On Characterizing the Data Access Complexity of Programs

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Outline

1. Motivation
2. Prior work & Challenges
3. Static analysis of affine programs

Outline

What is a Good Algorithm?

- Computational cost: number of operations executed by the algorithm
  - Objective: reduce the operation complexity

- Execution time: time to execute the operations
- Actually, time to execute the operations **and** time to move the operands in the system
  - Example: moving data from disk to RAM at 3Gb/s
  - Example: moving data from RAM to CPU at 17Gb/s
  - ...

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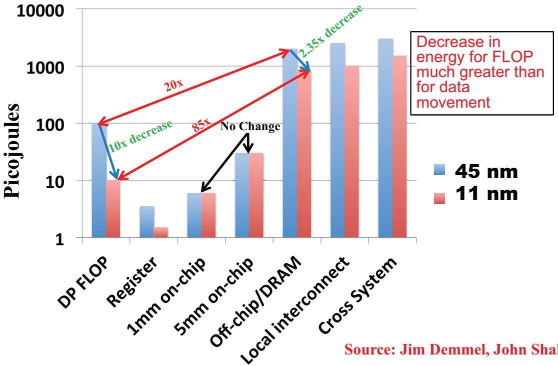
➡ Good algorithm: good execution time (computation + data movement)

A Look at Architectural Trends

- The relative cost of data movement vs. computation keeps increasing
  - Ex: Intel 80286: 2 MIPS, 13 MB/s for transfer RAM->CPU
  - Ex: Intel core i7: 50,000 MIPS, 16,000 MB/s for transfer RAM->CPU
- The relative energy cost of data movement vs. computation keeps increasing



Source: Jim Demmel, John Shalf

➡ Computational complexity alone is not sufficient. Data movement complexity matters!

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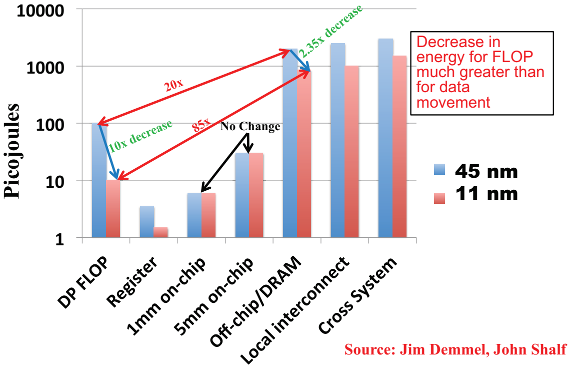

➡ **Computational complexity alone is not sufficient. Data movement complexity matters!**

Data Movement vs. Computational Complexity

Untiled

```
for(i=1; i<N-1; i++)
 for(j=1; j<N-1; j++)
  A[i][j] = A[i][j-1]\
    + A[i-1][j];
```

