THE LAPLACIAN ON p-FORMS ON THE HEISENBERG GROUP

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Abstract. The Novikov-Shubin invariants for a non-compact Riemannian manifold M can be defined in terms of the large time decay of the heat operator of the Laplacian on L^2 p-forms, $\triangle_p$, on M.

For the $(2n+1)$-dimensional Heisenberg group H^{2n+1}, the Laplacian $\triangle_p$ can be decomposed into operators $\triangle_{p,n}(k)$ in unitary representations $\bar{\beta}_k$ which, when restricted to the centre of H, are characters (mapping ω to $\exp(-ik\omega)$). The representation space is an anti-Fock space $(\mathcal{F}_n^{-k})$, of anti-holomorphic functions F on $\mathbb{C}^n$ such that $\int_{\mathbb{C}^n} |F(\bar{z})|^2 e^{-1/4k|z|^2} dz < +\infty$.

In this paper, the eigenvalues of $\triangle_{p,n}(k)$ are calculated, for all n and p, using operators which commute with the Laplacian; this information determines the pth Novikov-Shubin invariant of H^{2n+1}. Further, some eigenvalues of operators connected with nilpotent Lie groups of Heisenberg type are calculated in the later sections.

1. Introduction

The Novikov-Shubin invariants for a non-compact Riemannian manifold M can be defined in terms of the large time decay of the heat operator of the Laplacian on L^2 p-forms, on M, which we'll denote by $\triangle_p$.

For the $(2n+1)$-dimensional Heisenberg group H^{2n+1}, the Laplacian can be decomposed into operators $\triangle_p(k)$ in irreducible unitary representations $\bar{\beta}_k$ which, when restricted to the centre of H, are characters (mapping ω to $\exp(-ik\omega)$). In this paper, the eigenvalues of $\triangle_p(k)$ are calculated; these determine the Novikov-Shubin invariants of H^{2n+1}.

Novikov-Shubin invariants are a relatively new set of topological invariants, usually defined analytically, of certain non-compact Riemannian manifolds. They were first defined in [27, 28], but are most comprehensively discussed in [17], from which the below definition is taken.

They are related to the L^2 Betti numbers, in the following way. We can define a function $\theta_p(t)$, depending on the manifold M, which is a positive function of $t \in \mathbb{R}_+$. Then the pth L^2 Betti number $b_p^{(2)}$ is equal to the limit as $t \to +\infty$ of $\theta_p(t)$, while the pth Novikov-Shubin number α_p measures the (degree of the inverse polynomial) rate at which this limit is approached.

The theory of L^2 torsion (see for example [4]) is also closely linked to that of Novikov-Shubin invariants; for example, if all the Novikov-Shubin invariants of a manifold are positive, then the L^2 torsion of that manifold is defined. Further, Novikov-Shubin invariants have been placed in a more abstract, categorical setting and more naturally linked with torsion and L^2 cohomology by Farber in [12].

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These invariants do not exist for all manifolds, but generalised invariants to cover the exceptions have been defined. (See, again, [17], and also [5, 24], while a combinatorial definition is given in [11], and the Novikov-Shubin invariants of complexes of Hilbert spaces are defined in [17, 23].)

For d the usual exterior derivative on square-integrable p-forms, and d^* its adjoint with respect to the Riemannian metric, we define the Laplacian on L^2 p-forms to be

$$\triangle_p = dd^* + d^*d.$$

Note that the Laplacian is a self-adjoint, positive, second order elliptic differential operator; the Laplacian on functions, $\triangle_0$, is the typical such elliptic differential operator. The Laplacian on forms is somewhat more complex, but differs from $\triangle_0 \otimes \mathrm{Id}$ only in first and zeroth order terms.

We first define the heat operator $e^{-t\triangle_p}$ for all $t > 0$ using the spectral theorem for self-adjoint operators. Then for Γ a discrete subgroup of the isometry group of M, such that M/Γ is a compact manifold, we can define a certain von Neumann trace Tr_Γ on the Γ-invariant operators of $B(L^2(M))$, and thus the function $\theta_p(t) := \mathrm{Tr}_\Gamma(e^{-t\triangle_p})$, mentioned above, can be defined for all positive t. Then $\theta_p(t)$ approaches $b_p^{(2)}$, the pth L^2 Betti number of M, for large t. If furthermore $\theta_p(t) - b_p^{(2)}$ is of order $t^{-\alpha_p}$ and $t^{-\alpha_p}$ is of order $\theta_p(t) - b_p^{(2)}$ as $t \to \infty$, then we say that α_p is the pth Novikov-Shubin invariant (see [17]).

It is known that α_p is independent of the choice of Γ-invariant metric on M; other properties of α_p are discussed in the main text.

In this paper, we calculate the Novikov-Shubin invariants of the $(2n+1)$-dimensional Heisenberg group H^{2n+1}. The method chosen is to examine not only the Laplacian on L^2 p-forms on H^{2n+1}, but also this operator in an irreducible, unitary representation of H^{2n+1}. We study the spectrum of this latter operator and thereby derive all the Novikov-Shubin invariants for each Heisenberg group.

Recall that the Heisenberg group of dimension $(2n + 1)$ (hereafter denoted by H^{2n+1}, or H if the dimension is clear) is a 2-step nilpotent Lie group. (It arises naturally in Quantum Mechanics; it is also, in some sense, the simplest non-abelian nilpotent Lie group.)

We choose a left-invariant metric on H; then the Laplacian $\triangle_p$ defined with respect to this metric is (left) H-invariant.

