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Characteristic-free bounds for the Castelnuovo–Mumford regularity

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Abstract

We study bounds for the Castelnuovo–Mumford regularity of homogeneous ideals in a polynomial ring in terms of the number of variables and the degree of the generators. In particular, our aim is to give a positive answer to a question posed by Bayer and Mumford in *What can be computed in algebraic geometry*? (*Computational algebraic geometry and commutative algebra*, Symposia Mathematica, vol. XXXIV (1993), 1–48) by showing that the known upper bound in characteristic zero holds true also in positive characteristic. We first analyse Giusti's proof, which provides the result in characteristic zero, giving some insight into the combinatorial properties needed in that context. For the general case, we provide a new argument which employs the Bayer–Stillman criterion for detecting regularity.

Introduction

The Castelnuovo–Mumford regularity is an important invariant in commutative algebra and algebraic geometry, which gives an estimate of the complexity of computing a minimal free resolution. It is common in the literature to attempt to find bounds for this invariant and, in general, the expected results range quite widely, from the well-behaved examples coming from algebraic geometry, as suggested by the Eisenbud–Goto conjecture [EG84], to the worst case provided by the example of Mayr and Meyer [MM82]. Clearly, when the assumptions are quite unrestrictive, the regularity can be very large. If one works with a homogeneous ideal I in a polynomial ring $R = K[X_1, \dots, X_n]$ over a field K, a very natural question is to ask whether the regularity can be limited just by knowing that the ideal is generated in degree less than or equal to some positive integer d. What was known to this point were bounds depending on the characteristic of the base field K. If $\operatorname{char} K = 0$, as observed in [BM93, Proposition 3.8], from the work of Giusti [Giu84] and Galligo [Gal79, Gal73], one can derive the bound

$$\operatorname{reg}(I) \leqslant (2d)^{2^{n-2}}, \tag{A}$$

which seems to be sharp (see again [MM82]).

On the other hand, in any characteristic, it has been proved in [BM93] using 'straightforward cohomological methods' that

$$\operatorname{reg}(I) \leqslant (2d)^{(n-1)!}, \tag{B}$$

but in the same paper it is asked whether (A) holds in general independently of the characteristic. The main purpose of this paper is to give a positive answer to this question. The main effort in extending the result to positive characteristic is that this proof utilises the combinatorial structure of the *generic initial ideal* in characteristic zero.

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The generic initial ideal was introduced in [Gal73], where it was defined with the assumption that the base field has characteristic zero. The definition was then generalised a few years later for base fields of any characteristic in [BS87], and grew in importance, as many recent results demonstrate. One of the points of major interest in considering the generic initial ideal $\mathrm{Gin}_{\mathrm{rlex}}(I)$ with respect to the (degree) reverse lexicographic order of a homogeneous ideal I is that this is a monomial ideal with the same Hilbert function, projective dimension and Castelnuovo–Mumford regularity as I. Furthermore, a generic initial ideal is Borel-fixed, i.e. it is invariant under the action of the Borel group, which is the subgroup of $GL_n(K)$ consisting of all non-singular upper-triangular $n \times n$ matrices with coefficients in K. According to the characteristic of the underlying field, whether it is zero or positive, Borel-fixed ideals have a more or less manageable combinatorial structure. It may be convenient to recall some of the most interesting notions used in this context and to fix some terminology since this is not unique in the literature. We refer the interested reader to the detailed treatise in [Par94] and [Par96] for further information.

Let K be an infinite field (which is not a restrictive hypothesis for our purposes). Given a monomial u we denote $\max\{i \colon X_i \mid u\}$ by $m(u)$. Now let p be a prime number and k a non-negative integer. The *p-adic expansion* of k is the expression of k as $\sum_i k_i p^i$, with $0 \leqslant k_i \leqslant p-1$. If $k = \sum_i k_i p^i$ and $l = \sum_i l_i p^i$ are the p-adic expansions of the two integers k and l respectively, one sets $k \leqslant_p l$ if and only if $k_i \leqslant l_i$ for all i.

