Choosability of Toroidal Graphs Without Short Cycles

Leizhen Cai,1 Weifan Wang,2 and Xuding Zhu3

1DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING
THE CHINESE UNIVERSITY OF HONG KONG, SHATIN
NEW TERRITORIES, HONG KONG SAR, CHINA
E-mail: lcai@cse.cuhk.edu.hk

2DEPARTMENT OF MATHEMATICS, ZHEJIANG
NORMAL UNIVERSITY, JINHUA 321004, CHINA
E-mail: wwf@zjnu.cn

3DEPARTMENT OF APPLIED MATHEMATICS
NATIONAL SUN YAT-SEN UNIVERSITY
KAOSHIUNG 804, TAIWAN
E-mail: zhu@math.nsysu.edu.tw

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Abstract: Let G be a toroidal graph without cycles of a fixed length k, and χ_l(G) the list chromatic number of G. We establish tight upper bounds

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of $\chi_l(G)$ for the following values of $k$:

$$\chi_l(G) \leq \begin{cases} 
4 & \text{if } k \in \{3, 4, 5\} \\
5 & \text{if } k = 6 \\
6 & \text{if } k = 7 
\end{cases}$$

1. INTRODUCTION

Motivated by the four color problem, coloring problems on graphs embedded on various surfaces have been studied extensively in the literature. In this article, we study list colorings of toroidal graphs, i.e., graphs that can be drawn on the torus without crossing edges. The concept of list coloring, a generalization of vertex coloring, was introduced by Vizing [23] and independently by Erdős et al. [6]. For each vertex of a graph $G$, let $L(v)$ denote a list of available colors for $v$. An $L$-coloring of $G$ is a vertex coloring $\phi$ that colors each vertex $v$ by a color $\phi(v) \in L(v)$ so that no adjacent vertices receive the same color. A graph $G$ is $L$-colorable if it admits an $L$-coloring, and $k$-choosable (or $k$-list-colorable) if it is $L$-colorable for every assignment of $k$-element lists to the vertices. An $L$-coloring $\phi$ for $G$ with every vertex $v$ satisfying $|L(v)| = k$ is also called a $k$-list-coloring. The list chromatic number (or the choice number) $\chi_l(G)$ of a graph $G$ is the smallest $k$ for which $G$ is $k$-choosable.

Thomassen [18] proved that every planar graph is 5-choosable, whereas Voigt [24] gave a planar graph that is not 4-choosable. Lam et al. [16, 17], independently Wang and Lih [26, 27], showed that for any $k \in \{3, 4, 5, 6\}$, every planar graph $G$ without $k$-cycles is 4-choosable. More recently, Farzad [7] showed that a planar graph without 7-cycles is 4-choosable. Several researchers [3, 4, 9, 12, 15, 22, 28] have also studied the choosability of graphs embedded on various surfaces.

In this article, we study list chromatic numbers of toroidal graphs without cycles of specific lengths. We note that there are quite a few papers in the literature [5, 10, 11, 13, 14, 19] dealing with vertex colorings of toroidal graphs. Let $k \geq 3$ be an integer and $G$ a toroidal graph without $k$-cycles. In this article, we establish tight upper bounds of $\chi_l(G)$ for the following values of $k$:

$$\chi_l(G) \leq \begin{cases} 
4 & \text{if } k \in \{3, 4, 5\} \\
5 & \text{if } k = 6 \\
6 & \text{if } k = 7 
\end{cases}$$

**Keywords:** choosability; list coloring; toroidal graph; cycle; discharging method
Our main tool is the discharging method, which is used to obtain structural properties of toroidal graphs without $k$-cycles in Section 2. We use these results to obtain the upper bounds of $\chi_l(G)$ in Section 3, and discuss open problems in Section 4.

2. STRUCTURAL PROPERTIES

In this section, we consider structural properties of toroidal graphs without $k$-cycles, which will be used in the next section to establish upper bounds of list chromatic number of such toroidal graphs.

A graph $G$ is $k$-degenerate if every subgraph $H$ of $G$ contains a vertex of degree at most $k$. Equivalently, a $k$-degenerate graph admits a linear ordering such that the forward degree of every vertex is at most $k$. Obviously, a $k$-degenerate graph is $(k+1)$-choosable. It is well-known that every planar graph is 5-degenerate, every toroidal graph is 6-degenerate, and every planar graph without 3-cycles is 3-degenerate. Wang and Lih [26] proved that every planar graph without 5-cycles is 3-degenerate, and Fijavž [8] et al. showed that every planar graph without 6-cycles is 3-degenerate. Here we will investigate degeneracies of toroidal graphs when some short cycles are excluded from the graphs.

An embedding of $G$ is a 2-cell embedding if each face of the embedded graph is homeomorphic to the open unit disc. A cycle $C$ of $G$ is noncontractible if it separates the torus into two components with one being homeomorphic to a disc; otherwise, $C$ is noncontractible. We assume that all embeddings considered in this article are 2-cell embeddings, and use $F(G)$ to denote the set of faces of $G$ for a given embedding of $G$ on the torus.

Suppose that $G$ is a graph embedded on the torus. For $x \in V(G) \cup F(G)$, let $d_G(x)$ (simply $d(x)$) denote the degree of $x$ in $G$. A vertex (or a face) of degree $k$ is called a $k$-vertex (or $k$-face). Let $\Delta(G)$ and $\delta(G)$ denote the maximum degree and the minimum degree of the vertices of $G$, respectively. For a face $f \in F(G)$, we use $b(f)$ to denote the closed boundary walk of $f$ and write $f = [u_1u_2...u_n]$ if $u_1, u_2, ..., u_n$ are the vertices on the boundary walk in the clockwise order. The set of boundary vertices of $f$ is denoted by $V(f)$. A face $f$ of $G$ is called a simple face if $b(f)$ forms a cycle. Obviously, when $\delta(G) \geq 2$, for $k \leq 5$, each $k$-face is a simple face. For $v \in V(G)$, let $N(v)$ denote the set of neighbors of $v$ in $G$. Furthermore, let $T(v)$ and $Q(v)$ denote, respectively, the set of 3-faces and 4-faces incident with $v$. If $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$, and simply write $G - S = G[V(G) \setminus S]$.

