = \bigcup_{i=1}^{k} \{v_{1i}, v_{2i}, \ldots, v_{pi}\}$ and $W = \bigcup_{i=1}^{k} \{w_{1i}, w_{2i}, \ldots, w_{pi}\}$; we assume that the vertices having same subscript and superscript are the corresponding vertices of the partite sets. From the graph $C_3 \times K_m$ we obtain the graph $C_3 \ast \overline{K}_k$ as in the proof of Subcase 1.1 of Theorem 2.5 with partite sets $\{u_1, u_2, \ldots, u_k\}$, $\{v_1, v_2, \ldots, v_k\}$ and $\{w_1, w_2, \ldots, w_k\}$ by suitably identifying the vertices of $C_3 \times K_m$. Since $k \neq 2, 6$, the graph $C_3 \ast \overline{K}_k$ has a $C_3$-factorization $\mathcal{F}$ in which $F = \{(u_i, v_i, w_i) | 1 \leq i \leq k\}$ is a $C_3$-factor, by Theorem 4.1. The subgraph of $C_3 \times K_m$ corresponding to $F$ of $C_3 \ast \overline{K}_k$ is $k$ vertex disjoint copies of $C_3 \times K_p$. Now $C_p \parallel C_3 \times K_p$, by Lemma 2.1. Similarly consider the subgraphs of $C_3 \times K_m$ corresponding to each of the other $C_3$-factors in $\mathcal{F}$; each of these subgraphs is isomorphic to $k$ vertex disjoint copies of $C_3 \ast \overline{K}_p$. Now $C_p \parallel C_3 \ast \overline{K}_p$, by Theorem A, and this completes the proof of the theorem. \hfill $\Box$

**Remark 4.4.** Let $U_i = \{u_{1i}, u_{2i}, \ldots, u_{mi}\}$, $1 \leq i \leq 2k + 1$, be the partite sets of the graph $C_{2k+1} \ast \overline{K}_m$, $m \neq 2, k \geq 2$. We know that $C_{2k+1} \parallel C_{2k+1} \ast \overline{K}_m$, by Theorem H. By suitably relabeling the vertices, if necessary, we get another $C_{2k+1}$-factorization of $C_{2k+1} \ast \overline{K}_m$ in which $F = \{(u_i, v_i, w_i) | 1 \leq i \leq m\}$ is a $C_{2k+1}$-factor. Thus we have $C_{2k+1} \parallel (C_{2k+1} \ast \overline{K}_m - F)$. Since $C_{2k+1} \ast \overline{K}_m - F$ is isomorphic to $C_{2k+1} \times K_m$, $C_{2k+1} \parallel C_{2k+1} \times K_m$. We quote this as a theorem below.

**Theorem 4.5.** For $m \neq 2$ and $k \geq 2$, $C_{2k+1} \parallel C_{2k+1} \times K_m$.

**Theorem 4.6.** For any prime $p \geq 11$, $C_p \parallel C_5 \times K_m$ if and only if $m \equiv 0 \pmod{p}$ except possibly $m = 2p$.

**Proof.** The proof of the necessity is obvious. We prove the sufficiency. Let $m = pk$. Let the partite sets of the 5-partite graph $C_5 \times K_m$ be $U = \bigcup_{i=1}^{k} \{u_{1i}, u_{2i}, \ldots, u_{pi}\}$, $V = \bigcup_{i=1}^{k} \{v_{1i}, v_{2i}, \ldots, v_{pi}\}$ and $W = \bigcup_{i=1}^{k} \{w_{1i}, w_{2i}, \ldots, w_{pi}\}$; we assume that the vertices having same subscript and superscript are the corresponding vertices of the partite sets. From the graph $C_5 \times K_m$, obtain the graph $C_5 \ast \overline{K}_k$ as in the proof of Subcase 1.1 of Theorem 3.7 with partite sets $\{u_1, u_2, \ldots, u_k\}$, $\{v_1, v_2, \ldots, v_k\}$, $\{w_1, w_2, \ldots, w_k\}$, $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$, by suitably identifying the vertices of $C_5 \times K_m$.