Now since H is a Lie group, its tangent bundle is trivial; so $\triangle_p$ can be thought of as a matrix, with entries which are differential operators on $L^2(H)$. But we know from the abstract Plancherel theorem that $L^2(H)$ splits into a direct integral of Hilbert spaces:

$$L^2(H) = \int_{\mathbb{R}}^{\oplus} \mathcal{F}_n^k \otimes \mathcal{F}_n^{-k} |k|^n dk$$

where k corresponds to the Fock-Bargmann representation β_k with parameter k. The Laplacian $\triangle_p$ also splits under this direct integral, with the corresponding operator in each Hilbert space being denoted by $\triangle_p(k)$.

The central result of this paper is Theorem 7.1, which lists all the eigenvalues of $\triangle_{p,n}(k)$ for $p \leq n$ and $k > 0$ (though not, in general, their multiplicity). In particular, the lowest eigenvalue of $\triangle_{p,n}(k)$, again for $p \leq n$ and $k > 0$, is $k^2 + (n - p)k$, which has multiplicity $\binom{n}{p}$. Further, Theorem 7.1 implies that the spectrum of $\triangle_{p,n}(k)$ contains the spectrum of $\triangle_{p-1,n-1}(k)$ for all $p \geq 1, n \geq 2, p \leq n$.

Using this theorem, we calculate exactly the Novikov-Shubin invariants of the Heisenberg group in Corollary 8.1:

$$\alpha_p(H^{2n+1}) = \begin{cases} n+1, & p \neq n, n+1, \\ \frac{1}{2}(n+1), & p = n, n+1. \end{cases}$$

The results of Varopoulos in [37] determine $\alpha_0(M)$ explicitly for all manifolds M, which agree with the above for the case $p = 0$. Further, Corollary 8.1 refines the following inequalities for $\alpha_p(H^{2n+1})$ which were proved in [22]:

$$\alpha_p(H^{2n+1}) \leq \begin{cases} n+1, & p \neq n, n+1, \\ \frac{1}{2}(n+1), & p = n, n+1, \end{cases}$$

where our definition of α_p differs by a factor of 2 from Lott's.

A result analogous to, but weaker than, Theorem 7.1 can be found in [16], where the Laplacian on a quotient H/Γ of the Heisenberg group by a discrete, cocompact subgroup Γ is considered. There, the eigenvalues of the decomposition of this operator in characters of H are calculated, rather than the eigenvalues in an infinite-dimensional representation, as in this paper.

The algebraic methods which we use to simplify the problem of calculating eigenvalues of the Laplacian on the Heisenberg group have their analogues for other nilpotent Lie groups. In the final two sections of this paper, we generalise these methods to the case of Heisenberg-type groups and obtain some information on the spectrum of the Laplacian. In particular, we obtain estimates of the lowest eigenvalue of the Laplacian on 1-forms on a family of nilpotent Lie groups with two-dimensional centre, and thus calculate the first Novikov-Shubin invariant of these Lie groups.

The results of this paper form an extension of the results of my thesis [33] which was supervised by Alan Carey and Varghese Mathai; many thanks are due them for all their patience and encouragement.

2. Novikov-Shubin invariants

We define Novikov-Shubin invariants as in [17].

Let M be a non-compact oriented Riemannian manifold on which a discrete infinite group Γ acts freely, such that the quotient $X := M/\Gamma$ is a compact manifold.

Let $\mathcal{A}$ be the algebra of all bounded linear operators on $L^2(M)$ which commute with the action of Γ: it can be shown that $\mathcal{A}$ is a von Neumann algebra [1].

There is a von Neumann trace on $\mathcal{A}$, denoted by Tr_Γ, first defined by Atiyah in [1]. If an operator $A \in \mathcal{A}$ is of Γ-trace class, has smooth kernel $K_A(x, y)$ (which is a distribution on $M \times M$), and is positive and self-adjoint, then

$$\mathrm{Tr}_\Gamma A = \int_{\mathcal{F}} K_A(x, x) d\mu(x)$$

where $\mathcal{F}$ is the fundamental domain for the action of Γ on M and μ is Haar measure on $\mathcal{F}$.

Let $\triangle_p$ denote the Laplacian on smooth, compactly supported p-forms on M. The closure of this operator, $\tilde{\triangle}_p$, has domain the first (generalized) Sobolev space on p-forms, which is dense in the set of L^2 p-forms. This space is defined as the closure of smooth, compactly supported p-forms on M with respect to the norm

$$\|\omega\|_1 = \langle (\mathrm{Id} + \triangle_p)\omega, (\mathrm{Id} + \triangle_p)\omega \rangle_2$$

where $\langle .,. \rangle_2$ is the usual L^2 inner product on p-forms, here applied to ω in the sense of distributions (see [1] or [10]). Hereafter we write $\triangle_p$ instead of $\tilde{\triangle}_p$ and refer to this operator as acting on L^2 p-forms.

Using the spectral theorem for self-adjoint operators, we can form the operator $e^{-t\triangle_p}$ for all positive t. We can then define a function $\theta_p(t)$ for all $t > 0$, by

$$\theta_p(t) := \mathrm{Tr}_\Gamma e^{-t\triangle_p}.$$

It was shown in [17] that $\theta_p(t) \to \bar{b}_p$, the pth L^2 Betti number (first defined in [1]), as $t \to \infty$.

If for some constants C, t_0 and α, we have

$$C^{-1}t^{-\alpha} \leq \theta_p(t) - \bar{b}_p \leq Ct^{-\alpha}$$

for all $t > t_0$, we say that $\alpha = \alpha_p(M, \Gamma)$ is the pth Novikov-Shubin invariant of (M, Γ).

This was not the original definition of $\alpha_p(M)$, but it was proved to be equivalent in [17]. It is the most useful definition for our purposes.