Lemma 1. Let $G$ be a toroidal graph. If $G$ contains no 4-cycle and $\delta(G) \geq 4$, then it contains either

(a) an even cycle where each vertex is a 4-vertex, or
(b) a 6-cycle $x_1x_2...x_6x_1$ with a chord $x_1x_3$ such that $d(x_1) \leq 5$ and $d(x_i) = 4$ for every remaining vertex $x_i$. 

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**Proof.** By contradiction. Let $G$ be a connected counterexample to the lemma, i.e., $G$ is a connected toroidal graph such that $\delta(G) \geq 4$ and $G$ contains neither 4-cycles nor configurations in (a) or (b). Without loss of generality, we may assume that $G$ is embedded on the torus. We will show by the discharging method that such $G$ does not exist. Note that $G$ contains no 4-faces, and thus also no two 3-faces with a common edge.

We first define a few terms. A vertex $v$ is a *major vertex* if $d(v) \geq 5$, and a 5-vertex $v$ on a 5-face $f$ is a *special vertex for $f$* if we have the configuration shown in Figure 1. Note that a 5-vertex can be a special vertex for at most one face.

A face $f$ with $d(f) \geq 5$ is a *source* of a 5-face $f'$, and then $f'$ is a *sink* of $f$, if they have exactly the connection shown in Figure 2. The vertex $v$ in the figure is referred to as the *joint-vertex* from $f$ to $f'$. Note that $f$ can be multiple sources of $f'$ through different joint-vertices.

We define an initial weight $w$ for vertices and faces as follows:

$$w(v) = 2d(v) - 6$$ for each vertex $v$ and
$$w(f) = d(f) - 6$$ for each face $f$. 

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It follows from Euler’s formula that \( \sum_{x \in V(G) \cup F(G)} w(x) = 0 \). We now use the following discharging rules \((R1)\), \((R2)\), and \((R3)\) in this order to obtain a new weight \( w' \).

\( (R1) \) Each vertex \( v \) sends 1 to each incident 3-face, and a special vertex for a 5-face \( f \) sends 1 to \( f \).

\( (R2) \) Each vertex sends its remaining weight evenly to the remaining faces incident with it.

\( (R3) \) For each joint-vertex from a source to a sink, the source transfers \( \frac{1}{5} \) to the sink.

We now show that after discharging, the new weight \( w' \) satisfies \( w'(x) \geq 0 \) for all \( x \in V(G) \cup F(G) \), and there is a face \( f \) with \( w'(f) > 0 \).

This is a contradiction since the total weight, which is 0, is unchanged during the discharging, and hence the lemma holds.

Consider the weight \( w'(v) \) of an arbitrary vertex \( v \). Since no 3-faces in \( G \) share edges, the number of 3-faces incident with \( v \) is at most \( d(v)/2 \), and thus \( v \) sends at most \( d(v)/2 \) to 3-faces incident with it. Therefore the remaining weight of \( v \) after applying rule \((R1)\) is 1 if \( v \) is a special vertex and at least \( (2d(v) - 6) - d(v)/2 \geq 0 \) as \( d(v) \geq 4 \). It follows that \( w'(v) = 0 \).

We now consider the weight \( w'(f) \) of an arbitrary face \( f \). First we note the following three simple facts that will be used frequently in our discussions.

**Fact 1.** If a 4-vertex \( v \) is incident with at most one 3-face, then it sends at least \( \frac{1}{3} \) to every face incident with it.

**Fact 2.** A 5-vertex \( v \) sends every face incident with it at least \( \frac{1}{2} \), and at least \( \frac{2}{3} \) if \( v \) is not a special vertex.

**Fact 3.** A \( d \)-vertex with \( d \geq 6 \) sends at least 1 to each face incident with it.

If \( d(f) = 3 \), then \( w'(f) = -3 + 1 + 1 + 1 = 0 \) (rule \((R1)\)). Otherwise \( d(f) \geq 5 \), and we note that the number of major vertices in \( V(f) \) is no less than the number \( k \) of sinks of \( f \). It follows that \( w'(f) \geq (d(f) - 6) + (\frac{1}{2} - \frac{1}{5})k > 0 \) (rule \((R3)\) and Facts 2 and 3) unless \( d(f) = 6 \) and \( f \) has no sink. For this case, because \( G \) contains no configuration in (a), \( V(f) \) contains at least one major vertex and thus \( w'(f) \geq \frac{1}{2} > 0 \) (Facts 2 and 3).

If \( f \) is a 5-face, then \( w(f) = -1 \) and we consider three cases depending on the number of major vertices in \( V(f) \).

**Case 1.** \( V(f) \) contains no major vertex, i.e., every vertex of \( V(f) \) is a 4-vertex.

In this case, \( f \) is not a source. Consider an arbitrary vertex \( v \in V(f) \). If \( v \) is incident with at most one 3-face, then \( v \) sends at least \( \frac{1}{3} \) to \( f \) (Fact 1). Otherwise since \( G \) contains no configuration in (b), there is a source \( f' \) with \( v \) being a joint-vertex from \( f' \) to \( f \), and thus \( f' \) sends \( \frac{1}{5} \) to \( f \) through \( v \) (rule \((R3)\)). It follows that at least \( \frac{1}{5} \) will be sent to \( f \) through \( v \), and thus \( w'(f) \geq -1 + 5 \cdot \frac{1}{5} = 0 \). Furthermore, \( w'(f) = 0 \) if and only if \( f \) has the configuration shown in Figure 3.
Case 2. $V(f)$ contains exactly one major vertex $v$.

