Zero distribution of orthogonal polynomials with asymptotically periodic varying recurrence coefficients

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Abstract

Orthogonal polynomials on the real line satisfy a recurrence relation of the form $xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x)$, with $a_n > 0$ and $b_n \in \mathbb{R}$. Assuming that the recurrence coefficients depend on a parameter N and that $a_{n,N} \to a(t)$ and $b_{n,N} \to b(t)$ as $n, N \to \infty$ and $n/N \to t$, the asymptotic zero distribution was obtained by Kuijlaars and Van Assche [10] and Pastur [13]. In the present paper we assume that the recurrence coefficients are varying and asymptotically periodic with period $m \geq 1$, i.e., for $k, N \to \infty$ we have $a_{km+j,N} \to a_j(t)$ and $b_{km+j,N} \to b_j(t)$, where $km/N \to t$. We give the asymptotic distribution of the zeros and show that the zeros are dense on the union of at most m intervals and give an explicit expression of the density of the zeros. The results rely heavily on work of Ya. L. Geronimus who studied asymptotically periodic recurrence coefficients in [8] and [9].

1 Introduction

When dealing with a system of orthogonal polynomials on the real line

$$\int p_n(x)p_m(x)\,d\mu(x) = \delta_{m,n}, \qquad m, n \geq 0,$$

one can study these polynomials using information on the orthogonalizing measure μ or, alternatively, one may use explicit knowledge of the recurrence coefficients in the three-term recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \qquad n \geq 0, \tag{1}$$

with initial values $p_0 = 1$ and $p_{-1} = 0$ and $a_{n+1} > 0$, $b_n \in \mathbb{R}$ for $n \geq 0$. In this paper we will study the asymptotic distribution of the zeros of orthogonal polynomials that are given by the recurrence relation (1). A very important class of orthogonal polynomials consists of those systems which have converging recurrence coefficients

$$\lim_{n\to\infty} a_n = a \geq 0, \quad \lim_{n\to\infty} b_n = b \in \mathbb{R}, \tag{2}$$

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and this class will be denoted by $\mathrm{M}(a,b)$. It is well known (see, e.g., [12], [15], [14], [11]) that the zeros of orthogonal polynomials in $\mathrm{M}(a,b)$ are dense in the interval $[b-2a,b+2a]$ (this is known as Blumenthal's theorem); in fact the zeros are dense on $[b-2a,b+2a]\cup E$ where E is bounded and at most denumerable with accumulation points (if any) only at $b\pm 2a$. One has ratio asymptotics of the form

$$\lim_{n\to\infty} a_{n+1}\frac{p_{n+1}(z)}{p_n(z)} = \begin{cases} \frac{1}{2}\left(z-b+\sqrt{(z-b)^2-4a^2}\right) & \text{if } a>0, \\ z-b & \text{if } a=0, \end{cases} \tag{3}$$

uniformly on compact subsets of $\mathbb{C}\setminus([b-2a,b+2a]\cup E)$, and the support of the orthogonalizing measure μ for these orthogonal polynomials is $[b-2a,b+2a]\cup E$. From these properties one can obtain the asymptotic zero distribution: if $x_{1,n}<x_{2,n}<\ldots<x_{n,n}$ are the zeros of p_n, then for orthogonal polynomials in the class $\mathrm{M}(a,b)$ one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} f(x_{j,n}) = \int_{b-2a}^{b+2a} f(x)\, d\omega_{[b-2a,b+2a]}(x), \tag{4}$$

for every continuous function f, where the measure $\omega_{[\alpha,\beta]}$ is the measure on $[\alpha,\beta]$ with density

$$\frac{d\omega_{[\alpha,\beta]}(x)}{dx} = \frac{1}{\pi}\frac{1}{\sqrt{(x-\alpha)(\beta-x)}}, \qquad \alpha<x<\beta. \tag{5}$$