First of all notice that an ideal I which is fixed under the action of the Borel group (i.e. a *Borel*(*-fixed*) ideal) is monomial. A *standard Borel*(*-fixed*) (or *strongly stable*) ideal I is an ideal endowed with the following property: for each $u \in I$, if $X_i \mid u$ then $X_j u / X_i \in I$, for each $j < i$. The class of *stable* ideals is defined by a weaker exchange condition on the variables of the monomials: an ideal I is stable if and only if for each $u \in I$, $X_j u / X_{m(u)} \in I$, for each $j < m(u)$. Finally, an ideal I is said to be *p-Borel* if and only if for each monomial $u \in I$, if l is the maximum integer such that $X_i^l | u$, then $X_j^k u / X_i^k \in I$, for all $j < i$ and $k \leqslant_p l$. Standard Borel ideals are Borel. If $\operatorname{char} K = 0$ every Borel ideal is standard. If $\operatorname{char} K = p$ a monomial ideal is Borel if and only if it is p-Borel.

The crucial difference between characteristic zero and positive characteristic can be noticed at a first glance: the combinatorial structure of standard Borel ideals is easier than that of the non-standard ones, which relies on the p-adic expansion of non-negative integers. This difference in behaviour also results in the fact that there is a complete description of the minimal free graded resolution of a standard Borel ideal I in terms of the monomials of its minimal system of generators $G(I)$ (cf. [EK90], [AH96]), while the task of finding an analogue for non-standard Borel ideals still seems to be too difficult. In particular, the graded Betti numbers of a standard Borel ideal I can be computed explicitly in terms of $G(I)$ and it is easily deduced that its Castelnuovo–Mumford regularity equals the highest degree of an element in $G(I)$, i.e. the so-called *generating degree* of I. Thus, it is quite clear that the assumption of characteristic zero has made the task of investigating the generic initial ideal easier.

This paper is divided in two sections. The first section is dedicated to a better understanding of the arguments that lead to the inequality (A), which are due to Galligo [Gal79] and Giusti [Giu84]. For this purpose, we introduce and study certain ideals, which we call *weakly stable*. As a result we obtain a bound for the regularity which improves (A). In the second section we use these ideas and we prove a formula that relates the regularity of the original ideal with its generating degree and the regularity of its sum with an almost-regular sequence of linear forms (Theorem 2.4). As a consequence, we obtain the desired result (Corollary 2.6).

1. Weakly stable ideals

The purpose of this section is to study a certain class of monomial ideals which we call *weakly stable*.

We recall the definition of the Castelnuovo–Mumford regularity, whereas we refer the reader to [EG84], [Eis95] and [BS98] for further details on the subject.

DEFINITION 1.1. Let M be a finitely generated graded R-module and let $\beta_{ij}(M)$ denote the graded Betti numbers of M (i.e. the numbers $\dim_K \operatorname{Tor}_i(M,K)_j$). The *Castelnuovo–Mumford regularity* $\operatorname{reg}(M)$ of M is

$$\max_{i,j}\{j-i\colon \beta_{ij}(M)\neq 0\}.$$

Recall also another useful characterisation of regularity in terms of local cohomology modules of M, which we shall use in the following. Since the graded local cohomology modules $H^i_{\mathfrak{m}}(M)$ with support in the graded maximal ideal $\mathfrak{m}$ of R are Artinian, one defines $\operatorname{end}(H^i_{\mathfrak{m}}(M))$ to be the maximum integer k such that $H^i_{\mathfrak{m}}(M)_k \neq 0$. Then

$$\operatorname{reg}(M)=\max_i\{\operatorname{end}(H^i_{\mathfrak{m}}(M))+i\}.$$

Finally, a finitely generated R-module M is said to be *m-regular* for some integer m if and only if $\operatorname{reg}(M)\leqslant m$.

Let us outline the main steps of the proof of (A). For any field K and any homogeneous ideal I it is well-known that $\operatorname{reg}(I)$ equals $\operatorname{reg}(\operatorname{Gin}_{\mathrm{rlex}}(I))$, and if $\operatorname{char} K=0$ then $\operatorname{Gin}_{\prec}(I)$ is a standard Borel ideal for any term order $\prec$ for which $X_1\succ X_2\succ\cdots\succ X_n$, so that its regularity equals its generating degree D. Furthermore, if $\operatorname{char} K=0$, the following principle holds.

CRYSTALLISATION PRINCIPLE (CP). Let I be a homogeneous ideal generated in degrees $\leqslant d$. Assume that $\operatorname{Gin}_{\mathrm{rlex}}(I)$ has no generator in degree $d+1$. Then $D\leqslant d$ (cf. [Gre98, Proposition 2.28]).

Hence, due to the good properties of $\operatorname{Gin}_{\mathrm{rlex}}$ and an induction argument on the numbers of variables, one obtains bounds for D in terms of the generating degree d of I (cf. [Giu84], in particular the 'Proof of Theorem B'), and this completes the argument.