Comp. cost: $(N-1)^2$

Tiled

```
for(it=1; it<N-1; it+=B)
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Data movement cost

- Data movement cost different for two versions
- Also depends on cache size
- Question: What is data movement complexity?

➡ **Data movement complexity: Minimum data movement cost considering all possible valid schedules**

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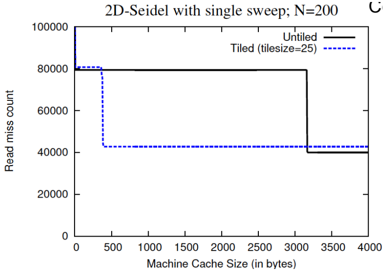

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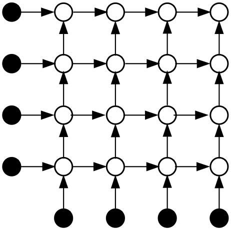

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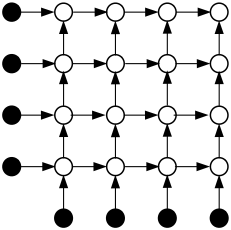

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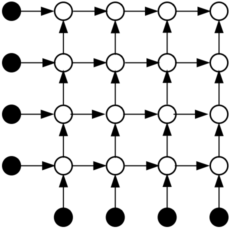
Graph based

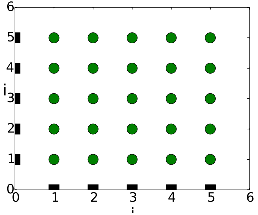
- Arbitrary CDAGs



- [Hong and Kung, 1981]: all valid schedules ⇝ all valid "$2S$-partitions" of CDAG
- (+) Generality
- (-) manual reasoning $\Longrightarrow$ challenge to automate

Geometric data footprint

- Linear algebra like algorithms



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- [Christ et al., 2013]: Automation based on Holder-Brascamp-Leib (HBL) ineq.
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- (-) Restricted model $\Longrightarrow$ weakness of bounds or inapplicability

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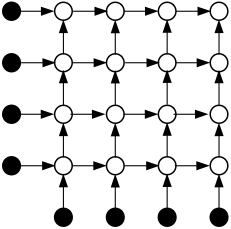

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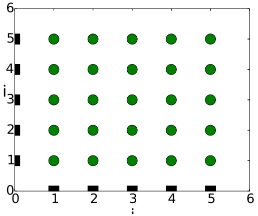

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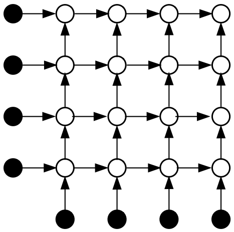

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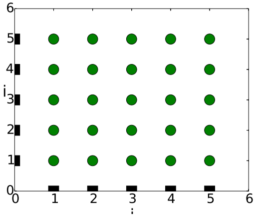

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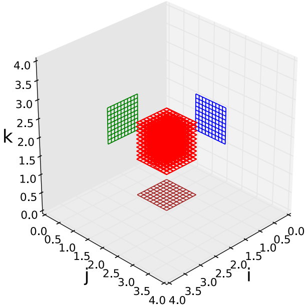
Our work: Static analysis to automate asymptotic parametric lower bounds analysis of affine codes for CDAG model.

Loomis-Whitney inequality

- $E \subset \mathbb{R}^d$
- $\phi_1(E), \ldots, \phi_d(E)$ its projections on the coordinates hyperplanes

Example ($d = 3$):



$$|E| \leq |\phi_1(E)|^{1/2} \times |\phi_2(E)|^{1/2} \times |\phi_3(E)|^{1/2}$$

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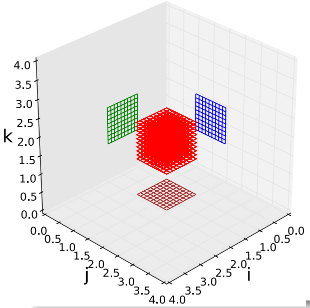

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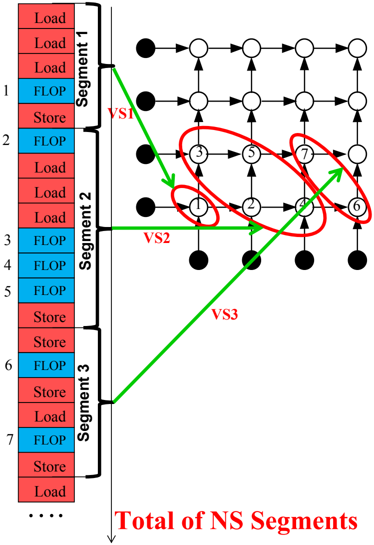
Hong & Kung $2S$-partioning



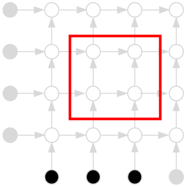
- Any valid schedule is asociated with a $2S$-partition

S-partition

Collection of h subsets $(V_1, \ldots, V_h)$ of $V \setminus I$ s.t:

- **P1** pairwise disjoint
- **P2** no cyclic dependence
- **P3** $\forall i, \ \ |\mathrm{In}(V_i)| \leq S$

- Largest vertex-set: P



Data movement complexity

$$Q \geq \left(\frac{|V|}{|P|} - 1\right) \times S$$

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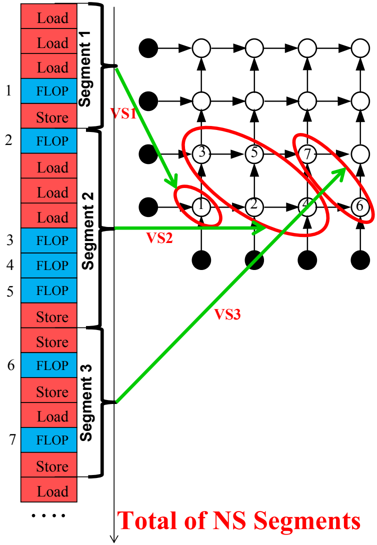

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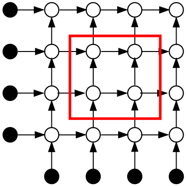

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Affine computations

Can be represented as (union of) $\mathcal{Z}$-polyhedra:

- Space: d-dimensional integer lattice ($\mathbb{Z}^d$).
- Points: Each instance of the statement.
- Arrows: True data dependencies.