When $[\alpha,\beta]=[-1,1]$ then this measure is the so-called arcsin measure, which is the orthogonalizing measure for the Chebyshev polynomials of the first kind $T_n(x)=\cos n\theta$, where $x=\cos\theta$. This measure is a very important measure in logarithmic potential theory, where it is known as the equilibrium measure for the set $[\alpha,\beta]$. Its logarithmic potential

$$\int_{b-2a}^{b+2a} \log\frac{1}{|z-x|}\, d\omega_{[b-2a,b+2a]}(x) = \begin{cases} \log 2 - \log|z-b+\sqrt{(z-b)^2-4a^2}| & \text{if } z\notin[b-2a,b+2a], \\ -\log a & \text{if } z\in[b-2a,b+2a], \end{cases}$$

is constant on the support $[b-2a,b+2a]$ and this constant (Robin's constant) gives the logarithmic capacity (transfinite diameter) of $[b-2a,b+2a]$, which is a. The measure $\omega_{[\alpha,\beta]}$ minimizes the logarithmic energy

$$I(\mu) = \int_\alpha^\beta\int_\alpha^\beta \log\frac{1}{|x-y|}\, d\mu(x)\, d\mu(y)$$

over all probability measures μ supported on $[\alpha,\beta]$.

2 Varying recurrence coefficients

Quite often the recurrence coefficients a_n, b_n depend on extra parameters and if we allow these parameters to depend on the degree n, then often the behavior of the zeros will change. In [10] we assumed that the orthogonal polynomials and their recurrence coefficients depend on an extra parameter N so that the recurrence relation becomes

$$xp_{n,N}(x) = a_{n+1,N}p_{n+1,N}(x) + b_{n,N}p_{n,N}(x) + a_{n,N}p_{n-1,N}(x), \qquad n \geq 0, \tag{6}$$

and we studied the asymptotic distribution of the zeros of $p_{n,N}$ when both n and N tend to infinity but in such a way that $n/N \to t > 0$. We will denote such a limit by $\lim_{n/N\to t}$. Our assumption on the recurrence coefficients is that

$$\lim_{n/N\to t} a_{n,N} = a(t), \quad \lim_{n/N\to t} b_{n,N} = b(t), \tag{7}$$

where a and b are now continuous functions on $(0,\infty)$. A similar problem was also considered by Pastur [13]. The ratio asymptotic behavior now becomes

$$\lim_{n/N\to t} a_{n+1,N}\frac{p_{n+1,N}(z)}{p_{n,N}(z)} = \frac{1}{2}\left(z - b(t) + \sqrt{(z-b(t))^2 - 4a^2(t)}\right), \tag{8}$$

which holds uniformly on compact subsets of $\mathbb{C} \setminus Z(t)$, where $Z(t)$ is a bounded interval that contains all the zeros of the polynomials $p_{n,N}$ as $n/N \to t$. This interval $Z(t)$ can be taken as $[\min_{0\leq s\leq t}(b(s)-2a(s)), \max_{0\leq s\leq t}(b(s)+2a(s))]$ if the functions a and b are continuous at 0. The ratio asymptotics implies also nth root asymptotics since

$$p_{n,N}(z) = \prod_{k=1}^{n} \frac{p_{k,N}(z)}{p_{k-1,N}(z)},$$

so that

$$\begin{aligned} \log|p_{n,N}(z)|^{1/n} &= \frac{1}{n}\sum_{k=1}^{n} \log\left|\frac{p_{k,N}(z)}{p_{k-1,N}(z)}\right| \\ &= \int_0^1 \log\left|\frac{p_{\lceil ns\rceil,N}(z)}{p_{\lceil ns\rceil-1,N}(z)}\right| ds, \end{aligned} \tag{9}$$

where $\lceil x \rceil$ is the smallest integer greater than or equal to x. From (8) we know that the integrand converges as $n/N \to t$ to

$$\log\left|\frac{z-b(st)+\sqrt{(z-b(st))^2-4a^2(st)}}{2a(st)}\right| \\ = \int_{\alpha(st)}^{\beta(st)} \log|z-x|\, d\omega_{[\alpha(st),\beta(st)]}(x) - \log a(st),$$

where

$$\alpha(t) = b(t) - 2a(t), \quad \beta(t) = b(t) + 2a(t).$$

One can justify the limit as $n/N \to t$ inside the integral in (9) when z is on a compact set in $\mathbb{C} \setminus Z(t)$ so that

$$\lim_{n/N\to t} \log|p_{n,N}(z)|^{1/n} = \int_0^1 \int_{\alpha(st)}^{\beta(st)} \log|z-x|\, d\omega_{[\alpha(st),\beta(st)]}(x)\, ds - \int_0^1 \log a(st)\, ds.$$