Since $k \neq 2$, the graph $C_5 \ast \overline{K}_k$ has a $C_5$-factorization $\mathcal{F}$ in which $F = \{(u_i, v_i, x_i, y_i) | 1 \leq i \leq k\}$ is a $C_5$-factor, by Remark 4.4. The subgraph of $C_5 \times K_m$ corresponding to $F$ of $C_5 \ast \overline{K}_k$ is $k$ vertex disjoint subgraphs isomorphic to $C_5 \times K_p$ and $C_p \parallel C_5 \times K_p$, by Lemma 3.1. Similarly consider the subgraphs of $C_5 \times K_m$ corresponding to each of the other $C_5$-factors of $\mathcal{F}$; each of these subgraphs is isomorphic to $k$ vertex disjoint copies of $C_5 \ast \overline{K}_p$ and by Theorem A, $C_p \parallel C_5 \ast \overline{K}_p$. \hfill $\Box$

**Theorem 4.7.** For any prime $p \geq 11$, $m, n \geq 3$, $C_p \parallel K_m \times K_n$ if and only if (1) either $m$ or $n$ is odd and (2) $p \mid mn$, except possibly for the following cases,

(a) $m = 7$ or $11$ and $n \equiv 0 \pmod{2p}$ or $n = 7$ or $11$ and $m \equiv 0 \pmod{2p}$,

(b) $m \notin \{7, 11\}$ and $n = 2p$ or $6p$ or $n \notin \{7, 11\}$ and $m = 2p$ or $6p$. 

Theorem 5.1. For any prime \( p \geq 11, m, n \geq 3 \), \( C_p \mid K_m \times K_n \) if and only if (1) \( p \mid mn(m-1)(n-1) \) and (2) either \( m \) or \( n \) is odd.

Proof. The proof of the necessity is obvious and we prove the sufficiency in two cases. Since the tensor product is commutative, we may assume that \( m \) is odd.

Case 1. \( m \equiv 0 \pmod{p} \).

Since \( m \) is odd and \( p \mid m \), we have \( m \equiv p \pmod{2p} \). By Theorem C, we have \( C_p \mid K_m \), that is \( K_m = F_1 \oplus F_2 \oplus \cdots \oplus F_{(m-1)/2} \), where each \( F_i \) is a \( C_p \)-factor of \( K_m \). Now \( K_m \times K_n = (F_1 \times K_n) \oplus (F_2 \times K_n) \oplus \cdots \oplus (F_{(m-1)/2} \times K_n) \).

Since \( F_1 \times K_n = (C_p \times K_n) \oplus (C_p \times K_n) \oplus \cdots \oplus (C_p \times K_n) \) and \( C_p \mid C_p \times K_n \), by Theorem 4.5, the result follows.

Case 2. \( m \not\equiv 0 \pmod{p} \).

From the necessary condition we have \( n \equiv 0 \pmod{p} \).

Subcase 2.1. \( m \not\in \{7, 11\} \).

By (b), \( n \not\in \{2, 6p\} \). By Theorem B, \( K_m = F_1 \oplus F_2 \oplus \cdots \oplus F_{(m-1)/2} \), where each \( F_i \) is a \( C_p \)-factor of \( K_m \) and each cycle in \( F_i \) is of length 3 or 5. Now \( K_m \times K_n = (F_1 \times K_n) \oplus (F_2 \times K_n) \oplus \cdots \oplus (F_{(m-1)/2} \times K_n) \) and \( F_i \times K_n = (C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_5 \oplus C_5 \oplus \cdots \oplus C_5) \). Since \( m \) is odd we have \( m \equiv 0 \pmod{6} \). By Theorem E, \( K_m \), and hence \( K_m \times K_n = (C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n) \). Since \( C_p \mid C_3 \times K_n \), by Theorem 2.5, \( C_p \mid K_m \times K_n \).

Subcase 2.2. \( m \equiv 5 \pmod{6} \).

By Theorem E, \( K_m = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_5 \oplus C_5 \).

By (m(m-1)−20)/6-times

Now \( K_m \times K_n = ((C_3 \times K_n) \oplus (C_3 \times K_n) \oplus \cdots \oplus (C_3 \times K_n)) \oplus ((C_5 \times K_n) \oplus (C_5 \times K_n)) \). Since \( C_p \mid C_3 \times K_n \), by Theorem 2.5, and \( C_p \mid C_5 \times K_n \), by Theorem 3.7, \( C_p \mid K_m \times K_n \).