One notices that in the proof the hypothesis $\operatorname{char} K=0$ is used solely to exploit the combinatorial structure of $\operatorname{Gin}_{\mathrm{rlex}}$. Furthermore, CP only holds true in characteristic zero. Consider, for instance the ideal (X^{2p},Y^{2p}) in $K[X,Y]$ with $\operatorname{char} K=p\neq 2$. Then $\operatorname{Gin}_{\mathrm{rlex}}(I)=(X^{2p},X^pY^p,Y^{3p})$. Here it is sufficient to observe that the ideal (X^{2p},Y^{2p}) is the ideal generated by the images of X^2 and Y^2 under the Frobenius map $R\to R$, $X\to X^p$. In fact the following more general result is well-known.

PROPOSITION 1.2. *Let I be a homogeneous ideal of $R=K[X_1,\dots,X_n]$ with $\operatorname{char} K=p$ and let F be the Frobenius map. Then for any term order τ one has*

$$\operatorname{Gin}_\tau(F(I))=F(\operatorname{Gin}_\tau(I)).$$

Proof. Note that the computation of the initial ideal of $F(I)$ can be performed in $K[X_1^p,\dots,X_n^p]$, i.e. the S-pairs of $F(I)$ are just the pth power of the S-pairs of I, so that $F(\operatorname{in}_\tau(I))=\operatorname{in}_\tau(F(I))$. This suffices, since by definition $\operatorname{Gin}_\tau(F(I))=\operatorname{in}_\tau(g(F(I)))=\operatorname{in}_\tau(F(g(I)))$, where g is a generic change of coordinates. □

Since CP fails in positive characteristic, one might wonder if the proof of (A) could be performed by making use of *lexicographic* (also called *lex-segment*) ideals instead of generic initial ideals. In fact, there is a natural counterpart of CP, the so-called Gotzmann's persistence theorem (see [Got78] or [Gre98, Theorem 3.8]): Given an ideal I with generating degree D the lexicographic ideal L associated with I cannot have generators in degree $k>h$ for any k if it has none in degree $h>D$.

This property is not strong enough to furnish our bounds since there is something missing: modulo the last variable the resulting lexicographic ideal does not fulfil Gotzmann's persistence theorem for the same D. So, proceeding recursively, one would obtain much higher bounds.

This makes it necessary to have a deeper understanding of Giusti's argument, and motivates the following definition. As before we let $m(u)$ denote $\max\{i\colon X_i \mid u\}$ for any monomial u.

Definition 1.3. A monomial ideal I is said to be *weakly stable* if for all $u \in I$ and for all $j < m(u)$ there exists a positive integer k such that $X_j^k u / X_{m(u)}^l \in I$, where l is the maximum integer such that $X_{m(u)}^l \mid u$.

It is straightforward from this definition that if I is weakly stable then so is $\bar{I}$, the quotient ideal modulo the last variable (any other variable would do after re-labelling). One easily verifies that finite intersections, sums and products of weakly stable ideals are weakly stable. It is worth pointing out that a monomial ideal is weakly stable if and only if its associated prime ideals are lexicographic, i.e. of the form $(X_1, \ldots, X_i)$ for some i. This and other combinatorial properties of weakly stable ideals have been proved in [Cav04]. We also recall that monomial ideals which have lexicographic associated prime ideals have been used in [BG05] for algorithmic computations of Castelnuovo–Mumford regularity.

Remark 1.4. Strongly stable, stable and p-Borel ideals are weakly stable.

Henceforth we let $D(I)$ denote the generating degree of the ideal I, i.e. the maximum of the degrees of a minimal set of generators of I. We next prove a bound on the cardinality of the minimal system of generators of a weakly stable ideal in terms of the generating degrees of its reductions modulo the last variables. In the following we shall denote by $I_{[i]}$ the image of the ideal I in $R/(X_{i+1}, \ldots, X_n)$, for $i = 1, \ldots, n-1$, and we let $I_{[n]} = I$. Given a monomial u in $K[X_1, \ldots, X_n]$, we set $\mathrm{Md}_i(u) \doteq \max\{j\colon X_i^j \mid u\}$, for $i = 1, \ldots, n$. Accordingly, if J is a monomial ideal we define $\mathrm{Md}_i(J) \doteq \max_{u \in G(J)}\{j\colon X_i^j \mid u\} = \max_{u \in G(J)}\{\mathrm{Md}_i(u)\}$, for $i = 1, \ldots, n-1$. We first prove the following lemma.

