The Independence Number in Graphs of Maximum Degree Three

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Abstract. We prove that a \(K_4\)-free graph \(G\) of order \(n\), size \(m\) and maximum degree at most three has an independent set of cardinality at least \(\frac{1}{7}(4n - m - \lambda - tr)\) where \(\lambda\) counts the number of components of \(G\) whose blocks are each either isomorphic to one of four specific graphs or edges between two of these four specific graphs and \(tr\) is the maximum number of vertex-disjoint triangles in \(G\). Our result generalizes a bound due to Heckman and Thomas (A New Proof of the Independence Ratio of Triangle-Free Cubic Graphs, Discrete Math. 233 (2001), 233-237).

Keywords. independence; triangle; cubic graph
We consider finite simple and undirected graphs $G = (V, E)$ of order $n(G) = |V|$ and size $m(G) = |E|$. The independence number $\alpha(G)$ of $G$ is defined as the maximum cardinality of a set of pairwise non-adjacent vertices which is called an independent set.

Our aim in the present note is to extend a result of Heckman and Thomas [6] (cf. Theorem 1 below) about the independence number of triangle-free graphs of maximum degree at most three to the case of graphs which may contain triangles. With their very insightful and elegant proof, Heckman and Thomas also provide a short proof for the result conjectured by Albertson, Bollobás and Tucker [1] and originally proved by Staton [9] that every triangle-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $\frac{5}{14} n(G)$ (cf. also [7]). (Note that there are exactly two connected graphs for which this bound is best-possible [2, 3, 5, 8] and that Fraughnaugh and Locke [4] proved that every cubic triangle-free graph $G$ has an independent set of cardinality at least $\frac{11}{30} n(G) - \frac{2}{15}$ which implies that, asymptotically, $\frac{5}{14}$ is not the correct fraction.)

In order to formulate the result of Heckman and Thomas and our extension of it we need some definitions.

A block of a graph is called difficult if it is isomorphic to one of the four graphs $K_3$, $C_5$, $K_4^*$ or $C_5^*$ in Figure 1, i.e., it is either a triangle, or a cycle of length five, or arises by subdividing two independent edges in a $K_4$ twice, or arises by adding a vertex to a $C_5$ and joining it to three consecutive vertices of the $C_5$. A connected graph is called bad if its blocks are either difficult or are edges between difficult blocks.

For a graph $G$ we denote by $\lambda(G)$ the number of components of $G$ which are bad and by $tr(G)$ the maximum number of vertex-disjoint triangles in $G$. Note that for triangle-free graphs $G$ our definition of $\lambda(G)$ coincides with the one given by Heckman and Thomas [6]. Furthermore, note that $tr(G)$ can be computed efficiently for a graph $G$ of maximum degree at most three as it equals exactly the number of non-trivial components of the graph formed by the edges of $G$ which lie in a triangle of $G$.

**Theorem 1 (Heckman and Thomas [6])** Every triangle-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7} (4n(G) - m(G) - \lambda(G))$.

Since every $K_4$ in a graph of maximum degree at most three must form a component and contributes exactly one to the independence number of the graph, we can restrict our attention to graphs that do not contain $K_4$’s.
**Theorem 2** Every $K_4$-free graph $G$ of maximum degree at most three has an independent set of cardinality at least $\frac{1}{7} (4n(G) - m(G) - \lambda(G) - tr(G))$.

**Proof:** For a graph $G$ we denote the quantity $4n(G) - m(G) - \lambda(G) - tr(G)$ by $\psi(G)$. We wish to show that $7\alpha(G) \geq \psi(G)$. For contradiction, we assume that $G = (V, E)$ is a counterexample to the statement such that $\psi(G)$ is smallest possible and subject to this condition the order $n(G)$ of $G$ is smallest possible. If $tr(G) = 0$, then the result follows immediately from Theorem 1. Therefore, we may assume $tr(G) \geq 1$. Since $\alpha(G)$ and $\psi(G)$ are additive with respect to the components of $G$, we may assume that $G$ is connected. Furthermore, we may clearly assume that $n(G) \geq 4$.