It has been shown that $\alpha_p(M)$ is invariant of choice of Γ-invariant metric, and furthermore is a homotopy invariant. This last statement was proved in [17], but the proof is complicated, and relies on assumptions that certain operators are bounded. For an alternative proof for closed manifolds, which uses standard topological techniques, see [3].

3. The Plancherel theorem for the Heisenberg group

The Heisenberg group of dimension $2n + 1$, which we'll denote by H^{2n+1} or H, is a connected, simply connected real nilpotent Lie group. It is modelled on $\mathbb{R}^{2n+1}$ with the group law

$$(x, y, w) \cdot (x', y', w') = (x + x', y + y', w + w' + \tfrac{1}{2}(x \cdot y' - y \cdot x'))$$

for $x, y \in \mathbb{R}^n, w \in \mathbb{R}$.

Its Lie algebra $\mathfrak{h}$ has basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, W\}$ and non-zero commutation relations $[X_j, Y_j] = W = -[Y_j, X_j]$.

Let X_j also denote the left-invariant vector field on H given by left translation of $X_j \in \mathfrak{h}$, which we identify with an element of the tangent space at the identity.

We define complex vector fields $Z_j, Z_{\bar{j}}$ on H by $Z_j := \frac{1}{\sqrt{2}}(X_j - iY_j), Z_{\bar{j}} := \frac{1}{\sqrt{2}}(X_j + iY_j)$. Alternatively, with the same definitions, we consider Z_j and $Z_{\bar{j}}$ to be elements of $u(\mathfrak{h})$, the universal enveloping algebra of $\mathfrak{h}$.

We choose a left-invariant metric on H such that $\{X_j, Y_j, W\}$ is an orthonormal basis for T_pH at each point p of H. Then $\{Z_1, \ldots, Z_n, Z_{\bar{1}}, \ldots, Z_{\bar{n}}, W\}$ is an orthonormal basis for the complexified tangent space at each point.

Let $\{\tau^1, \ldots, \tau^n, \tau^{\bar{1}}, \ldots, \tau^{\bar{n}}, \tau^w\}$ be the basis of 1-forms dual to $\{Z_1, \ldots, Z_n, Z_{\bar{1}}, \ldots, Z_{\bar{n}}, W\}$.

We define $\hat{G}$ to be the set of (unitary) equivalence classes of irreducible unitary representations of a locally compact group G.

If π is a unitary representation of a group G on a Hilbert space $\mathcal{H}_\pi$, then there is an induced representation of $L^1(G)$ on $\mathcal{H}_\pi$. That is, take any element f in $L^1(G)$: we define

$$\pi(f) := \int_G f(x)\pi(x)dx.$$

If we have any elements $u, v \in \mathcal{H}_\pi$, then

$$\langle \pi(f)u, v\rangle = \int_G f(x)\langle \pi(x)u, v\rangle dx.$$

The operator $\pi(f)$ is known as the *group Fourier transform* of f (see [8] or [13]).

Let $\mathcal{J}^1$ be $L^1(G) \cap L^2(G)$ and $\mathcal{J}^2$ be the set of finite linear combinations of elements of the form $f * g$, for $f, g \in \mathcal{J}^1$. (Note the similarities to Hilbert-Schmidt and nuclear or trace-class operators.)

The following theorem, the abstract Plancherel theorem, was first proved in [26], [34]; the formulation below is taken from [8] and [14].

Theorem 3.1. *Let G be a type I, unimodular, separable, locally compact group. Then there exists a measurable field of irreducible representations π_ζ over $\hat{G}$ such that π_ζ belongs to the equivalence class ζ. We identify π_ζ with ζ, and write $\mathcal{H}_\zeta$ for the Hilbert space which π_ζ acts on. Let t_ζ be the trace $T \otimes 1 \mapsto \mathrm{Tr}_\zeta(T)$ (the Hilbert-Schmidt trace) on the positive operators in $B(\mathcal{H}_\zeta) \otimes \mathbb{C}$.*

Let π_L and π_R be the left and right regular representations of G, and let $\mathcal{U}$ and $\mathcal{V}$ be the von Neumann algebras on $L^2(G)$ generated by $\pi_L(G)$ and $\pi_R(G)$. Let t be the trace on $\mathcal{U}^+$ defined as above.

Then there exists a positive measure μ on $\hat{G}$ and an isomorphism W from $L^2(G)$ to $\int_{\hat{G}}^{\oplus} (\mathcal{H}_\zeta \otimes \overline{\mathcal{H}}_\zeta)\, d\mu(\zeta)$ such that:

(i) *W transforms π_L into $\int_{\hat{G}}^{\oplus} (\zeta \otimes 1)\, d\mu(\zeta)$, π_R into $\int_{\hat{G}}^{\oplus} (1 \otimes \bar{\zeta})\, d\mu(\zeta)$, $\mathcal{U}$ into $\int_{\hat{G}}^{\oplus} (B(\mathcal{H}_\zeta) \otimes \mathbb{C})\, d\mu(\zeta)$, $\mathcal{V}$ into $\int_{\hat{G}}^{\oplus} (\mathbb{C} \otimes B(\overline{\mathcal{H}}_\zeta))\, d\mu(\zeta)$, and t into $\int_{\hat{G}}^{\oplus} t_\zeta\, d\mu(\zeta)$.*

(ii) *If $h \in \mathcal{J}^2$ and $x \in G$, then we have the Fourier inversion formula for G:*

$$h(x) = \int_{\hat{G}} \mathrm{Tr}(\zeta(x)\zeta(h))\, d\mu(\zeta). \tag{3.1}$$

In particular, if $u \in L^1(G) \cap L^2(G) = \mathcal{J}^1$, we have

$$\int_G |u(s)|^2\, ds = \int_{\hat{G}} \mathrm{Tr}(\zeta(u)\zeta(u)^*)\, d\mu(\zeta),$$

the Plancherel formula for G.