Lemma 1.5. *Let I be a weakly stable ideal. Then* $\mathrm{Md}_i(I_{[i]}) = \mathrm{Md}_i(I)$ *for all* $i = 1, \ldots, n-1$.

Proof. It is immediate that $\mathrm{Md}_i(I_{[i]}) \leqslant \mathrm{Md}_i(I)$. Suppose now that $\mathrm{Md}_i(I_{[i]}) < \mathrm{Md}_i(I)$ and let us find a contradiction. For the sake of simplicity, let $s \doteq \mathrm{Md}_i(I_{[i]})$. Then there exists $u \in G(I)$ such that $X_i^{s+1} \mid u$ and $m(u) > i$. Let us choose u such that $m(u)$ becomes minimal. As I is weakly stable there are integers k, l such that $uX_i^k / X_{m(u)}^l$ is an element of I. Hence there exists $n \in G(I)$ such that n divides $uX_i^k / X_{m(u)}^l$ and $m(n) < m(u)$. Therefore, $\mathrm{Md}_i(n) \leqslant s$, so that $n \mid uX_i^k$ and $n \nmid u$. But this implies the contradiction $s \geqslant \mathrm{Md}_i(n) \geqslant \mathrm{Md}_i(u) + 1 \geqslant s + 2$ and we are done. ☐

Proposition 1.6. *Let $I \subset R = K[X_1, \ldots, X_n]$, with $n \geqslant 2$, be a weakly stable ideal. Then*

$$|G(I)| \leqslant \prod_{i=1}^{n-1} (D(I_{[i]}) + 1).$$

Proof. Keeping in mind that we can decide whether two monomials of $G(I)$ are distinct just by looking at their first $n-1$ variables, it is clear that $|G(I)| \leqslant \prod_{i=1}^{n-1}(\mathrm{Md}_i(I) + 1)$. By Lemma 1.5, $\mathrm{Md}_i(I) = \mathrm{Md}_i(I_{[i]})$ for all $i = 1, \ldots, n-1$. Obviously $\mathrm{Md}_i(I_{[i]}) \leqslant D(I_{[i]})$ for all $i = 1, \ldots, n-1$. ☐

Let d be a non-negative integer. We consider weakly stable ideals which fulfil the following condition.

Condition 1.7. If $i \geqslant d$ and I has no minimal generator of degree i, then it also has none of degree $i + 1$.

Proposition 1.8. *Let $I \subset R = K[X_1, \ldots, X_n]$, with $n \geqslant 2$, be a weakly stable ideal for which Condition 1.7 holds with respect to d. Then*

$$D(I) \leqslant d - 1 + \prod_{i=1}^{n-1} (D(I_{[i]}) + 1).$$

Proof. The hypothesis on the generators implies that $D(I) - d + 1 \leqslant |G(I)|$ and since I is weakly stable the assertion follows directly from Proposition 1.6. □

Let us now look at the class of weakly stable ideals I, which satisfy the following condition.

Condition 1.9. $I_{[i]}$ verifies Condition 1.7 with respect to d for all $1 \leqslant i \leqslant n$.

Condition 1.9 is satisfied by the generic initial ideal $\mathrm{Gin}_{\mathrm{rlex}}(I)$ of a homogeneous ideal I in $K[X_1, \ldots, X_n]$, where the characteristic of K is zero and the generating degree of I is less than or equal to d. As noticed earlier, $\mathrm{Gin}_{\mathrm{rlex}}(I)$ is strongly stable and *a fortiori* weakly stable as observed in Remark 1.4. Condition 1.7 is verified for such an ideal by virtue of CP, whereas Condition 1.9 holds as $\mathrm{Gin}_{\mathrm{rlex}}(I)_{[i]} = \mathrm{Gin}_{\mathrm{rlex}}(I) + (X_{i+1}, \ldots, X_n) = \mathrm{Gin}_{\mathrm{rlex}}(I + (X_{i+1}, \ldots, X_n)) = \mathrm{Gin}_{\mathrm{rlex}}(I_{[i]})$, and CP applies since the generating degree of $I_{[i]}$ is obviously less than or equal to d.