In this case, $f$ is not a source either. If $d(v) \geq 6$ or $v$ is a special vertex for $f$, then $v$ sends at least 1 to $f$, implying $w'(f) \geq 0$. Otherwise, $v$ cannot be a special vertex for other 5-faces and thus it sends $\frac{2}{3}$ to $f$ (Fact 2). If both neighbors $x$ and $y$ of $v$ in $V(f)$ are incident with two 3-faces, then we have the situation in Figure 4. Since $v$ is not a special vertex for $f$, one of the two vertices $x'$ and $y'$, say $x'$, must be a 4-vertex; but this yields a configuration in (b), a contradiction. Therefore one of $x$ and $y$ is incident with at most one 3-face and hence sends at least $\frac{1}{3}$ to $f$ (Fact 1). Therefore $w'(f) \geq -1 + \frac{2}{3} + \frac{1}{3} \geq 0$.

Case 3. $V(f)$ contains at least two major vertices.

First note that $f$ can be a source of at most two sinks. If $V(f)$ has more than two major vertices, then $w'(f) \geq -1 + 3 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} > 0$ (Facts 2 and 3). Otherwise $V(f)$ has exactly two major vertices. If neither is a special vertex for a 5-face other than $f$, then each major vertex sends at least $\frac{2}{3}$ to $f$ and thus $w'(f) \geq -1 + 2 \cdot \frac{2}{3} - \frac{1}{3} > 0$ as $f$ has at most one sink. Otherwise one major vertex $v$ is a special vertex for a 5-face other than $f$, and thus the other major vertex is adjacent with $v$ and $f$ has no sink (see Figures 1 and 2). Let $u$ be the 4-vertex in $V(f)$ adjacent with $v$. Then edge $uv$ in not in the boundary of any 3-face and therefore $u$ is incident with at most one 3-face. It follows
that these two major vertices together send at least 1 to \( f \) (Facts 2 and 3) and \( u \) sends at least \( \frac{1}{3} \) to \( f \), and thus \( w'(f) > 0 \). We conclude that \( w'(f) > 0 \) for Case 3.

We have shown so far that \( w'(f) \geq 0 \) for every \( f \in V(G) \cup F(G) \), and we prove now that there is a face \( f \) with \( w'(f) > 0 \). Suppose that \( w'(f) = 0 \) for every face \( f \). From previous arguments we know that \( w'(f) > 0 \) for any 5-face with at least two major vertices (Case 3) or any face \( f \) with \( d(f) \geq 6 \). Therefore \( G \) has the following additional property:

Every face is either a 3-face or 5-face, which implies that \( G \) has no special vertex, neither source nor sink.

Let \( f \) be a 5-face of \( G \). If \( f \) has no major vertex, then \( f \) has the configuration in Figure 3 which is impossible as it contains sources. Therefore \( f \) has exactly one major vertex \( v \).

Note that if \( d(v) \geq 7 \) then \( v \) sends \( f \) more than 1, which makes \( w'(f) > 0 \). Therefore we may assume that \( d(v) \leq 6 \). If \( d(v) = 5 \), then \( v \) sends at least \( \frac{2}{3} \) to \( f \). By Fact 1 and the fact that \( w'(f) = 0 \), we deduce that all but one 4-vertex \( y \) in \( V(f) \) are incident with two 3-faces, which implies that \( y \) is adjacent with \( v \) and we have the situation in Figure 5. Since every 5-face has exactly one major vertex, vertex \( x' \) in Figure 5 is a 4-vertex, which yields configuration (b) in \( G \), a contradiction.

Otherwise \( d(v) = 6 \). Let \( v_0, v_1, v_2, v_3, v_4, v_5 \) be the neighbors of \( v \) ordered clockwisely, and let \( P_i \) be the \((v_i, v_{i+1})\)-path (modulo 6) along the boundary of the face containing vertices \( v_i, v, v_{i+1} \). Since every 5-face has exactly one major vertex and no 3-faces have common edges, all vertices in each \( P_i \) are 4-vertices. Furthermore, all internal vertices of \( P_i \)'s are distinct as \( \delta(G) \geq 4 \) and \( G \) has no 4-cycles, and therefore \( P_0P_1P_2P_3P_4P_5 \) is an even cycle consisting of 4-vertices only as each \( P_i \) contains an odd number of edges. This contradicts the assumption that \( G \) has no configuration (a). It follows that such \( G \) does not exist, and hence the truth of the lemma.

Lemma 2. Let \( G \) be a toroidal graph without 5-cycles. Then \( \delta(G) \leq 4 \); and \( \delta(G) = 4 \) if and only if \( G \) is 4-regular.

Proof. Suppose that the lemma is false. Let \( G \) be a connected counterexample which is embedded on the torus. Thus, \( \delta(G) \geq 4 \), \( \Delta(G) \geq 5 \) and without 5-cycles. In particular, \( G \) contains neither 5-face, nor 3-face adjacent with a 4-face, nor three consecutive
adjacent 3-faces. For each vertex \( x \in V(G) \), let \( w(x) = 2d(x) - 6 \) and, for each face \( x \in F(G) \), let \( w(x) = d(x) - 6 \). By applying Euler’s formula, \( \sum_{x \in V(G) \cup F(G)} w(x) = 0 \). Similarly as the proof of Lemma 1, we use the discharging method to derive a contradiction. The discharging rule is the following:

(R) If \( v \) is a vertex of degree at least 4, then we, from \( v \), transfer 1 to each incident 3-face and \( \frac{1}{3} \) to each incident 4-face.