**Claim 1.** Every vertex in a triangle has degree three.

**Proof of Claim 1:** Let $x$, $y$ and $z$ be the vertices of a triangle. We assume that $d_G(x) = 2$. Clearly, the graph $G' = G[V \setminus \{x, y, z\}]$ is no counterexample, i.e., $7\alpha(G') \geq \psi(G')$. Since for every independent set $I'$ of $G'$, the set $I' \cup \{x\}$ is an independent set of $G$, we have $\alpha(G) \geq \alpha(G') + 1$. The triangle $xyz$ is vertex-disjoint from all triangles in $G'$, and so $tr(G) \geq tr(G') + 1$.

Suppose $\min\{d_G(y), d_G(z)\} = 2$. Then $\max\{d_G(y), d_G(z)\} = 3$, since $G$ is not just a triangle. Furthermore, by the definition of a bad graph, we have $\lambda(G') = \lambda(G)$ and obtain

\[
7\alpha(G) \geq 7\alpha(G') + 7 \\
\geq \psi(G') + 7 \\
\geq 4n(G') - m(G') - \lambda(G') - tr(G') + 7 \\
\geq 4(n(G) - 3) - (m(G) - 4) - \lambda(G) - (tr(G) - 1) + 7 \\
\geq \psi(G) - 12 + 4 + 1 + 7 \\
= \psi(G),
\]

which implies a contradiction. Therefore, we may assume $d_G(y) = d_G(z) = 3$. Let $N_G(y) = \{x, y', z\}$ and $N_G(z) = \{x, y, z'\}$. Regardless of whether $y' = z'$ or not, we have $tr(G) \geq tr(G') + 1$.

If $y' = z'$, then $G'$ is connected, $y'$ is a vertex of degree one in $G'$ and thus $\lambda(G') = \lambda(G) = 0$. If $y' \neq z'$ and $\lambda(G') \geq 2$, then $\lambda(G') = 2$ and $G$ is a bad graph itself, i.e., $\lambda(G) = 1$. Therefore, in both cases $\lambda(G') \leq \lambda(G) + 1$ and we obtain

\[
7\alpha(G) \geq 7\alpha(G') + 7 \\
\geq \psi(G') + 7 \\
= 4n(G') - m(G') - \lambda(G') - tr(G') + 7 \\
\geq 4(n(G) - 3) - (m(G) - 5) - (\lambda(G) + 1) - (tr(G) - 1) + 7 \\
\geq \psi(G) - 12 + 5 - 1 + 1 + 7 \\
= \psi(G),
\]

which implies a contradiction and the proof of the claim is complete. □
Claim 4. There are two edges $\overrightarrow{e}$ and $\overrightarrow{f}$ in $G'$ such that $\lambda(G') + \overrightarrow{e} \leq \lambda(G) + 1$ and $\lambda(G' + \overrightarrow{f}) \leq \lambda(G) + 1$.

Proof of Claim 4: For contradiction, we assume that $\lambda(G') + \overrightarrow{e} \geq \lambda(G) + 2$. This implies that $G'$ consists exactly of two bad components and that $G$ itself is not a bad graph. Hence $\overrightarrow{e}$ can not be an edge between two difficult blocks, since otherwise $G$ would be a bad graph. Thus both $G' + \overrightarrow{z}'$ and $G' + \overrightarrow{y}'$ are connected and the claim follows for $\{\overrightarrow{e}, \overrightarrow{f}\} = \{\overrightarrow{z}', \overrightarrow{y}'\}$. □
Claim 5. If $\lambda(G' + e) = \lambda(G' + f) = \lambda(G) + 1$, then either $tr(G' + e) \leq tr(G) - 1$ or $tr(G' + f) \leq tr(G) - 1$.

Proof of Claim 5: We may assume that $e = x'z'$ and $f = y'z'$. For contradiction, we assume that $tr(G' + e), tr(G' + f) \geq tr(G)$. This implies that $x'$ and $z'$ have a common neighbour $x''$ in $G'$ and that $y'$ and $z'$ have a common neighbour $y''$ in $G'$. If possible, we choose $x'' = y''$. Clearly, this implies that $G'$ is connected. Furthermore, since the vertices $x, y, z, x', y', z', x'', y''$ all lie in one block of $G$ which cannot be a bad block, the graph $G$ can not be a bad graph. Since $\lambda(G' + e) = \lambda(G' + f) = \lambda(G) + 1$, both $G' + e$ and $G' + f$ must be bad graphs.