Corollary 1.10. *Let $I \subseteq K[X_1, \ldots, X_n]$, with $n \geqslant 2$, be a weakly stable ideal, which satisfies Condition 1.9 with respect to d. Then*

$$D(I) \leqslant (2d)^{2^{n-2}}.$$

Proof. According to Proposition 1.8, one has that $D(I_{[i]}) \leqslant B_i$, where we let $B_1 \doteq d$ and $B_i \doteq d - 1 + \prod_{j=1}^{i-1}(B_j + 1)$, for all $i > 1$. One easily verifies that $B_i = (B_{i-1} - (d-1))(B_{i-1} + 1) + d - 1 = B_{i-1}^2 - (d-2)B_{i-1}$ for all $i > 1$. Since we may assume that $d \geqslant 2$, we get $B_i \leqslant B_{i-1}^2$. Thus, for all $i \geqslant 2$, we have $B_i \leqslant (2d)^{2^{i-2}}$. In particular, we obtain $D(I) \leqslant B_n \leqslant (2d)^{2^{n-2}}$. □

The bound for the regularity expressed in (A) now follows easily under the assumption char $K = 0$.

Corollary 1.11. *Let I be an ideal of $R = K[X_1, \ldots, X_n]$ with $n \geqslant 2$ and $\mathrm{char}\, K = 0$. Let I be generated in degree $\leqslant d$. Then*

$$\mathrm{reg}(I) \leqslant (2d)^{2^{n-2}}.$$

Proof. Recall that I and $\mathrm{Gin}_{\mathrm{rlex}}(I)$ have the same regularity and that the latter is a stable ideal, so that its regularity equals its generating degree. By the observations preceding Corollary 1.10, this ideal satisfies the hypotheses of Corollary 1.10. □

2. Bounds for the regularity

In this section we show that the above bound holds independently of the characteristic. This improves [BM93, Theorem 3.7 and Proposition 3.8]. The techniques used here are based on general properties of local cohomology and almost-regular sequences of linear forms.

Henceforth, by flat base change property of local cohomology, we may assume without loss of generality that $|K| = \infty$.

We first notice a few easy facts which are used in the rest of the section. Given an arbitrary homogeneous ideal I of $R = K[X_1, \ldots, X_n]$, we denote by I^{sat} the *saturation* $I : \mathfrak{m}^\infty = \cup_{k \geqslant 0} I : \mathfrak{m}^k$ of I with respect to $\mathfrak{m}$. Let I, J be two arbitrary ideals. If $I \subseteq J \subseteq I^{\mathrm{sat}}$ then $J^{\mathrm{sat}} = I^{\mathrm{sat}}$. Recall that, given a finitely generated graded R-module M, a homogeneous element $l \in R_d$ is said to be *almost-regular for M* if and only if the multiplication map $M_k \xrightarrow{\cdot l} M_{k+d}$ is injective for all $k \gg 0$.

We say that $l_1, \ldots, l_r$ form an *almost-regular sequence for* M if l_1 is almost-regular for M and l_{i+1} is almost-regular for $M/(l_1, \ldots, l_i)M$ for all $i = 1, \ldots, r-1$. One can show that a homogeneous form is almost regular for a graded R-module M if and only if it is not contained in any associated prime ideal of M other than the homogeneous maximal ideal $\mathfrak{m}$. Recall also that, since $|K| = \infty$, a generic form in R is almost-regular for M.

Remark 2.1. Let I be a homogeneous ideal of R and let $l \in R_d$. Then the following statements are well known to be equivalent:

(i) l is almost regular for R/I;
(ii) l is regular for R/I^{sat};
(iii) $I : l^k \subset I^{\mathrm{sat}}$ for some $k > 0$;
(v) $I : l^k \subset I^{\mathrm{sat}}$ for all $k > 0$;
(iv) $I : l^k = I^{\mathrm{sat}}$ for some $k > 0$.

Let I and l satisfy these conditions. Then, the smallest integer k with $I : l^k = I^{\mathrm{sat}}$ is called the *index of saturation* of I with respect to l and is denoted by $k(I, l)$. It is not difficult to see that

$$(k(I,l) - 1)d \leqslant \max\{0, \operatorname{end}(H^0_{\mathfrak{m}}(R/I))\}.$$

In particular if $d = 1$ and $I \neq 0$, one obtains $k(I, l) \leqslant \operatorname{reg}(I)$.

Before giving the main result of this section we prove a useful lemma. In the following we let $\lambda(\cdot)$ denote the length function.

Lemma 2.2. *Let l be an element of R and I an ideal of R. For any integer $a \geqslant 0$ one has*

$$\lambda\left(\frac{I : l^a}{I : l^{a-1}}\right) = \lambda\left(\frac{(I : l^a) + (l)}{(I : l^{a-1}) + (l)}\right) + \lambda\left(\frac{I : l^{a+1}}{I : l^a}\right),$$

whenever all of the above lengths are finite.