Change variables $st = y$, then the double integral becomes

$$\frac{1}{t}\int_0^t \int_{\alpha(y)}^{\beta(y)} \log|z-x|\, d\omega_{[\alpha(y),\beta(y)]}(x)\, dy - \frac{1}{t}\int_0^t \log a(y)\, dy,$$

which, after changing the order of integration, becomes

$$\int_{\alpha^-(t)}^{\beta^+(t)} \log|z-x| \left(\frac{1}{t}\int_0^t d\omega_{[\alpha(y),\beta(y)]}(x)\, dy\right) - \frac{1}{t}\int_0^t \log a(y)\, dy,$$

where

$$\alpha^-(t) = \inf_{0<y\leq t} \alpha(y), \quad \beta^+(t) = \sup_{0<y\leq t} \beta(y).$$

If $\gamma_{n,N}$ is the leading coefficient of $p_{n,N}$, then this asymptotic behavior of $|p_{n,N}|^{1/n}$ implies

$$\lim_{n/N\to t} \gamma_{n,N}^{1/n} = \exp\left(-\frac{1}{t}\int_0^t \log a(y)\, dy\right), \tag{10}$$

and

$$\lim_{n/N\to t} \frac{1}{n}\sum_{k=1}^{n} \delta(x_{k,n}^N) = \frac{1}{t}\int_0^t \omega_{[\alpha(y),\beta(y)]}\, dy, \tag{11}$$

where $\delta(c)$ is the Dirac unit measure with mass one at the point c. Therefore the asymptotic distribution of the zeros $\{x_{j,n}^N : 1 \leq j \leq n\}$ of $p_{n,N}$ as n, N tend to infinity in such a way that $n/N \to t$, is given by the measure

$$\frac{1}{t}\int_0^t \omega_{[\alpha(y),\beta(y)]}\, dy$$

which is supported on $[\alpha^-(t), \beta^+(t)]$ and which is an average of the Chebyshev measures $\omega_{[\alpha(y),\beta(y)]}$ for $y \in [0,t]$. For a justification of all the steps we refer to [10], which also contains several explicit examples.

3 Asymptotically periodic recurrence coefficients

We will now extend the previously obtained result to the class of asymptotically periodic recurrence coefficients. Such orthogonal polynomials were investigated extensively by Ya. L. Geronimus (see, e.g., [8], [9]). We now deal with orthogonal polynomials for which the recurrence coefficients become periodic with a fixed period $m \geq 1$ as $n = km + j$ (with $0 \leq j < m$) tends to infinity, i.e.,

$$\lim_{k\to\infty} a_{km+j} = a_j^0, \quad \lim_{k\to\infty} b_{km+j} = b_j^0 \tag{12}$$

where a^0 and b^0 are periodic sequences with period m, i.e., $a_{j+m}^0 = a_j^0$ and $b_{j+m}^0 = b_j^0$ for every $j \in \mathbb{N}$. Geronimus found the asymptotic behavior of the zeros of the orthogonal polynomials, the ratio asymptotics, and the support of the orthogonalizing measure.

Later, Geronimo and Van Assche used perturbation techniques to study the asymptotic behavior of these polynomials (see, e.g., [5], [6], [7], but also [14], [11] and [15]). The ratio asymptotic behavior is given by

$$\lim_{n\to\infty} \frac{p_n(z)}{p_{n-m}(z)} = \frac{T_m(z) + \sqrt{T_m^2(z) - 4}}{2}, \tag{13}$$

which holds uniformly on compact sets of $\mathbb{C} \setminus Z$ (with Z a compact set that contains all the zeros of the polynomials p_n for $n \in \mathbb{N}$) where

$$T_m(z) = q_m(z) - (a_m^0)^2 q_{m-2}^{(1)}(z),$$

with q_n $(n = 0, 1, 2, \dots)$ the orthogonal polynomials with purely periodic recurrence coefficients

$$xq_n(x) = a_{n+1}^0 q_{n+1}(x) + b_n^0 q_n(x) + a_n^0 q_{n-1}(x),$$

and $q_n^{(1)}$ the associated polynomials, i.e., the orthogonal polynomials with recurrence coefficients shifted by one

$$xq_n^{(1)}(x) = a_{n+2}^0 q_{n+1}^{(1)}(x) + b_{n+1}^0 q_n^{(1)}(x) + a_{n+1}^0 q_{n-1}^{(1)}(x).$$