Let \( w' \) denote the resulting weight function after discharging. Suppose \( f \in F(G) \). Then \( d(f) \neq 5 \). If \( d(f) = 6 \), then \( w'(f) = w(f) = 0 \). If \( d(f) \geq 7 \), then \( w'(f) = w(f) \geq 1 \). If \( d(f) = 3 \), then \( f \) receives the amount 1 from each of its boundary vertices and hence \( w'(f) \geq -3 + 3 = 0 \). If \( d(f) = 4 \), \( f \) receives \( \frac{1}{2} \) from each of its boundary vertices and \( w'(f) \geq -2 + 4 \cdot \frac{1}{2} = 0 \).

Next suppose \( v \in V(G) \). Then \( d(v) \geq 4 \) by the assumption. Assume \( d(v) = 4 \). Then \( |T(v)| \leq 2 \) (as \( G \) has no three consecutive 3-faces). If \( |T(v)| = 0 \), then \( w'(v) \geq 2 - 4 \cdot \frac{1}{2} = 0 \). If \( |T(v)| = 1 \), then \( |Q(v)| \leq 1 \) (otherwise \( G \) contains a 3-face adjacent with a 4-face). Thus \( w'(v) \geq 2 - 1 - \frac{1}{2} = \frac{1}{2} \). If \( |T(v)| = 2 \), then \( |Q(v)| = 0 \) and hence \( w'(v) = 2 - 1 - 1 = 0 \). Assume \( d(v) = 5 \). Then \( |T(v)| \leq 3 \). If \( |T(v)| = 2 \), then \( w'(v) \geq 4 - 2 - 3 \cdot \frac{1}{2} = \frac{1}{2} \). If \( |T(v)| = 3 \), then \( |Q(v)| = 0 \) and hence \( w'(v) \geq 4 - 3 = 1 \). Finally, assume that \( d(v) \geq 6 \). Note that \( |T(v)| \leq \lfloor 2d(v)/3 \rfloor \leq 2d(v)/3 \) and \( |Q(v)| \leq d(v) - |T(v)| \). It follows that \( w'(v) \geq w(v) - |T(v)| - \frac{1}{2} |Q(v)| \geq 2d(v) - 6 - |T(v)| - \frac{1}{2} (d(v) - |T(v)|) = \frac{3}{2} d(v) - \frac{1}{2} |T(v)| - 6 \geq \frac{3}{2} d(v) - \frac{1}{2} \frac{2}{3} d(v) - 6 = \frac{7}{6} d(v) - 6 > 0 \).

The preceding argument shows actually that \( w'(x) > 0 \) if \( x \) is either a face of degree at least 7 or a vertex of degree at least 5, and \( w'(x) \geq 0 \) for all other vertices and faces \( x \). Since \( G \) is not a 4-regular graph and \( \delta(G) \geq 4 \), it follows that

\[
\sum_{x \in V(G) \cup F(G)} w(x) = \sum_{x \in V(G) \cup F(G)} w'(x) > 0,
\]

a contradiction.

Lemma 3. If \( G \) is a toroidal graph without 6-cycles, then \( \delta(G) \leq 4 \).

Proof. Assume to the contrary that \( G \) is a connected graph embedded on the torus without 6-cycles such that \( \delta(G) \geq 5 \). Let \( w(x) = d(x) - 4 \) for all \( x \in V(G) \cup F(G) \). Then \( \sum_{x \in V(G) \cup F(G)} w(x) = 0 \). Apply the following discharging rule:

(R) Every vertex \( v \) transfers \( \frac{1}{2} \) to each incident 3-face.

Let \( w' \) denote the new weight function after discharging. Suppose \( v \in V(G) \). Then \( d(v) \geq 5 \). Since \( G \) contains no 6-cycles, \( v \) is not incident with four consecutive adjacent 3-faces. If \( d(v) \geq 6 \), then \( |T(v)| \leq d(v) - 2 \) and hence \( w'(v) = d(v) - 4 - \frac{1}{3} |T(v)| \geq d(v) - 4 - \frac{1}{3} (d(v) - 2) = \frac{2}{3} (d(v) - 5) > 0 \). Assume \( d(v) = 5 \), so \( w'(v) = 1 \) and \( |T(v)| \leq 3 \). When \( |T(v)| \leq 2 \), \( w'(v) \geq 1 - 2 \cdot \frac{1}{3} = \frac{1}{3} \). When \( |T(v)| = 3 \), \( w'(v) = 1 - 3 \cdot \frac{1}{3} = 0 \).

Suppose \( f \in F(G) \). If \( d(f) = 3 \), then \( w'(f) \geq -1 + 3 \cdot \frac{1}{3} = 0 \). If \( d(f) = 4 \), then \( w'(f) = w(f) = 0 \). If \( d(f) \geq 5 \), then \( w'(f) = w(f) = d(f) - 4 \geq 1 \).

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Therefore, it follows that \( w'(x) \geq 0 \) for every \( x \in V(G) \cup F(G) \). Since \( \sum_{x \in V(G) \cup F(G)} w'(x) = \sum_{x \in V(G) \cup F(G)} w(x) = 0 \), we conclude that \( w'(x) = 0 \) for every \( x \in V(G) \cup F(G) \). The above argument implies that \( G \) is a 5-regular graph with each vertex being incident with exactly three 3-faces and two 4-faces. We shall show such graphs always have 6-cycles, which contradicts our assumption.