Note that we write $\int_{\hat{G}}^{\oplus} (\zeta \otimes 1)\, d\mu(\zeta)$ rather than $\int_{\hat{G}}^{\oplus} \zeta\, d\mu(\zeta)$ and $\int_{\hat{G}}^{\oplus} (B(\mathcal{H}_\zeta) \otimes \mathbb{C})\, d\mu(\zeta)$ rather than $\int_{\hat{G}}^{\oplus} B(\mathcal{H}_\zeta)\, d\mu(\zeta)$; this is to clarify the action of these operators on $\int_{\hat{G}}^{\oplus} (\mathcal{H}_\zeta \otimes \overline{\mathcal{H}}_\zeta)\, d\mu(\zeta)$.

The measure μ is known as the Plancherel measure of $\hat{G}$ (associated with the Haar measure of G).

For the Heisenberg group, the Plancherel measure μ is zero except on representations β_k, where β_k is the Fock-Bargmann representation of H with parameter k (for $k \in \mathbb{R}^*$). This representation is irreducible and acts on the Fock space $\mathcal{F}_n^k$ defined by

$$\mathcal{F}_n^k = \{F : F \text{ is entire on } \mathbb{C}^n \text{ and } \int_{\mathbb{C}^n} |F(z)|^2 e^{-kz\cdot\bar{z}/4} dz < \infty\}.$$

In fact, we're more interested in the conjugate representation $\bar{\beta}_k$, which acts on the anti-Fock space $\mathcal{F}_n^{-k}$, where $F \in \mathcal{F}_n^{-k}$ iff $\bar{F} \in \mathcal{F}_n^k$. This representation is defined by

$$\bar{\beta}_k(p, q, w)F(\bar{z}) = e^{-ikw - \frac{1}{4}k(p^2+q^2) - \frac{1}{2}k\bar{z}\cdot(p+iq)} F(\bar{z} + p - iq).$$

Thus, the Plancherel theorem for H implies that

$$L^2(H) \cong \int_{k\in\mathbb{R}}^{\oplus} \mathcal{F}_n^k \otimes \mathcal{F}_n^{-k} |k|^n dk. \tag{3.2}$$

Under this decomposition, the right regular representation π_R of H on $L^2(H)$ is given by

$$\pi_R = \int_{k\in\mathbb{R}}^{\oplus} (\mathrm{Id} \otimes \bar{\beta}_k)|k|^n dk.$$

From the representation $\bar{\beta}_k$ of H, we have a representation (also denoted by $\bar{\beta}_k$) of $u(\mathfrak{h})$ on the C^∞ vectors of $\mathcal{F}_n^{-k}$ (see [6, 15]), given by

$$\bar{\beta}_k(Z_j) = -\tfrac{1}{\sqrt{2}} k\bar{z}_j,\ \bar{\beta}_k(Z_{\bar{j}}) = \sqrt{2}\partial_{\bar{z}_j},\ \bar{\beta}_k(W) = -ik.$$

For any multi-index $\beta \in \mathbb{Z}_+^n$, we define a function $\psi_\beta(k)$ by

$$\psi_\beta(k) := \left(\frac{k}{2\pi}\right)^{n/2} \left(\frac{ik}{2}\right)^{|\beta|/2} \frac{\bar{z}^\beta}{\sqrt{\beta!}}.$$

Then the set $\{\psi_\beta(k) : \beta \in \mathbb{Z}_+^n\}$ is a complete orthonormal basis of $\mathcal{F}_n^{-k}$ (see [13]).

The action of the above operators on this basis is given by

$$\begin{aligned}
\bar{\beta}_k(Z_j)(\psi_\beta(k)) &= -i\sqrt{k}\sqrt{\beta_j+1}\,\psi_{\beta+e_j}(k), \\
\bar{\beta}_k(Z_{\bar{j}})(\psi_\beta(k)) &= -i\sqrt{k}\sqrt{\beta_j}\,\psi_{\beta-e_j}(k)
\end{aligned}$$

where e_j is the multi-index with 1 in the jth place and zeros elsewhere.

We define creation and annihilation operators a_j, a_j^* which act on $\mathcal{F}_n^{-k}$. Let a_j be the operator $ik^{-1/2}\bar{\beta}_k(Z_{\bar{j}})$, and a_j^* the operator $ik^{-1/2}\bar{\beta}_k(Z_j)$. Then $[a_j, a_j^*] = \mathrm{Id}$. We call a_j^* a creation operator and a_j an annihilation operator. Note that

$$\begin{aligned}
a_j^*\psi_\beta(k) &= \sqrt{\beta_j+1}\,\psi_{\beta+e_j}(k), \\
a_j\psi_\beta(k) &= \sqrt{\beta_j}\,\psi_{\beta-e_j}(k)
\end{aligned}$$

4. An explicit formula for the Laplacian

In this section, we begin to explicitly analyse the action of the Laplacian.

For $d : \Lambda_{(2)}^p H \otimes \mathbb{C} \to \Lambda_{(2)}^{p+1} H \otimes \mathbb{C}$ the (complexified) exterior derivative on L^2 p-forms and $d^* : \Lambda_{(2)}^p H \otimes \mathbb{C} \to \Lambda_{(2)}^{p-1} H \otimes \mathbb{C}$ its adjoint, the Laplacian on p-forms is defined to be

$$\triangle = dd^* + d^*d : \Lambda_{(2)}^p H \otimes \mathbb{C} \to \Lambda_{(2)}^p H \otimes \mathbb{C}.$$

It will also be denoted by $\triangle_p$ or $\triangle_{p,n}$, when the degree of the forms and/or the dimension of the group that the Laplacian is acting on is important. Note that the domain of the Laplacian is the first Sobolev space of p-forms; since this is dense in the space of L^2 p-forms, we assume for the purposes of this discussion that the Laplacian acts on L^2 p-forms. (For more on this, see [1, 10].)