Observe that T_m is a polynomial of degree m. Denote by $\alpha_1 < \beta_1 \leq \alpha_2 < \beta_2 \leq \cdots \leq \alpha_m < \beta_m$ the zeros of $T_m \pm 2$ then the support of the orthogonalizing measure μ consists essentially of the m intervals $\bigcup_{k=1}^m [\alpha_k, \beta_k]$, which are at most m disjoint intervals on the real axis (see Figure 1). Some of the intervals may touch, in which case there are less than m intervals. In addition, the support may also contain a bounded set E outside these intervals, which is at most denumerable and for which the accumulation points are in the set $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_m, \beta_m\}$. The case where $m = 1$ coincides with the class $\mathrm{M}(a, b)$ for $a = a_1^0$ and $b = b_0^0$.

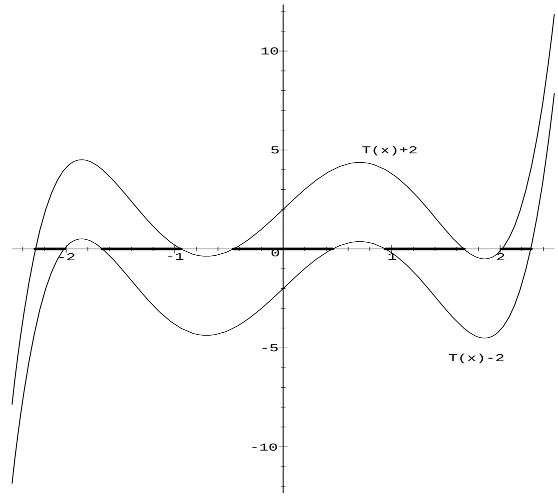

Figure 1: Essential spectrum for asymptotically periodic recurrence coefficients ($m = 5$)

The asymptotic distribution of the zeros can be found in a similar way as for the class $\mathrm{M}(a,b)$. Let $n = km + j$ with $0 \le j < m$, then

$$p_n(z) = p_j(z) \prod_{i=1}^k \frac{p_{j+im}(z)}{p_{j+(i-1)m}(z)},$$

so that

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = 2^{-1/m} |T_m(z) + \sqrt{T_m^2(z) - 4}|^{1/m},$$

and this holds uniformly on compact sets of $\mathbb{C} \setminus Z$. Since we know that

$$\int_{-2}^{2} \log \frac{1}{|z-x|} \, d\omega_{[-2,2]}(x) = \log 2 - \log |z + \sqrt{z^2-4}|, \qquad z \notin [-2,2],$$

the change of variables $z \to T_m(z)$ gives for $z \notin E_m$

$$\int_{E_m} \log \frac{1}{|z-x|} \, d\omega_{E_m}(x) = \log(2c_m)^{1/m} - \log |T_m(z) + \sqrt{T_m^2(z)-4}|^{1/m}, \tag{14}$$

with $E_m = T_m^{-1}([-2,2]) = \bigcup_{k=1}^m [\alpha_k, \beta_k]$, $T_m(z) = c_m z^m + \cdots$, where $c_m = (a_1^0 a_2^0 \cdots a_m^0)^{-1}$, and

$$d\omega_{E_m}(x) = \frac{1}{\pi m} \frac{|T_m'(x)|}{\sqrt{4 - T_m^2(x)}} \, dx$$

the equilibrium measure for the set E_m. The asymptotic distribution of the zeros of orthogonal polynomials with recurrence coefficients satisfying (12) is thus given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n f(x_{j,n}) = \frac{1}{\pi m} \int_{E_m} \frac{|T_m'(x)|}{\sqrt{4-T_m^2(x)}} f(x)\, dx, \tag{15}$$

which holds for every bounded and continuous function f.