Let \( f = [x_1x_2x_3x_4] \) be a 4-face. For \( i = 1, 2, 3, 4 \), let \( f_i \) denote the adjacent face of \( f \) with \( x_i x_{i+1} \) as a boundary edge, where the indices are taken modulo 4. If \( f \) is adjacent with two consecutive 3-faces, say \( f_1, f_2 \), then it is easy to see that the vertices in \( V(f) \cup V(f_1) \cup V(f_2) \) give rise to a 6-cycle. If \( f \) is adjacent with two consecutive 4-faces, say \( f_3, f_4 \), then the vertex \( x_3 \) is incident with at least three 4-faces, contradicting the previous paragraph. Without loss of generality, we may assume that \( f_1 = [x_1y_1x_2] \) and \( f_2 = [x_3y_2x_4] \) are 3-faces, and \( f_3 = [x_1y_1x_2] \) and \( f_4 = [x_2u_1u_2x_3] \) are 4-faces. If \( y_1 \neq y_2 \), then \( x_1y_1x_2x_3y_2x_4x_1 \) is a 6-cycle. If \( y_1 = y_2 \), then either \( u_1 \neq x_4 \) or \( u_2 \neq x_1 \). When \( u_1 \neq x_4 \) and \( u_2 \neq x_1 \), \( x_1y_1x_2x_3y_2x_4x_1 \) is a 6-cycle. When \( u_1 = x_4 \) and \( u_2 \neq x_1 \), \( x_1y_1x_2x_3u_2x_4x_1 \) is a 6-cycle. When \( u_1 \neq x_4 \) and \( u_2 = x_1 \), \( x_1u_1x_3y_2x_4x_1 \) is a 6-cycle. \( \square \)

**Lemma 4.** Let \( G \) be a toroidal graph without 7-cycles. If \( \Delta(G) \geq 6 \) and \( K_6 \not\subseteq G \), then \( \delta(G) \leq 4 \).

**Proof.** By contradiction. Assume that \( G \) is a connected counterexample, which is embedded on the torus. Again, let \( w(x) = d(x) - 4 \) for all \( x \in V(G) \cup F(G) \), so \( \sum_{x \in V(G) \cup F(G)} w(x) = 0 \). Before introducing the discharging rules, we need a new concept. Let \( f \) be a face adjacent with a 3-face \( f' = [xyz] \) such that \( x \in b(f) \cap b(f') \). Then the vertex \( z \) is called a leaf of the face \( f \). The discharging rules are as follows:

\((R1)\) Every vertex \( v \) transfers \( \frac{1}{4} \) to each incident 3-face.

\((R2)\) Every face \( f \) of degree at least 6 transfers \( \frac{1}{5} \) to each of its leaves.

Let \( w' \) be the new weight function. Let \( f \in F(G) \). If \( d(f) = 3 \), then \( f \) receives \( \frac{1}{4} \) from each of its boundary vertices by \((R1)\), and thus \( w'(f) \geq -1 + 3 \cdot \frac{1}{4} = 0 \). If \( 4 \leq d(f) \leq 5 \), then \( w'(f) = w(f) \geq 0 \). If \( d(f) \geq 6 \), then, since \( f \) has at most \( d(f) \) leaves, \( w'(f) \geq d(f) - 4 - \frac{1}{5}d(f) = \frac{2}{5}(d(f) - 6) \geq 0 \).

Next let \( v \in V(G) \). If \( d(v) \geq 6 \), then \( f \) is incident with at most \( d(v) - 3 \) 3-faces for otherwise the induced subgraph \( G[N(v) \cup \{v\}] \) contains a 7-cycle. Therefore, \( w'(v) \geq d(v) - 4 - \frac{1}{4}(d(v) - 1) = \frac{1}{5}(2d(v) - 11) > 0 \). Finally assume that \( d(v) = 5 \). Then \( w(v) = 1 \). If \( |T(v)| \leq 3 \), then \( w'(v) \geq 1 - 3 \cdot \frac{1}{4} = 0 \) by \((R1)\). Suppose \( |T(v)| \geq 4 \). Let \( x_1, x_2, \ldots, x_5 \) denote the neighbors of \( v \) in clockwise order and let \( f_1 = [vx_1x_2], f_2 = [vx_2x_3], f_3 = [vx_3x_4], f_4 = [vx_4x_5], \) and \( f_5 \) denote the faces of \( G \) incident with the vertex \( v \). For \( i = 1, 2, 3, 4 \), let \( g_i \) denote the adjacent face of \( f_i \) with \( x_i x_{i+1} \) in \( b(f_i) \cap b(g_i) \). If \( |T(v)| = 5 \), we further denote by \( g_5 \) the adjacent face of \( f_5 \) with \( x_5x_1 \) in \( b(f_5) \cup b(g_5) \).

**Claim 1.** \( d(g_i) \neq 5 \) for all \( i \in \{1, 2, 3, 4\} \).

Assume to the contrary that some \( g_i \) is a 5-face. We shall show that \( G \) contains a 7-cycle, contrary to our assumption. Note that the boundary of a 5-face is a simple
cycle. By symmetry, it suffices to consider two cases as follows.

(i) \( g_1 = [x_1y_1y_2y_3x_2]. \) Since \( d(G) \geq 5, \) it is easy to derive that \( v \notin \{y_1, y_2, y_3\} \) and \( \{y_1, y_2, y_3\} \cap \{x_3, x_4, x_5\} \leq 1. \) In particular, since \( d(x_2) \geq 5, \) we have \( x_3 \neq y_3. \) If \( x_3 \notin \{y_1, y_2\}, \) then \( G \) contains a 7-cycle \( vx_1y_1y_2y_3x_2x_3v. \) If \( x_3 = y_2, \) then \( G \) contains a 7-cycle \( vx_1x_2y_3x_4x_5v. \) If \( x_3 = y_1, \) then \( G \) contains a 7-cycle \( vx_1x_2y_3y_2x_3x_4v. \) A contradiction is always established.

(ii) \( g_2 = [x_2y_1y_2y_3x_3]. \) Similarly, we first note that \( v \neq y_1, y_2, y_3. \) Furthermore, at most one of \( x_1, x_4 \) belongs to \( \{y_1, y_2, y_3\}. \) Consequently, a 7-cycle is contained in the induced subgraph \( G((V(g_2) \cup N(v)) \setminus \{x_5\}). \) This proves Claim 1.