If the triangle $x'z'x''$ forms a difficult block in $G' + e$, the edge $x'x''$ forms a block in $G' + f$ which does not connect two difficult blocks. This implies that $G' + f$ can not be bad which is a contradiction. Therefore, by symmetry, we may assume that the triangle $x'z'x''$ is contained in a difficult block $B_e$ in $G' + e$ which is isomorphic to $C_5^*$ and that also the triangle $y'z'y''$ is contained in a difficult block $B_f$ in $G' + f$ which is isomorphic to $C_5^*$.

![Figure 2](image)

Figure 2

First, we assume $x'' = y''$. If $e = x'z'$ is not the edge shared by the two triangles of $B_e$, then either $x'$ and $x''$ or $z'$ and $x''$ have a common neighbour in $G'$. This implies that $y'$ is adjacent to either $x'$ or $z'$ which contradicts Claim 3. Hence the edge $e = x'z'$ must be the edge shared by the two triangles of $B_e$. Now, $G'$ contains the configuration shown in Figure 2. Clearly, all six vertices in Figure 2 belong to one block of $G' + f$ which can not be a difficult block. Therefore, $G' + f$ can not be a bad graph which is a contradiction.

Next, we assume that $x'' \neq y''$. By the choice of $x''$ and $y''$, this implies that no vertex in $G'$ is adjacent to all of $x'$, $y'$ and $z'$. If $e = x'z'$ is the edge shared by the two triangles of $B_e$, then $x'$ and $z'$ must have a common neighbour in $G'$ different from $x''$. This implies that $y''$ is adjacent to all of $x'$, $y'$ and $z'$ which is a contradiction. Hence $x'z'$ is not the edge shared by the two triangles of $B_e$. If $x'x''$ is the edge shared by the two triangles of $B_e$, then the block of $G' + f$ which contains $x'$ contains two vertex-disjoint triangles. Therefore, $G' + f$ can not be a bad graph which is a contradiction. We obtain that $z'x''$ is the edge shared by the two triangles of $B_e$ which implies the existence of a vertex $z''$ such that $G$ contains the configuration shown in Figure 3.
Figure 3

Since \( G[\{x, y, z, x', y', z', x'', y'', z''\}] \) is not a counterexample, the vertex \( z'' \) has degree three. Now the graph \( G'' = G[V \setminus \{x, y, z, x', y', z', x'', y'', z''\}] \) satisfies \( \alpha(G) \geq \alpha(G'') + 3 \), \( n(G) = n(G'') + 9 \), \( m(G) = m(G'') + 14 \), \( \lambda(G'') \leq \lambda(G) + 1 \) and \( tr(G) \geq tr(G'') + 2 \) which implies a similar contradiction as before and completes the proof of the claim. □

Note that \( tr(G' + x'z') \leq tr(G') + 1 = tr(G) \). Therefore, by Claims 4 and 5, we can assume that either \( \lambda(G' + x'z') \leq \lambda(G) \) and \( tr(G' + x'z') \leq tr(G) \) or \( \lambda(G' + x'z') = \lambda(G) + 1 \) and \( tr(G' + x'z') \leq tr(G) - 1 \) both of which imply that \( \lambda(G' + x'z') + tr(G' + x'z') \leq \lambda(G) + tr(G) \).

Similarly as above, for every independent set \( I' \) of \( G' + x'z' \) either \( I' \cup \{x\} \) or \( I' \cup \{z\} \) is an independent set of \( G \) which implies \( \alpha(G) \geq \alpha(G') + 1 \). Since \( n(G' + e) = n(G) - 3 \) and \( m(G' + e) = m(G) - 5 \), we obtain a similar contradiction as above which completes the proof. □

Note that Theorem 2 is best-possible for all bad graphs, all graphs which arise by adding an edge to a bad graph and further graphs such as for instance the graph in Figure 4.

Figure 4

In [5] Heckman characterized the extremal graphs for Theorem 1. Similarly, it might be an interesting yet challenging task to characterize the extremal graphs for Theorem 2.

References