We will now extend the results for varying recurrence coefficients given in Section 2 to the asymptotically periodic case. This means that we will consider orthogonal polynomials $p_{n,N}$ with recurrence coefficients that depend on an extra parameter N and for which

$$\lim_{km/N \to t} a_{km+j,N} = a_j(t), \qquad \lim_{km/N\to t} b_{km+j,N} = b_j(t), \tag{16}$$

where $a_0, \ldots, a_{m-1}$ are non-negative continuous functions and $b_0, \ldots, b_{m-1}$ are real continuous functions on $(0,\infty)$.

Theorem 1 *Suppose $p_{n,N}$ $(n, N \in \mathbb{N})$ are orthonormal polynomials on the real line satisfying a three-term recurrence relation*

$$x p_{n,N}(x) = a_{n+1,N} p_{n+1,N}(x) + b_{n,N} p_{n,N}(x) + a_{n,N} p_{n-1,N}(x), \qquad n \ge 0,$$

with $p_{-1,N} = 0$ and $p_{0,N} = 1$. If the recurrence coefficients satisfy (16), then

$$\lim_{n/N\to t} \frac{p_{n,N}(z)}{p_{n-m,N}(z)} = \frac{T_m(z,t) + \sqrt{T_m^2(z,t) - 4}}{2}, \tag{17}$$

uniformly on compact subsets of $\mathbb{C} \setminus Z(t)$*, where* $Z(t)$ *is a compact set containing all the zeros of* $p_{n,N}$ *as* $n, N \to \infty$ *and* $n/N \to t > 0$*. The polynomial* $T_m(z,t)$ *is given by* $q_m(z,t) - [a_m(t)]^2 q_{m-2}^{(1)}(z,t)$*, where*

$$xq_k(x,t) = a_{k+1}(t)q_{k+1}(x,t) + b_k(t)q_k(x,t) + a_k(t)q_{k-1}(x,t),$$

with $q_0 = 1$ *and* $q_{-1} = 0$*.*

From this one can now use the same reasoning as in Section 2 and observe that for $n = km + j$

$$\begin{aligned} \log |p_{n,N}(z)|^{1/n} &= \log |p_{j,N}|^{1/n} + \log \prod_{i=1}^{k} \left| \frac{p_{j+im,N}(z)}{p_{j+(i-1)m,N}(z)} \right|^{1/n} \\ &= \frac{1}{n} \log |p_{j,N}| + \frac{k}{n} \int_0^1 \log \left| \frac{p_{j+m\lceil ks \rceil,N}(z)}{p_{j+m(\lceil ks \rceil -1),N}(z)} \right| ds, \end{aligned}$$

and if we use the asymptotic behavior given in Theorem 1 then this gives

$$\begin{aligned} \lim_{n/N \to t} \log |p_{n,N}(z)|^{1/n} &= \frac{1}{m} \int_0^1 \log \frac{|T_m(z,st) + \sqrt{T_m^2(z,st) - 4}|}{2} \, ds \\ &= \frac{1}{mt} \int_0^t \log \frac{|T_m(z,s) + \sqrt{T_m^2(z,s) - 4}|}{2} \, ds. \end{aligned}$$

Now use (14) to conclude that

$$\begin{aligned} \lim_{n/N \to t} \log |p_{n,N}(z)|^{1/n} &= \frac{1}{t} \int_0^t \int_{E_m(s)} \log |z-x| \, d\omega_{E_m(s)} \, ds + \frac{1}{mt} \int_0^t \log c_m(s) \, ds \\ &= \int_{E_m^*(t)} \log |z-x| d \left(\frac{1}{t} \int_0^t \omega_{E_m(s)}(x) \, ds \right) + \frac{1}{mt} \int_0^t \log c_m(s) \, ds, \end{aligned}$$

where $c_m(s) = [a_1(s)a_2(s) \cdots a_m(s)]^{-1}$. Hence the asymptotic distribution of the zeros will now be given by the following theorem:

Theorem 2 *Suppose the recurrence coefficients of the orthogonal polynomials* $p_{n,N}(x) = \gamma_{n,N} x^n + \cdots$ *satisfy (16), then*

$$\lim_{n/N \to t} \gamma_{n,N}^{1/n} = \exp \left(- \sum_{j=1}^{m} \frac{1}{mt} \int_0^t \log a_j(s) \, ds \right),$$

and for every bounded and continuous function f *one has*

$$\lim_{n/N \to t} \frac{1}{n} \sum_{j=1}^{n} f(x_{j,n}^N) = \int_{E_m^*(t)} f(x) \, d\omega_t(x), \tag{18}$$

where

$$E_m^*(t) = \bigcup_{0 \le s \le t} E_m(s),$$

with $E_m(s) = T_m^{-1}([-2,2],s)$ *and*

$$\omega_t = \frac{1}{t}\int_0^t \omega_{E_m(s)}\,ds$$

is an average of equilibrium measures of the sets $E_m(s)$ *when* s *ranges from* 0 *to* t*.*

Each set $E_m(s)$ consists of at most m intervals $\bigcup_{k=1}^m [\alpha_k(s),\beta_k(s)]$. Let

$$\alpha_k^-(t) = \inf_{0<s\le t} \alpha_k(s), \quad \beta_k^+(t) = \sup_{0<s\le t} \beta_k(s),$$

then the set $E_m^*(t)$ can also be described as

$$E_m^*(t) = \bigcup_{k=1}^m [\alpha_k^-(t),\beta_k^+(t)],$$

so that $E_m^*(t)$ consists of at most m intervals. However, these m intervals may overlap. The functions α_k^- and β_k^+ are monotonic, so that the intervals $[\alpha_k^-(t),\beta_k^+(t)]$ are nested as t increases. Hence the number of intervals in $E_m^*(t)$ is at most m and at least 1, and as t increases there are at most a $m-1$ values $t_1 \le t_2 \le \cdots \le t_{m-1}$ where the number of intervals decreases.

4 Examples

Let us consider the case $m = 2$ and suppose that

$$\lim_{n/N\to t} a_{2n,N} = a_0(t), \quad \lim_{n/N\to t} a_{2n+1,N} = a_1(t),$$
$$\lim_{n/N\to t} b_{2n,N} = b_0(t), \quad \lim_{n/N\to t} b_{2n+1,N} = b_1(t),$$

then $E_2(s) = [\alpha_1(s),\beta_1(s)] \cup [\alpha_2(s),\beta_2(s)]$ with

$$\begin{aligned}
\alpha_1(s) &= \frac{b_0(s)+b_1(s)}{2} - \sqrt{\left(\frac{b_0(s)-b_1(s)}{2}\right)^2 + (a_0(s)+a_1(s))^2} \\
\beta_1(s) &= \frac{b_0(s)+b_1(s)}{2} - \sqrt{\left(\frac{b_0(s)-b_1(s)}{2}\right)^2 + (a_0(s)-a_1(s))^2} \\
\alpha_2(s) &= \frac{b_0(s)+b_1(s)}{2} + \sqrt{\left(\frac{b_0(s)-b_1(s)}{2}\right)^2 + (a_0(s)-a_1(s))^2} \\
\beta_2(s) &= \frac{b_0(s)+b_1(s)}{2} + \sqrt{\left(\frac{b_0(s)-b_1(s)}{2}\right)^2 + (a_0(s)+a_1(s))^2}
\end{aligned}$$

As a first example, we consider the case where

$$a_0(t) = a_0 t^\alpha,\ a_1(t) = a_1 t^\alpha, \quad b_0(t) = b_0 t^\alpha,\ b_1(t) = b_1 t^\alpha, \tag{19}$$

then $E_2(s) = E_2(1)s^\alpha$, so that each $E_2(s)$ consists of two intervals. The set $E_2^*(t) = \bigcup_{0\le s\le t} E_2(s)$ is the interval $[\min(0,\alpha_1(1))t^\alpha, \max(0,\beta_2(1))t^\alpha]$ which is only one interval. This situation occurs, for example, when dealing with Stieltjes-Carlitz polynomials C_n and D_n, which satisfy

$$C_{n+1}(x) = xC_n(x) - a_n^2 C_{n-1}(x),$$

where $a_{2n} = 2nk$ and $a_{2n+1} = (2n+1)$, with $0 < k < 1$, and

$$D_{n+1}(x) = xD_n(x) - A_n^2 D_{n-1}(x),$$

with $A_{2n} = 2n$ and $A_{2n+1} = (2n+1)k$, again with $0 < k < 1$, see, e.g., [3], [2], [14, II]. The orthonormal polynomials $c_n(x) = C_n(x)/(a_1 \ldots a_n)$ then satisfy the recurrence