**Claim 2.** \( d(g_i) \neq 4 \) for \( i = 2, 3. \)

Assume the Claim is false, and without loss of generality, assume that \( g_2 = [x_2y_1y_2x_3] \) is a 4-face. Note that \( v \neq y_1, y_2 \) and at most one of \( y_1, y_2 \) is identical to some vertex in \( \{x_1, x_4, x_5\}. \) If \( y_1 = x_4, \) then \( vx_1x_2y_3x_4x_5v \) is a 7-cycle. If \( y_1 = x_5 \) or \( y_2 = x_5, \) then \( vx_1x_2y_1y_2x_3x_4v \) is a 7-cycle. If \( y_2 = x_1, \) then \( vx_1y_1x_2x_3x_4x_5v \) is a 7-cycle. This proves Claim 2.

The remaining part of the proof is split into two cases.

**Case 1.** \( |T(v)| = 5. \)

By Claims 1 and 2, \( d(g_i) = 3 \) or \( d(g_i) \geq 6 \) for all \( i = 1, 2, \ldots, 5. \) First consider the case that there exist four \( g_i \)'s of degree 3, say \( g_i = [x_i z_i x_i-1] \) for \( i = 1, 2, 3, 4. \) If there is some \( z_i \notin N(v), \) then \( vx_i z_i x_i+1 x_{i+2} \ldots x_{i-1} x_{i+1}v \) is a 7-cycle. Thus assume that \( \{z_1, z_2, z_3, z_4\} \subseteq N(v). \) It is easy to observe that \( z_1 = x_4, z_2 = x_5, z_3 = x_1, \) and \( z_4 = x_2. \) It follows that the edges \( x_1x_4, x_4x_2, x_2x_5, x_5x_3, x_3x_1 \in E(G) \) and \( G[\{v, x_1, x_2, x_3, x_4, x_5\}] \) induces a \( K_6, \) contrary to our assumption. Thus we may assume that there exist \( k \neq j \) such that \( d(g_k) \geq 6 \) and \( d(g_j) \geq 6. \) Since both \( g_k \) and \( g_j \) have the same leaf \( v, \) \( w'(v) \geq 1 + 2 - \frac{1}{3} - 5 - \frac{1}{3} = 0 \) by (R1) and (R2).

**Case 2.** \( |T(v)| = 4. \)

In this case, \( d(f_5) \geq 4. \) If either \( d(g_2) \geq 6 \) or \( d(g_3) \geq 6, \) then \( v \) receives at least \( \frac{1}{3} \) from \( v \) and \( g_3 \) by (R2) and hence \( w'(v) \geq 1 + \frac{1}{3} - 4 - \frac{1}{3} = 0. \) Otherwise, by Claims 1 and 2, we may assume that \( d(g_2) = d(g_3) = 3. \) Similarly to the proof of Case 1, we derive that \( g_2 = [x_2x_3x_5] \) and \( g_3 = [x_1x_4] \). It follows that \( x_1x_3, x_1x_4, x_2x_5, x_3x_5 \in E(G). \) This implies that the 3-cycles \( vx_2x_3v, vx_3x_5v, vx_1x_3v, \) and \( vx_1x_4v \) are all noncontractible. By Claim 1, \( d(g_1) \neq 3. \) Also \( d(g_1) \neq 3 \) by the embedding property of \( G \) on the torus. Assume that \( g_1 = [x_1u_1u_2x_2] \) is a 4-face. Thus \( v \neq u_1, u_2 \) and at most one of \( u_1, u_2 \) is identical to some of \( x_3, x_4, x_5. \) If \( u_1, u_2 \notin \{x_3, x_4, x_5\}, \) \( G \) has a 7-cycle \( vx_1u_1u_2x_3x_4v. \) If \( u_1 = x_4, \) \( G \) has a 7-cycle \( vx_1x_2u_2x_4x_5v. \) If \( u_1 = x_5, \) \( G \) has a 7-cycle \( vx_1x_2x_3x_4v. \) If \( u_2 = x_4, \) \( G \) has a 7-cycle \( vx_1u_1x_4x_5x_3v. \) If \( u_2 = x_5, \) \( G \) has a 7-cycle \( vx_1u_1x_5x_4x_3v. \) So we have \( d(g_1) \geq 6, \) i.e., \( v \) is a leaf of \( g_1. \) By (R2), \( w'(v) \geq 1 + \frac{1}{3} - 4 - \frac{1}{3} = 0. \)

We have proved that \( w'(x) \geq 0 \) for all \( x \in V(G) \cup F(G). \) Hence \( w'(x) = 0 \) for all \( x \in V(G) \cup F(G). \) This implies that \( \Delta(G) \leq 5, \) contrary to our assumption. This completes the proof of the lemma.  

\[ \square \]
Obviously, if a graph $G$ embedded on the torus satisfies $d(x) = 4$ for all $x \in V(G) \cup F(G)$, then $G$ can be expressed as a $p \times q$ grid, where $p, q \geq 3$.

**Lemma 5.** If $G$ is a toroidal graph without 3-cycles, then $\delta(G) \leq 4$; and moreover $\delta(G) = 4$ if and only if $G$ is a grid.

This lemma is easily proved by taking the weight assignment $w(x) = d(x) - 4$ for all $x \in V(G) \cup F(G)$.

It is easy to see that every subgraph of a graph $G$ without $k$-cycles is a graph without $k$-cycles. Note that each of the subgraphs (a) and (b) in Lemma 1 contains a vertex of degree 4 in the original graph $G$. These facts, altogether with Lemmas 1 to 5, imply the following result.