In particular, the Laplacian on functions is given by

$$\triangle_{0,n} = \sum_{j=1}^{n} (-Z_j Z_{\bar{j}} - Z_{\bar{j}} Z_j) - W^2,$$

which implies that, acting on $\mathcal{F}_n^{-k}$, $\triangle_{0,n}(k) = \sum_{j=1}^n (2ka_j^* a_j + k) + k^2$. In particular, on the basis elements, $\triangle_{0,n}(k)\psi_\beta(k) = (2k|\beta| + nk + k^2)\psi_\beta(k)$ where $|\beta| = \beta_1 + \ldots + \beta_n$.

By inspection, the lowest eigenvalue of $\triangle_{0,n}(k)$ is $nk + k^2$. Furthermore, the eigenvalue corresponding to $\psi_\beta(k)$ depends only on $|\beta|$, and not on any other function of β.

We begin by calculating explicitly the form of d and d^* acting on p-forms.

Lemma 4.1. *The actions of d and d^* on p-forms on H^{2n+1} are given by*

$$\begin{aligned} d &= \left(\sum_{j=1}^n e(\tau^j)Z_j + e(\tau^{\bar{j}})Z_{\bar{j}}\right) + e(\tau^w)W - i\sum_{j=1}^n e(\tau^j)e(\tau^{\bar{j}})i(W) \\ d^* &= -\left(\sum_{j=1}^n i(Z_{\bar{j}})Z_j + i(Z_j)Z_{\bar{j}}\right) - i(W)W + i\sum_{j=1}^n e(\tau^w)i(Z_{\bar{j}})i(Z_j) \end{aligned}$$

where $e(\tau)$ denotes exterior multiplication by the 1-form τ and $i(V)$ denotes contraction by the vector field V.

The proof of this lemma uses the Leibnitz rule (giving the first few terms in the above formula for d, which are the same as those for d on functions) and the fact that for any 1-form η and vector fields X, Y, $d\eta(X,Y) = X\eta(Y) - Y\eta(X) - 1/2\eta([X,Y])$ (see for example [35]). The action for d on 2-forms is unremarkable since the Heisenberg group is a 2-step nilpotent Lie group.

Using these formulae for d and d^*, we can explicitly calculate the form of $\triangle_{p,n}$ (again in terms of $e(*)$ and $i(*)$'s). (The details of this calculation are given in an appendix.) Here we write $\triangle_{p,n}$ (and $\triangle_{p,n}(k)$) as a matrix, considering a p-form to be a $\binom{2n+1}{p}$ vector - again using the triviality of the tangent bundle of H^{2n+1}. Recall that for any operator A acting on $\Lambda^*(H) \otimes \mathbb{C}$ or on $L^2(H) \otimes \mathbb{C}$, we denote the decomposition in the representation $\bar{\beta}(k)$ by $A(k)$.

The Laplacian on p-forms, acting on H^{2n+1}, is given by:

$$\begin{aligned} \triangle_{p,n} = -W^2 + \sum_{j=1}^n \Big(&-2Z_jZ_{\bar{j}} + iW(i(Z_j)e(\tau^j) + e(\tau^{\bar{j}})i(Z_{\bar{j}})) \\ &+ ie(\tau^w)(i(Z_{\bar{j}})Z_j - i(Z_j)Z_{\bar{j}}) - i(e(\tau^j)Z_j - e(\tau^{\bar{j}})Z_{\bar{j}})i(W) \\ &+ \sum_{k=1,k\neq j}^n e(\tau^j)e(\tau^{\bar{j}})i(Z_{\bar{k}})i(Z_k) \\ &+ e(\tau^j)i(Z_j)e(\tau^{\bar{j}})i(Z_{\bar{j}})i(W)e(\tau^w) + i(Z_j)e(\tau^j)i(Z_{\bar{j}})e(\tau^{\bar{j}})e(\tau^w)i(W)\Big) \end{aligned} \tag{4.1}$$

(This formula is derived in Appendix A.)

After the transform corresponding to the conjugate Fock-Bargmann representation with parameter k, this operator becomes:

$$\begin{aligned}\triangle_{p,n}(k) = k^2 + \sum_{j=1}^{n} \Big(& 2ka_j^* a_j + ki(Z_j)e(\tau^j) + ke(\tau^{\bar{j}})i(Z_{\bar{j}}) \\ & + \sqrt{k}e(\tau^w)(i(Z_{\bar{j}})a_j^* - i(Z_j)a_j) + \sqrt{k}i(W)(e(\tau^j)a_j^* - e(\tau^{\bar{j}})a_j) \\ & + \sum_{k=1,k\neq j}^{n} e(\tau^j)e(\tau^{\bar{j}})i(Z_{\bar{k}})i(Z_k) \\ & + e(\tau^j)i(Z_j)e(\tau^{\bar{j}})i(Z_{\bar{j}})i(W)e(\tau^w) + i(Z_j)e(\tau^j)i(Z_{\bar{j}})e(\tau^{\bar{j}})e(\tau^w)i(W)\Big)\end{aligned} \tag{4.2}$$

Using this last formula, we could explicitly calculate all the eigenvalues of $\triangle_{p,n}(k)$ for certain (small) values of n and p, writing the Laplacian globally as a matrix (since the tangent space of H^{2n+1} is trivial). However, the size of this matrix is $\binom{2n+1}{p}$, as implied above, and so will grow polynomially as n and p increase.

We note instead that we can define the following operators.