$$xc_n(x) = a_{n+1}c_{n+1}(x) + a_n c_{n-1}(x),$$

and the orthonormal polynomials $d_n(x) = D_n(x)/(A_1 \cdots A_n)$ satisfy

$$xd_n(x) = A_{n+1}d_{n+1}(x) + A_n d_{n-1}(x).$$

The recurrence coefficients in both cases are unbounded, so that the zeros will be unbounded as n tends to infinity. In order to keep the zeros from going to infinity, we consider the polynomials $c_n(Nx)$ and $d_n(Nx)$ and let $n, N \to \infty$ with $n/N \to t$. The case where $t = 1$ gives the distribution of the zeros of $c_n(nx)$ and $d_n(nx)$. Theorem 2 applies here with $m = 2$, and $\alpha = 1$ and $b_0 = b_1 = 0$, $a_0 = k$, $a_1 = 1$ in (19) for $c_n(Nx)$ and $a_0 = 1$ and $a_1 = k$ for $d_n(Nx)$. See Figure 2 for the curves $\alpha_1, \alpha_2, \beta_1, \beta_2$, where we indicate the set $E_2(0.6)$ (two vertical pieces) and $E_2^*(0.8)$ (one vertical piece).

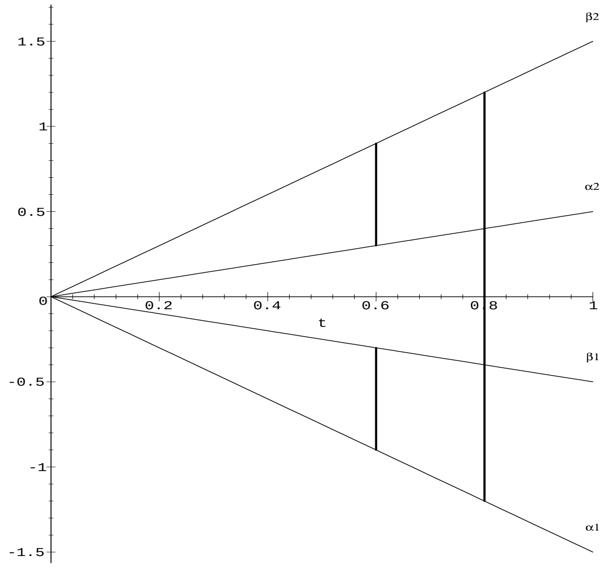

Figure 2: The sets $E_2(0.6)$ and $E_2^*(0.8)$ for Stieltjes-Carlitz polynomials $c_n(Nx)$

As a second example we consider the case where

$$b_0(t) = b_1(t) = bt^\alpha, \quad a_0(t) = a_0, \ a_1(t) = a_1, \tag{20}$$

where $a_0 \neq a_1$. In this case

$$\alpha_1(t) = bt^\alpha - (a_0 + a_1), \quad \beta_1(t) = bt^\alpha - |a_0 - a_1|,$$
$$\alpha_2(t) = bt^\alpha + |a_0 - a_1|, \quad \beta_2(t) = bt^\alpha + (a_0 + a_1).$$

The set $E_2(0)$ consists of the two intervals $[-(a_0+a_1), -|a_0-a_1|] \cup [|a_0-a_1|, a_0+a_1]$ and as t increases these intervals are shifted with the amount bt^α. Suppose that $b > 0$, then the set $E_2^*(t)$ consists of the two intervals $[-(a_0+a_1), -|a_0-a_1|+bt^\alpha] \cup [|a_0+a_1|, a_0+a_1+bt^\alpha]$ as long as $-|a_0 - a_1| + bt^\alpha < |a_0 - a_1|$, hence for $t < (2|a_0 - a_1|/b)^{1/\alpha} := t_1$. For $t \geq t_1$ the set $E_2^*(t)$ becomes one interval $[-(a_0+a_1), a_0+a_1+bt^\alpha]$. See Figure 3 where we took $\alpha = 1$ and $t_1 = 0.6$ and indicated the sets $E_2^*(0.4)$ and $E_2^*(0.8)$.