Case 2. \( n \not\equiv 0 \pmod{p} \) and \( n \not\equiv 1 \pmod{p} \).

Since \( n(n-1) \not\equiv 0 \pmod{p} \), condition (1) implies that \( m \equiv 0 \pmod{p} \). Since \( m \) is odd we have \( m \equiv 1 \pmod{p} \). By Theorem D, \( K_m \times K_n = (C_p \times K_n) \oplus (C_p \times K_n) \oplus \cdots \oplus (C_p \times K_n) \). Now \( C_p \mid C_p \times K_n \), by Theorem 4.5. Thus \( C_p \mid K_m \times K_n \). This completes the proof.

Theorem 5.2. For any prime \( p \geq 11, m \geq 3 \), \( C_p \mid K_m \star \overline{K_n} \) if and only if (1) \( n(m-1) \) is even and (2) \( p \mid (m-1)n^2 \).

Proof. The necessary conditions are obvious and we prove the sufficiency in two cases.

Case 1. \( m \) is odd.

Subcase 1.1. \( m \equiv 1 \pmod{p} \).

Since \( C_p \mid K_n \), by Theorem D, \( K_m \star \overline{K_n} = (C_p \star \overline{K_n}) \oplus (C_p \star \overline{K_n}) \oplus \cdots \oplus (C_p \star \overline{K_n}) \). Now \( C_p \mid C_p \star \overline{K_n} \), by Theorem 3.5, and hence \( C_p \mid K_m \star \overline{K_n} \).

Subcase 1.2. \( m \not\equiv 1 \pmod{2p} \) and \( m \not\equiv p \pmod{2p} \).

Condition (2) implies that \( n \equiv 0 \pmod{p} \). If \( m \equiv 1 \) or \( 3 \pmod{6} \), then \( C_3 \mid K_m \), by Theorem E, and so \( K_m \star \overline{K_n} = (C_3 \star \overline{K_n}) \oplus (C_3 \star \overline{K_n}) \oplus \cdots \oplus (C_3 \star \overline{K_n}) \). Now \( C_p \mid C_3 \star \overline{K_n} \), by Theorem 2.6. Thus \( C_p \mid K_m \star \overline{K_n} \). If \( m \equiv 5 \pmod{6} \),
then by Theorem E, $K_m = C_3 \oplus C_3 \oplus \cdots \oplus C_3 \oplus C_5 + C_5$ and hence $K_m \ast K_n = ((C_3 \ast K_n) \oplus (C_3 \ast K_n) \oplus \cdots \oplus (m-1-20)/6$-times $(C_3 \ast K_n)) \oplus ((C_5 \ast K_n) \oplus (C_5 \ast K_n))$. Now $C_p \mid C_3 \ast K_n$, by Theorem 2.6, and $C_p \mid C_5 \ast K_n$, by Theorem 3.8 and consequently, $C_p \mid K_m \ast K_n$.

Case 2. $m$ is even.

Condition (1) implies that $n$ is even.

Subcase 2.1. $m \equiv 0$ or 1 (mod $p$).