**Theorem 6.** Let $G$ be a toroidal graph without $k$-cycles. Then the following statements hold.

(a) For $k = 3$, $G$ is 3-degenerate unless it contains an induced grid.
(b) For $k = 5$, $G$ is 3-degenerate unless it contains an induced 4-regular subgraph.
(c) For $k \in \{4, 6\}$, $G$ is 4-degenerate.
(d) For $k = 7$, $G$ is 5-degenerate.
(e) For $k \geq 8$, $G$ is 6-degenerate.

For a fixed $n \in \{4, 5, 6, 7\}$, $K_n$ is a toroidal graph with $\delta(K_n) = n - 1$ and without $(n + 1)$-cycles. Observe that the line graph of a dodecahedron is a 4-regular toroidal graph without 4-cycles. Indeed, this graph also is a planar graph. In addition, there exist infinitely many toroidal graphs without 3-cycles and having the minimum degree 3. These examples illustrate that Theorem 6 is best possible in certain sense.

### 3. CHOOSABILITY

In this section, we give tight upper bounds of $\chi_l(G)$ when the toroidal graph $G$ contains no $k$-cycles for some $3 \leq k \leq 7$. First we note the following two results due to Erdős et al. [6] and Böhme et al. [4], respectively.

**Lemma 7.** If a connected graph $G$ is neither a complete graph nor an odd cycle, then $\chi_l(G) \leq \Delta(G)$.

**Theorem 8.** For any toroidal graph $G$, $\chi_l(G) \leq 7$ with $\chi_l(G) = 7$ if and only if $K_7 \subseteq G$.

Our main result in the paper is the following tight upper bounds for $\chi_l(G)$ when the toroidal graph $G$ contains no $k$-cycles for some $3 \leq k \leq 7$. For the tightness of the bound, we note that the Mycielski graph of a 5-cycle is a (nonplanar) toroidal graph without 3-cycles and having list chromatic number 4. Aksionov and Mel’nikov [1] constructed many planar graphs without 4-cycles that have list chromatic number 4. For $n = 4, 5, 6$, $K_n$ is a toroidal graph with $\chi_l(K_n) = n$ but without $(n + 1)$-cycles. Moreover, it should
be noted that excluding cycles of length eight does not forbid the complete graph $K_7$ and thus the values of $k$ between 3 and 7 are of the most interest.

**Theorem 9.** Let $k \geq 3$ be a fixed integer and $G$ a toroidal graph without $k$-cycles. Then the following statements hold.

1. If $3 \leq k \leq 5$, then $\chi_l(G) \leq 4$.
2. If $k = 6$, then $\chi_l(G) \leq 5$.
3. If $k = 7$, then $\chi_l(G) \leq 6$ with $\chi_l(G) = 6$ if and only if $K_6 \subseteq G$.

**Proof.** First assume that $k = 6$. By Theorem 6(c), $G$ is 4-degenerate and thus is 5-choosable. This proves (2).

Assume next that $k = 7$. By Theorem 6(d), $G$ is 5-degenerate and thus $\chi_l(G) \leq 6$. If $G$ contains a subgraph $K_6$, then it is obvious that $\chi_l(G) = \chi(K_6) = 6$. Conversely, we suppose that $K_6 \not\subseteq G$. To prove that $\chi_l(G) \leq 5$, we use induction on the vertex number $|V(G)|$. The result holds clearly if $|V(G)| \leq 5$. Let $G$ be a toroidal graph without 7-cycles that satisfies $K_6 \not\subseteq G$ and $|V(G)| \geq 6$. If $\Delta(G) \leq 5$, it is easy to derive $\chi_l(G) \leq 5$ by Lemma 7. If $\Delta(G) \geq 6$, then $\delta(G) \leq 4$ by Lemma 4. According to the induction hypothesis, $G - v$ is 5-choosable, where $v$ is a vertex of minimum degree in $V(G)$. Since $d(v) \leq 4$, every 5-list-coloring of $G - v$ can be extended to a 5-list-coloring of $G$. This proves (3).

Assume now that $k \in \{3, 5\}$. If $|V(G)| \leq 4$, it is evident that $\chi_l(G) \leq 4$. Let $G$ be a toroidal graph without 3-cycles or 5-cycles and with $|V(G)| \geq 5$. We note that $G \not= K_5$. If $G$ is 4-regular, then Lemma 7 asserts that $G$ is 4-choosable. Otherwise, $G$ contains a vertex $v$ of degree at most 3 by Lemma 2 or Lemma 5. By the induction hypothesis, $G - v$ is 4-choosable. Obviously, any 4-list coloring of $G - v$ can be easily extended to a 4-list coloring of $G$.

Finally assume that $k = 4$. The result holds trivially when $|V(G)| \leq 4$. Suppose that $G$ is a toroidal graph without 4-cycles such that $|V(G)| \geq 5$. Let $L$ denote an assignment for $G$ such that $|L(v)| = 4$ for all $v \in V(G)$. If $\delta(G) \leq 3$, let $u$ be a vertex of minimum degree in $G$. By the induction hypothesis, $G - u$ is $L$-colorable. Obviously, we can extend any $L$-coloring of $G - u$ to an $L$-coloring of $G$. If $\delta(G) \geq 4$, $G$ contains the configurations (a) or (b) by Lemma 1. Thus the proof is divided into the following two cases.

**Case 1.** $G$ contains an even cycle $C = u_1u_2 \ldots u_nu_1$ such that $d(u_i) = 4$ for all $i = 1, 2, \ldots, n$.

Let $H = G - V(C)$ and $B = G[V(C)]$. By the induction hypothesis, $H$ has an $L$-coloring $\phi$. For every vertex $u_i \in V(C)$, define a list $L'(u_i) = L(u_i) \setminus \{\phi(z) | z \in V(H) \text{ and } z u_i \in E(G)\}$. Since $|L(u_i)| = 4$, it follows that $|L'(u_i)| \geq d_G(u_i) - d_H(u_i) = d_B(u_i) \geq 2$ for all $u_i \in V(C)$. Since $n \geq 4$ is even and $B$ contains no 4-cycles, $B$ is neither a complete graph nor an odd cycle. By Lemma 7, $B$ is $L'$-colorable and furthermore $G$ is $L$-colorable.