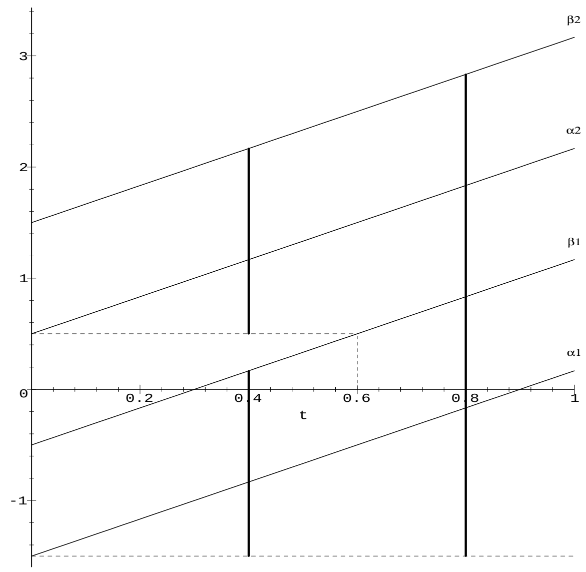

Figure 3: The sets $E_2^*(0.4)$ and $E_2^*(0.8)$ for the case considered in (20) with $t_1 = 0.6$

References

[1] S. Albeverio, L. Pastur, M. Shcherbina, *On asymptotic properties of certain orthogonal polynomials*, manuscript

[2] W. A. Al-Salam, *Characterization of certain classes of orthogonal polynomials related to elliptic functions*, Ann. Mat. Pura Appl. **67** (1965), 75–94.

[3] L. Carlitz, *Some orthogonal polynomials related to elliptic functions, I, II*, Duke Math. J. **27** (1960), 443–458; **28** (1961), 107–124.

[4] P. Deift, K. T-R McLaughlin, *A Continuum Limit of the Toda Lattice*, Memoirs Amer. Math. Soc. **624**, Providence, RI, 1998.

[5] J. S. Geronimo, W. Van Assche, *Orthogonal polynomials with asymptotically periodic recurrence coefficients*, J. Approx. Theory **46** (1986), 251–283.

[6] J. S. Geronimo, W. Van Assche, *Orthogonal polynomials on several intervals via a polynomial mapping*, Trans. Amer. Math. Soc. **308** (1988), 559–581.

[7] J. S. Geronimo, W. Van Assche, *Approximating the weight function for orthogonal polynomials on several intervals*, J. Approx. Theory **65** (1991), 341–371.

[8] Ya. L. Geronimus, *On the character of the solutions of the moment problem in the case of a limit-periodic associated fraction*, Izv. Akad. Nauk SSSR **88** (1953), 587–599 (in Russian).

[9] Ya. L. Geronimus, *On some finite difference equations and corresponding systems of orthogonal polynomials*, Zap. Mat. Otd. Fiz.-Mat. Fak i Kharkov Mat. Obsc. (4) **25** (1957), 87–100 (in Russian).

[10] A. B. J. Kuijlaars, W. Van Assche, *The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients*, J. Approx. Theory (to appear).

[11] A. Máté, P. Nevai, W. Van Assche, *The supports of measures associated with orthogonal polynomials and the spectra of the related self adjoint operators*, Rocky Mountain J. Math. **21** (1991), 501–527.

[12] P. G. Nevai, *Orthogonal Polynomials*, Memoirs Amer. Math. Soc. **213**, Providence, RI, 1979.

[13] L. A. Pastur, *Spectral and probabilistic aspects of matrix models*, in 'Algebraic and geometric models in mathematical physics' (A. Boutet de Monvel, V. Marchenko, eds.), Kluwer, Dordrecht, 1996, pp. 207–242.

[14] W. Van Assche, *Asymptotic properties of orthogonal polynomials from their recurrence formula, I, II*, J. Approx. Theory **44** (1985), 258–276; **52** (1988), 322–338.

[15] W. Van Assche, *Asymptotics for Orthogonal Polynomials*, Lecture Notes in Mathematics **1265**, Springer-Verlag, Berlin, 1987.