Definition 4.2. For $j = 1, \ldots, n$, we define θ_j to be a map from $\Lambda^p_{(2)}(H) \otimes \mathbb{C}$ and θ_j^* to be its adjoint, given by the following formulae:

$$\begin{aligned}\theta_j &= e(\tau^j)Z_j + e(\tau^{\bar{j}})Z_{\bar{j}} - ie(\tau^j)e(\tau^{\bar{j}})i(W) \\ \theta_j^* &= -i(Z_j)Z_{\bar{j}} - i(Z_{\bar{j}})Z_j + ie(\tau^w)i(Z_{\bar{j}})i(Z_j).\end{aligned}$$

We can then rewrite d and d^* as $d = \sum_j \theta_j + e(\tau^w)W$ and $d^* = \sum_j \theta_j^* - i(W)W$.

Writing $\triangle_{p,n}(k)$ in terms of the operators $\theta_j(k), \theta_j^*(k), e(\tau^w)k$ and $i(W)k$ gives us further information about the spectrum of $\triangle_{p,n}(k)$; in particular, we find a lower bound on the spectrum for all p and n, which is achieved for $p = n$.

Lemma 4.3. *The operator $\triangle_{p,n}(k)$ satisfies the inequality:*

$$\triangle_{p,n}(k) \geq k^2 Id.$$

In particular, $\triangle_{n,n}(k)$ has lowest eigenvalue k^2.

Proof. Since $e(\tau^w)$ and θ_j^* anticommute, as do $i(W)$ and θ_j, for all j, we have that

$$\begin{aligned}\triangle &= (\sum_j \theta_j)(\sum_m \theta_m)^* + (\sum_m \theta_m)^*(\sum_j \theta_j) - W^2 \\ \implies \triangle(k) &\geq k^2\end{aligned}$$

This is a lower bound on the eigenvalues of $\triangle_{p,n}(k)$ for all n and p. However, if $n = p$, we know (from [22]) that there is an eigenvector v of $\triangle_{n,n}(k)$,

$$v := f\tau^1 \wedge \ldots \wedge \tau^n,$$

where $f \in \ker Z_{\bar{1}}(k) \cap \ldots \cap \ker Z_{\bar{n}}(k)$, such that $\triangle_{n,n}(k)v = k^2 v$; thus k^2 is in fact the lowest eigenvalue of $\triangle_{n,n}(k)$ for all n. □

5. Commuting operators

In this section, we define a partition of $\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*)$ into subspaces, using a collection of commuting operators.

For $j = 1, \ldots, n$, we define U_{jj} to be the operator on $\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*)$ given by

$$U_{jj} := a_j^* a_j - e(\tau^j)i(Z_j) + e(\tau^{\bar{j}})i(Z_{\bar{j}}).$$

It should be clear from this definition that $[U_{jj}, U_{ll}] = 0$ for all $j \neq l$, and that this operator is self-adjoint: $U_{jj}^* = U_{jj}$.

Define the set $S := \{\gamma \in \mathbb{Z}^n : \gamma_j \geq -1, j = 1, \ldots, n$, and at most p of the indices γ_j are equal to $-1\}$. For multi-indices γ in S, we define the subspace $V^{p,n,\gamma}$ to be the simultaneous eigenspace of $U_{11}, \ldots, U_{nn}$, with eigenvalues $\gamma_1, \ldots, \gamma_n$. (We sometimes omit the mention of n.) That is, if we write $E_\lambda A$ for the eigenspace of an operator A corresponding to the eigenvalue λ, then $V^{p,n,\gamma}$ is given by

$$V^{p,n,\gamma} := E_{\gamma_1} U_{11} \cap \ldots E_{\gamma_n} U_{nn} \cap (\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*)).$$

For example, the p-form $\psi_{\gamma+I-J} \tau^I \wedge \tau^{\bar{J}}$ is in $V^{p,n,\gamma}$, where I and J are both multi-indices, with entries either 0 or 1, $|I| + |J| = p$ and if $I = e_{i_1} + \ldots + e_{i_m}$, then $\tau^I = \tau^{i_1} \wedge \ldots \wedge \tau^{i_m}$ (and similarly for $\tau^{\bar{J}}$).

Note that we can have $\gamma_j = -1$ for some $j = 1, \ldots, n$, but this means that every element of $V^{p,n,\gamma}$ would have to be of the form $\tau^j \wedge v$ for some $v \in V^{p-1,n,\gamma+e_j}$; thus at most p of the γ_j's can be -1. The remainder of the indices of γ must be non-negative.

In fact, the collection of the subspaces $V^{p,n,\gamma}$ for all values of γ in S is a partition:

$$\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*) = \oplus_{\gamma \in S} V^{p,n,\gamma}.$$

The subspace $V^{0,n,\gamma}$ consists of (complex) scalar multiples of $\psi_\gamma(k)$; the subspace $V^{p,n,\gamma}$ also corresponds to $\psi_\gamma(k)$ in some sense, but with dimension $\binom{2n+1}{p}$.

The usefulness of this definition is due to the following theorem.