Proof. Consider the following exact sequence,

$$0 \longrightarrow \frac{I : l^{a+1}}{I : l^a} \xrightarrow{\cdot l} \frac{I : l^a}{I : l^{a-1}} \longrightarrow \frac{I : l^a}{(I : l^{a-1}) + l(I : l^{a+1})} \longrightarrow 0\,,$$

where the third term is

$$\frac{I : l^a}{(I : l^{a-1}) + (l) \cap (I : l^a)} \simeq \frac{(I : l^a) + (l)}{(I : l^{a-1}) + (l)}.$$

The conclusion follows immediately from the additivity of length. □

Remark 2.3. If $\dim R/I = 0$ and I is generated in degree $\leqslant d$, then I contains a regular sequence $f_1, \ldots, f_n$ of forms of degree at most d; therefore,

$$\begin{aligned}\operatorname{reg}(I) &= \operatorname{reg}(R/I) + 1 = \operatorname{end}(R/I) + 1 \leqslant \operatorname{end}(R/(f_1, \ldots, f_n)) + 1 = \operatorname{reg}(R/(f_1, \ldots, f_n)) + 1 \\ &= n(d-1) + 1.\end{aligned}$$

We adopt the standard agreement that a product over the empty set is 1.

Theorem 2.4. *Let I be a homogeneous ideal of $K[X_1, \ldots, X_n]$ of height $c < n$ and generated in degree $\leqslant d$. Then, if $l_n, \ldots, l_{c+1}$ is an almost-regular sequence of linear forms for R/I, one has*

$$\operatorname{reg}(I) \leqslant \max\{d, \operatorname{reg}(I + (l_n))\} + d^c \prod_{i=c+2}^{n} \operatorname{reg}(I + (l_n, \ldots, l_i)).$$

Proof. The proof consists essentially in proving two separate inequalities:

$$\operatorname{reg}(I) \leqslant \max\{d, \operatorname{reg}(I+(l_n))\} + \lambda\left(\frac{I:l_n}{I}\right); \tag{2.1}$$

for all $i \geqslant c+2$,

$$\lambda\left(\frac{(I+(l_n,\dots,l_{i+1})):l_i}{I+(l_n,\dots,l_{i+1})}\right) \leqslant \lambda\left(\frac{(I+(l_n,\dots,l_i))^{\mathrm{sat}}+(l_{i-1})}{I+(l_n,\dots,l_{i-1})}\right) K_{i-1}, \tag{2.2}$$

where for all $i=1,\dots,n$ the integer $K_i = k(I+(l_n,\dots,l_{i+1}),l_i)$ denotes the index of saturation of $I+(l_n,\dots,l_{i+1})$ with respect to l_i (cf. Remark 2.1).

Proof of (2.1). For the sake of notational simplicity we set $r \doteq \max\{d, \operatorname{reg}(I+(l_n))\}$ and $\lambda \doteq \lambda((I:l_n)/I)$. We want to prove that I is $(r+\lambda)$-regular. We do this by (a special case of) the regularity criterion ([BS87, Theorem 1.10] or [BM93, Theorem 3.3]), which says that an ideal J which is generated in degree $\leqslant d$ is m-regular, if $((J:l)/J)_m = 0$ and $m \geqslant \max\{d, \operatorname{reg}(J+(l))\}$, where l is an almost-regular linear form for R/J. Thus, we only have to prove that $((I:l_n)/I)_{r+i} = 0$ for some $i \leqslant \lambda$. Consider the chain

$$((I:l_n)/I)_{\geqslant r} \supset ((I:l_n)/I)_{\geqslant r+1} \supset \cdots \supset ((I:l_n)/I)_{\geqslant r+\lambda} \supset ((I:l_n)/I)_{\geqslant r+\lambda+1}$$

and observe that, if one of the inclusion is not strict, then one has $((I:l_n)/I)_{r+i} = 0$ for some integer $0 \leqslant i \leqslant \lambda$, so that I is $(r+\lambda)$-regular. If all of the above inclusions were strict, one would obtain the contradiction $\lambda \geqslant \lambda+1$.