**Case 2.** $G$ contains a 6-cycle $C = x_1x_2 \ldots x_6x_1$ with a chord $x_1x_3$ such that $d(x_1) \leq 5$ and $d(x_i) = 4$ for $i = 2, 3, \ldots, 6$. 

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Since $G$ contains no 4-cycles, $C$ has no other chords except $x_1x_3$. Let $H=G-V(C)$ and $B=G[V(C)]$. By the induction hypothesis, $H$ has an $L$-coloring $\phi$. For each vertex $x_i \in V(C)$, we define the similar list $L'(x_i)$ as in the proof of Case 1. It is easy to show that $|L'(x_i)| \geq 3$ and $|L'(x_i)| \geq 2$ for all $i \neq 3$. If $|L'(x_3)| = 4$, we color $x_1$, $x_2$, $x_6$, $x_5$, $x_4$ and $x_3$ successively. So assume that $|L'(x_3)| = 3$.

If there exists a color $x \in L'(x_4) \setminus L'(x_3)$, we assign $x$ to $x_4$, then color $x_5$, $x_6$, $x_1$, $x_2$ and $x_3$ successively. If $L'(x_2) \setminus L'(x_3) \neq \emptyset$, we have a similar proof. If $|L'(x_2)| \geq 3$ for some $k \neq 3$, an $L'$-coloring of $B$ is constructed easily. Thus assume that $|L'(x_i)| = 2$ for all $i \in \{1, 2, 4, 5, 6\}$. If $L'(x_4) \neq L'(x_5)$, we color $x_5$ with a color from $L'(x_5) \setminus L'(x_4)$, then color $x_6$, $x_1$, $x_2$, $x_3$ and $x_4$ successively. If either $L'(x_5) \neq L'(x_6)$ or $L'(x_6) \neq L'(x_1)$, we can form a similar coloring. Hence, in view of the above argument, we now assume that $L'(x_3) = \{1, 2, 3\}$ and $L'(x_1) = L'(x_4) = L'(x_5) = L'(x_6) = \{1, 2\}$. Notice that at least one of the colors 1 and 2 belongs to $L'(x_2)$, say $1 \in L'(x_2)$. We color $x_2$, $x_4$, $x_6$ with 1, $x_1$, $x_5$ with 2, and $x_3$ with 3. We have succeeded in obtaining an $L'$-coloring of $B$ in each possible case. Therefore $G$ is $L$-colorable. ■

4. OPEN PROBLEMS

In this section, we give some open problems about the choosability of toroidal graphs. As mentioned in Section 1, for any $k \in \{3, 4, 5, 6\}$, every planar graph without $k$-cycles is 4-choosable [16, 17, 26, 27]. In this paper, we show that for any $k \in \{3, 4, 5\}$, toroidal graphs without $k$-cycles are also 4-choosable. Note that toroidal graphs without 6-cycles (e.g., $K_5$) may not be 4-choosable. However, $K_5$ seems to be the only obstacle for such graphs to be 4-choosable, and we propose the following conjecture, whose truth implies the above result for planar graphs.

**Conjecture 1.** Let $G$ be a toroidal graph without 6-cycles. If $G$ is $K_5$-free, then $\chi_l(G) \leq 4$.

It is known [6, 24, 25] that for every $2 \leq k \leq 4$, there exist planar graphs (and hence toroidal graphs) which are $k$-colorable but not $k$-choosable. On the other hand, every 6-colorable toroidal graph is also 6-choosable, which leads to the following question:

**Problem 2.** Is there a toroidal graph $G$ with $\chi(G) \leq 5$ and $\chi_l(G) = 6$?

The *girth* of a graph $G$ is the length of a shortest cycle in $G$. Thomassen [21] proved that every planar graph of girth at least 5 is 3-choosable, and Voigt [25] constructed a planar graph of girth 4 that is not 3-choosable. Alon and Tarsi [2] proved that every bipartite planar graph is 3-choosable, and Kronk and White [14] showed that every toroidal graph of girth at least 6 is 3-colorable (the girth requirement was later reduced to 5 by Thomassen [20]). Here we propose the following conjecture in connection with girth.

**Conjecture 3.** Every toroidal graph of girth at least 5 is 3-choosable.
Note that the above conjecture holds if we change the girth requirement to 6. A graph \( G \) embedded on the torus is a *regular tessellation* if every vertex has degree 3 and every face has degree 6.

**Theorem 10.** Every toroidal graph \( G \) of girth at least 6 is 3-choosable.

**Proof.** We first show the following claim:

(*) If \( G \) is not a regular tessellation, then \( \delta(G) \leq 2 \).

Let \( w(x) = d(x) - 4 \) for each vertex or face \( x \), and carry out the following discharging rule on \( G \): Every face \( f \) sends \( \frac{1}{3} \) to each of its boundary vertices.

It is easy to verify that the new weight function \( w' \) and each vertex or face \( x \) satisfy \( w'(x) \geq 0 \) with \( w'(x)=0 \) only if \( G \) is a regular tessellation, which implies that \( \delta(G) \leq 2 \) when it is not.

Lemma 7 and statement (*) together give us the theorem. ■

Finally, we consider toroidal \( p \times q \) grids \( G \), where \( p, q \geq 3 \). It is easy to prove by induction that \( G \) is 3-colorable for all \( p, q \geq 3 \). Furthermore, using the technique of Alon and Tarsi [2], we can prove that \( G \) is 3-choosable if either \( p=q=3 \) or both \( p \) and \( q \) are even. However, the following question remains open.

**Problem 4.** Is every toroidal grid 3-choosable?

**REFERENCES**