Theorem 5.1. *Let $d(k)$ and $d^*(k)$ represent the exterior differential and its adjoint respectively in the representation $\bar{\beta}_k$. Then $d(k)$ maps $V^{p,n,\gamma}$ to $V^{p+1,n,\gamma}$ for $p < 2n+1$, and $d^*(k)$ maps $V^{p,n,\gamma}$ to $V^{p-1,n,\gamma}$, for $p \geq 1$. So $V^{p,n,\gamma}$ is a $\triangle_{p,n}(k)$-invariant subspace of $\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*)$.*

Proof. We prove that $[U_{jj}, \theta_j(k)] = 0$, and thus that $[U_{jj}, d(k)] = 0 = [U_{jj}, d^*(k)]$ for all j, which means that $[U_{jj}, \triangle_{p,n}(k)] = 0$ for all j.

$$\begin{aligned}
[\theta_j(k), U_{jj}] &= \sqrt{-1}([k^{-1/2} e(\tau^j) a_j^*, -e(\tau^j) i(Z_j)] + [k^{-1/2} e(\tau^{\bar{j}}) a_j, a_j^* a_j] \\
&\quad + [-e(\tau^j) e(\tau^{\bar{j}}) i(W), -e(\tau^j) i(Z_j)] \\
&\quad + [k^{-1/2} e(\tau^j) a_j^*, a_j^* a_j] + [k^{-1/2} e(\tau^{\bar{j}}) a_j, e(\tau^{\bar{j}}) i(Z_{\bar{j}})] \\
&\quad + [-e(\tau^j) e(\tau^{\bar{j}}) i(W), e(\tau^{\bar{j}}) i(Z_{\bar{j}})]) \\
&= \sqrt{-1}(k^{-1/2} e(\tau^j) a_j^* + k^{-1/2} e(\tau^{\bar{i}}) a_j - e(\tau^j) e(\tau^{\bar{j}}) i(W) \\
&\quad - k^{-1/2} e(\tau^j) a_j^* - k^{-1/2} e(\tau^{\bar{j}}) a_j - e(\tau^{\bar{j}}) e(\tau^j) i(W)) \\
&= 0.
\end{aligned}$$

Clearly, U_{jj} also commutes with $\theta_l(k)$ (for $l \neq j$), since different operators are involved. So U_{jj} commutes with $d(k)$; then since $U_{jj}^* = U_{jj}$, this means that U_{jj} also commutes with $d^*(k)$, and thus that U_{jj} commutes with $\triangle_{p,n}(k)$.

This means that any eigenspace of U_{jj} will be preserved by $\triangle_{p,n}(k)$. But this is true for all j, so the subspace $V^{p,n,\gamma}$ is $\triangle_{p,n}(k)$-invariant. ∎

We can now study the eigenvalues of $\triangle_{p,n}(k)$ restricted to $V^{p,\gamma}$. In fact, we'll also be interested in even smaller subspaces. For this, the following definition will be useful.

Definition 5.2. Fix $k > 0$. Let V be a subspace of $\mathcal{F}_n^{-k} \otimes \Lambda^p((\mathfrak{h}^{2n+1})^*)$ and let W be a subspace of $\mathcal{F}_n^{-k} \otimes \Lambda^q((\mathfrak{h}^{2m+1})^*)$ for some n, m, p, q (such that $p \leq 2n+1$ and $q \leq 2m+1$), so that $\triangle_{p,n}(k)$ acts on V and $\triangle_{q,m}(k)$ acts on W. Suppose also that V is $\triangle_{p,n}(k)$-invariant and W is $\triangle_{q,m}(k)$-invariant. Then we say that V and W are **spectrally equivalent** if $\triangle_{p,n}(k)$ acting on V has the same eigenvalues (including multiplicity) as $\triangle_{q,m}(k)$ acting on W.

Remark 5.3. This is true if and only if there is a linear isomorphism j from V to W which commutes with $\triangle(k)$, i.e. such that $j\triangle_{p,n}(k) = \triangle_{q,m}(k)j$. From either of these conditions, we can see that spectral equivalence is indeed an equivalence relation.

We now introduce an operator on p-forms, as a first step in calculating the eigenvalues of $\triangle_{p,n}(k)$.

Definition 5.4. The $(1, 2)$ transposition operator is an operator on $\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*)$, denoted by U_{12} and defined to be

$$U_{12} := a_1^* a_2 - e(\tau^2)i(Z_1) + e(\tau^{\bar{1}})i(Z_{\bar{2}}).$$

Similarly, we define U_{ij}, the (i, j) transposition operator, (for $i \neq j, i, j = 1, \dots, n$) to be the operator given by

$$U_{ij} := a_i^* a_j - e(\tau^j)i(Z_i) + e(\tau^{\bar{i}})i(Z_{\bar{j}}).$$

From the above definition, we see that the (j, i) transposition operator U_{ji} is the adjoint of U_{ij}. Also, U_{ij} is "usually" an isomorphism, as proved in the following lemma.

Lemma 5.5. (i) *If $v \in \ker U_{ij} \cap V^{p,n,\gamma}$ for some γ, then $\gamma_j = -1, 0$ or 1.*

(ii) *If $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, $\gamma_i \geq 1$ and $\gamma_j \geq 2$ (for $i \neq j$), then the restriction of U_{ij} to $V^{p,n,\gamma}$ is a linear isomorphism between $V^{p,n,\gamma}$ and $V^{p,n,\gamma+e_i-e_j}$.*

Proof. The proof of (i) can be found in the appendix. To prove (ii), we note from (i) that U_{ij} restricted to $V^{p,n,\gamma}$ is 1-1 (since $\gamma_j \geq 2$). Now the orthogonal complement of the image of U_{ij} is the kernel of the adjoint map. But the adjoint of U_{ij} on $V^{p,n,\gamma}$ is U_{ji} restricted to $V^{p,n,\gamma+e_i-e_j}$, which has kernel $\{0\}$, again by (i) (since $\gamma_i + 1 \geq 2$). So this map is onto and thus an isomorphism. □

Remark 5.6. Note that $U_{ij}U_{ji}$ is not the identity; however, it is an automorphism on $V^{p,n,\gamma}$ for "generic" γ, and since it commutes with $\triangle(k)$, it preserves eigenspaces.