Proof of (2.2). We first prove that, for all $i \geqslant c+1$, one has

$$\lambda\left(\frac{(I+(l_n,\dots,l_{i+1})):l_i}{I+(l_n,\dots,l_{i+1})}\right) = \lambda\left(\frac{(I+(l_n,\dots,l_{i+1}))^{\mathrm{sat}}+(l_i)}{I+(l_n,\dots,l_i)}\right). \tag{2.3}$$

To do so fix an integer i with $c+1 \leqslant i \leqslant n$ and let $J \doteq I+(l_n,\dots,l_{i+1})$. A repeated application of Lemma 2.2 yields

$$\begin{aligned}
\lambda\left(\frac{J:l_i}{J}\right) &= \lambda\left(\frac{(J:l_i)+(l_i)}{J+(l_i)}\right) + \lambda\left(\frac{J:l_i^2}{J:l_i}\right)\\
&= \lambda\left(\frac{(J:l_i)+(l_i)}{J+(l_i)}\right) + \lambda\left(\frac{(J:l_i^2)+(l_i)}{(J:l_i)+(l_i)}\right) + \lambda\left(\frac{J:l_i^3}{J:l_i^2}\right)\\
&= \lambda\left(\frac{(J:l_i)+(l_i)}{J+(l_i)}\right) + \lambda\left(\frac{(J:l_i^2)+(l_i)}{(J:l_i)+(l_i)}\right) + \cdots + \lambda\left(\frac{(J:l_i^{K_i})+(l_i)}{(J:l_i^{K_i-1})+(l_i)}\right)\\
&= \lambda\left(\frac{J^{\mathrm{sat}}+(l_i)}{J+(l_i)}\right),
\end{aligned}$$

as required.

For all $i \geqslant c+2$ we now prove that

$$\lambda\left(\frac{(I+(l_n,\dots,l_{i+1}))^{\mathrm{sat}}+(l_i)}{I+(l_n,\dots,l_i)}\right) \leqslant \lambda\left(\frac{(I+(l_n,\dots,l_i))^{\mathrm{sat}}+(l_{i-1})}{I+(l_n,\dots,l_{i-1})}\right) K_{i-1}. \tag{2.4}$$

Since $(I+(l_n,\dots,l_{i+1}))^{\mathrm{sat}}+(l_i) \subset (I+(l_n,\dots,l_i))^{\mathrm{sat}} = (I+(l_n,\dots,l_i)):l_{i-1}^{K_{i-1}}$ and in view of the K_{i-1} inclusions

$$I+(l_n,\dots,l_i) \subset (I+(l_n,\dots,l_i)):l_{i-1} \subset \cdots \subset (I+(l_n,\dots,l_i)):l_{i-1}^{K_{i-1}}$$

we may restrict ourselves to showing that, for all positive integers a and $i \geqslant c+2$, one has

$$\lambda\left(\frac{(I+(l_n,\dots,l_i)) : l_{i-1}^{a}}{(I+(l_n,\dots,l_i)) : l_{i-1}^{a-1}}\right) \leqslant \lambda\left(\frac{(I+(l_n,\dots,l_i))^{\mathrm{sat}}+(l_{i-1})}{I+(l_n,\dots,l_{i-1})}\right). \tag{2.5}$$

But this follows by repeated use of Lemma 2.2 applied to the ideal $I+(l_n,\dots,l_i)$. By (2.3) and (2.4) we get (2.2).

We now complete the proof of the theorem. By repeated application of (2.3) and (2.2) we obtain

$$\begin{aligned}\lambda\left(\frac{I : l_n}{I}\right) &\leqslant \lambda\left(\frac{(I+(l_n,\dots,l_{c+2}))^{\mathrm{sat}}+(l_{c+1})}{I+(l_n,\dots,l_{c+1})}\right) K_{c+1}\cdot K_{c+2}\cdots K_{n-1}\\ &\leqslant d^c \prod_{i=c+1}^{n-1} K_i.\end{aligned}$$

The last inequality is due to the fact that

$$\frac{(I+(l_n,\dots,l_{c+2}))^{\mathrm{sat}}+(l_{c+1})}{I+(l_n,\dots,l_{c+1})} \subset R/(I+(l_n,\dots,l_{c+1})).$$

The latter is a subring of $S=K[X_1,\dots,X_c]/(f_1,\dots,f_c)$, where $f_1,\dots,f_c$ is a regular sequence of c elements of degree $\leqslant d$ so that S is of length at most d^c.