First we suppose that $n \equiv 0$ (mod 4). Let $V(K_m) = \{x_1, x_2, \ldots, x_m\}$ and $V(K_n) = \{y_1, y_2, \ldots, y_n\}$. Let $V(K_m \ast K_n) = \bigcup_{i=1}^{m} \{x_i\} \times V(K_n) = \bigcup_{i=1}^{m} \{x_i, y_j\}$ where $x_j$ stands for $(x_i, y_j)$. For our convenience, let $X_i = \{x^i_j \mid 1 \leq j \leq n\}$ be the $i$th layer of $K_m \ast K_n$. In each layer, pair the consecutive vertices as follows: $\{x_i^1, x_i^2\}, \{x_i^3, x_i^4\}, \ldots, \{x_i^{n-1}, x_i^n\}$, $1 \leq i \leq 3$. Clearly, the subgraph of $K_m \ast K_n$, say, $G_k$, induced by $\bigcup_{i=1}^{m} \{x_i^1, x_i^2\}, 1 \leq k \leq n/2$, is isomorphic to $K_{2m} - I$, where $I$ is a 1-factor of $K_{2m}$. By Theorem F, $K_{2m} - I$ has a $C_p$-decomposition, that is $G_k$ has a $C_p$-decomposition. Next we prove that $K_m \ast K_n - \bigcup_{k=1}^{n/2} \{G_k\}$ admits a $C_p$-decomposition. From $H = (K_m \ast K_n) - \bigcup_{k=1}^{n/2} \{G_k\}$, we obtain a new graph as follows: identify each pair of vertices $\{x_i^1, x_i^2\}$, $1 \leq i \leq m$, $1 \leq k \leq n/2$, into a single vertex, say, $x_k^i$ and join $x_k^i$ and $x_k^j$ by an edge if and only if the corresponding pairs of vertices, namely, $\{x_i^{k-1}, x_i^k\}$ and $\{x_j^{k-1}, x_j^k\}$ induce the complete bipartite graph $K_{2,2}$ in $H$. Then the new graph is isomorphic to $K_m \times K_{n/2}$. Since $n \equiv 0$ (mod 4), $n/2$ is an odd integer. Because $m \geq 0$ and $n/2 \geq 3$ is an odd integer, $K_m \times K_{n/2}$ admits a $C_p$-decomposition, by Theorem 5.1. But each $p$-cycle of this decomposition corresponds to a subgraph of $H$ isomorphic to $C_p \ast K_2$. But $C_p \mid C_p \ast K_2$, see [14].

Next we suppose that $n \equiv 0$ (mod 4). In this case, we prove the result by induction on $n$. Let $n = 4k$. If $k = 1$, then $K_m \ast K_n = K_m \ast K_4 = (K_m \ast K_2) \ast K_2 = (K_{2m} - I) \ast K_2$, where $I$ is a 1-factor of $K_{2m}$. Now $C_p \mid K_{2m} - I$, by Theorem F, and $K_m \ast K_4 = (K_m \ast K_2) \ast K_2 = (K_{2m} - I) \ast K_2 \ast (C_p \ast K_2) \ast (C_p \ast K_2) \ast (C_p \ast K_2) \ast (C_p \ast K_2) \ast (C_p \ast K_2)$. Since $C_p \mid C_p \ast K_2$, we [14], we conclude that $C_p \mid K_m \ast K_4$. Assume that the result is true for all $n = 4t$, $1 < t < k$. Clearly, $K_m \ast K_4k = (K_m \ast K_{2k}) \ast K_2$. If $k$ is odd, then $C_p \mid K_m \ast K_{2k}$, by the first part of the proof this case. If $k$ is even, then $C_p \mid K_m \ast K_{2k}$, by induction hypothesis. Therefore irrespective of the parity of $k$, $K_m \ast K_{2k} = (C_p \ast K_2) \ast (C_p \ast K_2) \ast (C_p \ast K_2) \ast (C_p \ast K_2)$. Since $C_p \mid C_p \ast K_2$, we have $C_p \mid K_m \ast K_{2k}$.

Subcase 2.2. $m \equiv 0$ (mod $p$) and $m \equiv 1$ (mod $p$).

Condition (2) implies that $n \equiv 0$ (mod 2p) and, $C_p \mid K_m \ast K_n$, by Theorem G. This completes the proof.

In [12,13] it has been shown that for $p = 5, 7$, the necessary and sufficient conditions for the existence of $C_p$-decompositions of $K_m \times K_n$ and $K_m \ast K_n$ are proved to be sufficient. These results, in conjunction with Theorems 5.1 and 5.2 completes the characterization of the $C_p$-decompositions of $K_m \times K_n$ and $K_m \ast K_n$ for all primes $p \geq 5$.

Acknowledgements

This research is partially supported by Department of Science and Technology (DST), Government of India, New Delhi; Project Grant no: SR/S4/MS: 217/03. The authors thank the DST for financial support. The authors also would like to thank the referees for their careful reading of the manuscript and suggestions and also for bringing to our notice the references [5,10].

References