```
for (i=0; i<N; i++)
S1:  A[i] = I[i];
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  for (i=1; i<N-1; i++)
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S3:  A[i] = B[i];
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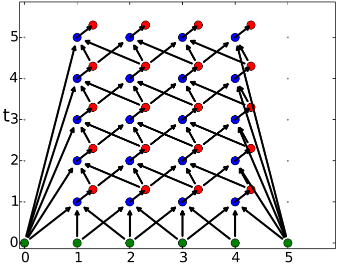

➡ Apply geometric reasoning on $\mathcal{Z}$-polyhedra to bound $|P|$

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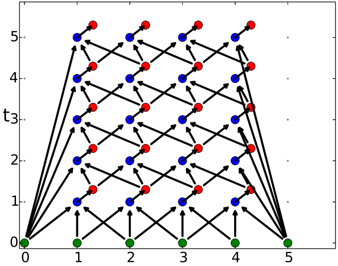

➡ **Apply geometric reasoning on $\mathcal{Z}$-polyhedra to bound $|P|$**

Example 1: Jacobi 1D

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Parameters: N, T; Inputs: I[N]; Outputs: A[N]
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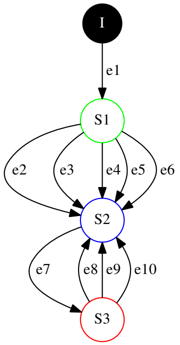
DFG:



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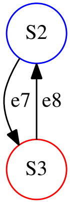
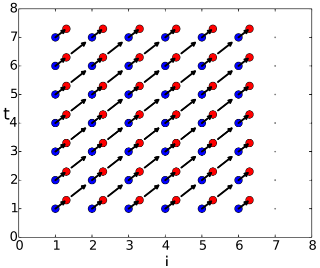
Injective (DFG) circuit:



Set of disjoint (CDAG) paths:



From disjoint paths to projections

- For any $P \rightsquigarrow$ at most one element per disjoint path in $\mathrm{In}(P)$.
- $\vec{b}$ as projection vector for ϕ_i $\rightsquigarrow |\phi_i(P)| \leq |\mathrm{In}(P)| \leq 2S$.

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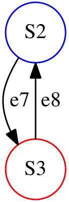

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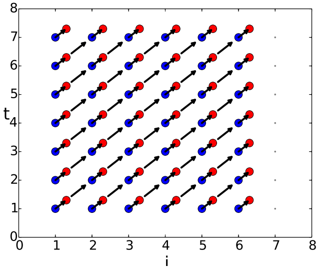

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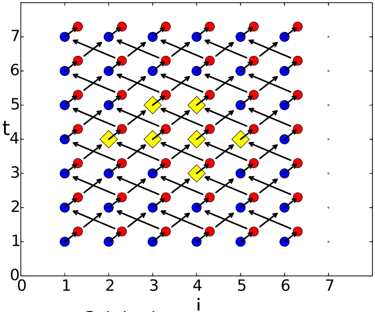
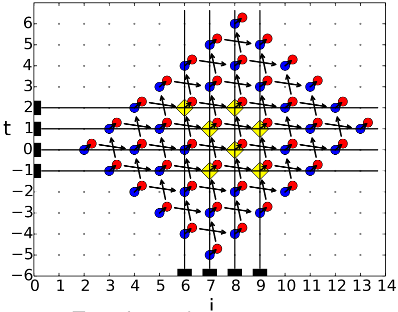


Original space

⇝



Transformed space

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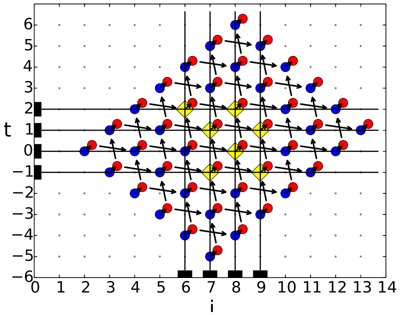

Apply Loomis-Whitney inequality:

$$|P| \leq (2S)^2 \qquad \rightsquigarrow \qquad Q = \Omega\left(\frac{NT}{S}\right) - (N + T)$$

High-Level algorithm

1. Extract data flow graph (DFG) from source code.
2. Identify paths of interest in DFG
3. Obtain projections that satisfy $|\phi_j(P)| \leq |\mathrm{In}(P)|$.
4. Apply geometric reasoning to obtain the lower bounds

more in the paper...

- use of “broadcast” paths to find projection directions
- use of generalized geometric Holder-Brascamp-Leib inequality
- Inherent multi-regime parametric characterization

Example: Rectangular matmult $(m \times n \times p)$

If $m, n, p \gg \sqrt{2S}$:

$$Q = \Omega\left(\frac{mnp}{\sqrt{S}}\right)$$

else if $m, n \gg \sqrt{2S}$ and $p \ll \sqrt{2S}$ (mat-vect):

$$Q = \Omega(mn)$$

else if $\ldots$:

$$\ldots$$

Conclusion

- Challenge: Computational complexity of algorithms is well understood, but data movement complexity is not.
- Applications:
  - **Algorithm analysis:** Which currently popular algorithms need rethinking due to high inherent data movement complexity?
  - **Compiler assessment:** Is further improvement of data locality possible?
  - **Algorithm-architecture co-design:** How to provision future architectures for the minimal data movement demands of algorithms?
- Ongoing / future work:
  - Methodologies
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  - handling irreglar CDAGs
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Thank you

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