By virtue of (2.1) we now have $\operatorname{reg}(I) \leqslant \max\{d, \operatorname{reg}(I+(l_n))\} + d^c \prod_{i=c+1}^{n-1} K_i$ and it is sufficient to prove that $K_i = k(I+(l_n,\dots,l_{i+1}), l_i) \leqslant \operatorname{reg}(I+(l_n,\dots,l_{i+1}))$ for all $i=c+1,\dots,n-1$. But this inequality has been observed in Remark 2.1. $\square$

Remark 2.5. Let $I \subset K[X_1,\dots,X_n]$ be a homogeneous ideal generated in degree $\leqslant d$. If the height of I is n, we have (cf. Remark 2.3) that $\operatorname{reg}(I) \leqslant n(d-1)+1$. Furthermore, if $\operatorname{ht} I = 1$ then there exists a homogeneous polynomial f of degree $0 < a \leqslant d$ such that $I=(f)J$ and J is an ideal generated in degree $\leqslant d-a$. Thus, the ideal I is a shifted copy of J and $\operatorname{reg}(I)=\operatorname{reg}(J)+a$.

We are now in a position to prove the requested bounds.

Corollary 2.6. *Let $I \subset K[X_1,\dots,X_n]$ be a homogeneous ideal of height $c<n$ generated in degree $\leqslant d$. Then*

$$\operatorname{reg}(I) \leqslant (d^c+(d-1)c+1)^{2^{n-c-1}}.$$

Proof. Let $l_n,\dots,l_{c+1}$ be an almost-regular sequence of linear forms for R/I. By virtue of Theorem 2.4 we are able to compute a bound for the regularity of $I+(l_n,\dots,l_i)$, $i \geqslant c+1$, in the following way. First we observe that the regularity of $I+(l_n,\dots,l_i)$ equals that of its image $\bar{I}$ in $R/(l_n,\dots,l_i) \simeq K[X_1,\dots,X_{i-1}]$. Moreover, the quotient algebra $R/(I+(l_n,\dots,l_{c+1})) \simeq K[X_1,\dots,X_c]/\bar{I}$ is Artinian so that $\operatorname{reg}(I+(l_n,\dots,l_{c+1})) = \operatorname{reg}(\bar{I}) \leqslant c(d-1)+1 \doteq B_0$ (cf. Remark 2.3). Now we apply Theorem 2.4 to the image of $I+(l_n,\dots,l_{c+2})$ in $K[X_1,\dots,X_{c+1}]$ and we obtain that $\operatorname{reg}(I+(l_n,\dots,l_{c+2})) \leqslant (d-1)c+1+d^c \doteq B_1$. For all $i \geqslant 2$ we define $B_i \doteq B_{i-1}+d^c\prod_{j=1}^{i-1} B_j$. Then $B_i=(B_{i-1}-B_{i-2})B_{i-1}+B_{i-1} \leqslant (B_{i-1})^2$. Hence $B_i \leqslant (B_1)^{2^{i-1}}$ for all $i \geqslant 1$. Moreover, by Theorem 2.4, we get by induction on i that $\operatorname{reg}(I+(l_n,\dots,l_{c+i+1})) \leqslant B_i$ for all $i \leqslant n-c$. So $\operatorname{reg}(I) \leqslant B_{n-c} \leqslant ((d-1)c+1+d^c)^{2^{n-c-1}}$, as desired. $\square$

Corollary 2.7. *Let $I \subset K[X_1,\dots,X_n]$ be an ideal generated in degree $\leqslant d$. If $n=2$ then $\operatorname{reg}(I) \leqslant 2d-1$. If $n \geqslant 3$, we have*

$$\operatorname{reg}(I) \leqslant (d^2+2d-1)^{2^{n-3}} \leqslant (2d)^{2^{n-2}}.$$

Proof. The case $n=2$ is easy. So, let $n \geqslant 3$. Since the bound of Corollary 2.6 is decreasing as a function of c, our statement is clear if $\operatorname{ht}(I)>1$. The case $\operatorname{ht}(I)=1$ is obvious by Remark 2.5. $\square$

Example 2.8. One could be interested in a slightly better estimate for the regularity and for this purpose one could follow step-by-step the proof of Corollary 2.6.

Consider, for instance, the case $n = 4$. As we have seen in the proof of Corollary 2.6, the worst possible case is provided by an ideal of height 2. Since $B_2 = (B_1 - B_0)B_1 + B_1$, the regularity of a homogeneous ideal in $K[X_1, X_2, X_3, X_4]$ is bounded by $((d^2 + 2d - 1) - (2d - 1))(d^2 + 2d - 1) + (d^2 + 2d - 1) = d^4 + 2d^3 + 2d - 1$.

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