We also have:

Lemma 5.7. (i) *For all i, j and p, $[d(k), U_{ij}] = 0$; also $[d^*(k), U_{ij}] = 0$ and thus $[\triangle_p(k), U_{ij}] = 0$. That is, the (i, j) transposition operator commutes with the Laplacian on p-forms in the representation $\bar{\beta}_k$.*

(ii) *If $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, $\gamma_i \geq 1$ and $\gamma_j \geq 2$, then $V^{p,n,\gamma}$ and $V^{p,n,\gamma+e_i-e_j}$ are spectrally equivalent.*

(iii) *For any β, γ multi-indices such that $|\beta| = |\gamma|$ and $\beta_i \geq 1$, $\gamma_i \geq 1$ for all $i = 1, 2, \dots, n$, the subspaces $V^{p,n,\beta}$ and $V^{p,n,\gamma}$ are spectrally equivalent.*

Proof. In proving (i), note that for any $l \neq i, l \neq j$, we have that $[\theta_l(k), U_{ij}] = 0$, so that we only need prove that $[\theta_i(k) + \theta_j(k), U_{ij}] = 0$. Now

$$\begin{aligned}
& [\theta_i(k) + \theta_j(k), U_{ij}] \\
= \; & \sqrt{-1}([k^{-1/2}e(\tau^i)a_i^*, -e(\tau^j)i(Z_i)] + [k^{-1/2}e(\tau^{\bar{i}})a_i, a_i^* a_j] \\
& +[-e(\tau^i)e(\tau^{\bar{i}})i(W), -e(\tau^j)i(Z_i)] \\
& +[k^{-1/2}e(\tau^j)a_j^*, a_i^* a_j] + [k^{-1/2}e(\tau^{\bar{j}})a_j, e(\tau^{\bar{i}})i(Z_{\bar{j}})] \\
& +[-e(\tau^j)e(\tau^{\bar{j}})i(W), e(\tau^{\bar{i}})i(Z_{\bar{j}}]) \\
= \; & \sqrt{-1}(k^{-1/2}e(\tau^j)a_i^* + k^{-1/2}e(\tau^{\bar{i}})a_j - e(\tau^j)e(\tau^{\bar{i}})i(W) \\
& -k^{-1/2}e(\tau^j)a_i^* - k^{-1/2}e(\tau^{\bar{i}})a_j - e(\tau^{\bar{i}})e(\tau^j)i(W)) \\
= \; & 0.
\end{aligned}$$

One can prove that $[d^*(k), U_{ij}] = 0$ in a similar way, or take the adjoint of the equation $[d(k), U_{ji}] = 0$. It then follows that $[\triangle(k), U_{ij}] = 0$.

(ii) follows from (i) and from Lemma 5.5 (i); (iii) is easily proved by repeated use of (ii) for selected values of i and j. □

Thus the eigenvalues of $\triangle_{p,n}(k)$ on different $V^{p,\gamma}$ also depend only on $|\gamma|$ for generic γ, as is the case when $p = 0$ for any γ.

The following definitions rely on the fact that $\mathfrak{h}_{2n+1}$ is symmetric with respect to the basis elements $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n$: i.e., if X_2 and Y_2 are interchanged with X_1 and Y_1, then the commutation relations are unchanged.

Definition 5.8. We define an action of S_n (the permutation group on n symbols) on $\mathbb{Z}^n$ by:

$$\sigma \cdot (\beta_1, \beta_2, \dots, \beta_n) = (\beta_{\sigma(1)}, \beta_{\sigma(2)}, \dots, \beta_{\sigma(n)})$$

for $\sigma \in S_n$ and $\beta_1, \dots, \beta_n \in \mathbb{Z}$. For example, for $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}^n$, we have $(12) \cdot \beta = (\beta_2, \beta_1, \dots, \beta_n)$.

We next define an isometry for all pairs of distinct numbers i, j, utilising the symmetry of the Lie algebra of the Heisenberg group.

Definition 5.9. Let χ_{ij} be the Lie algebra isomorphism on $\mathfrak{h}^{2n+1}$ (for $i \neq j, i, j \leq n$) defined by linearity and action $\chi_{ij} : Z_i \mapsto Z_j, Z_j \mapsto Z_i, Z_{\bar{i}} \mapsto Z_{\bar{j}}, Z_{\bar{j}} \mapsto Z_{\bar{i}}$, and $V \mapsto V$ if V is orthogonal to $Z_i, Z_j, Z_{\bar{i}}$ and $Z_{\bar{j}}$.

This map χ_{ij} is an isometry with respect to the inner product that we have chosen on $\mathfrak{h}^{2n+1}$.

It induces a map $\tilde{\chi}_{ij}$ on L^2 p-forms on H, and this factors through the representation to give a map on $\mathcal{F}_n^{-k} \otimes \Lambda^p(\mathfrak{h}^*)$, which we'll also denote by χ_{ij}. This map is linear, multiplicative, and has action

$$\begin{array}{lll}
\tau^i \mapsto \tau^j; & \tau^j \mapsto \tau^i; & \tau^m \mapsto \tau^m \text{ if } m \neq i, j; \\
\tau^{\bar{i}} \mapsto \tau^{\bar{j}}; & \tau^{\bar{j}} \mapsto \tau^{\bar{i}}; & \tau^{\bar{m}} \mapsto \tau^{\bar{m}} \text{ if } m \neq i, j; \\
\tau^w \mapsto \tau^w; & \psi_\beta(k) \mapsto \psi_{(ij)\cdot\beta}(k) &
\end{array}$$

So the operator χ_{ij} is an isometry from $V^{p,n,\gamma}$ to $V^{p,n,(ij)\cdot\gamma}$ and commutes with the Laplacian $\triangle_{p,n}(k)$.

We call χ_{ij} the (i,j) symmetry operator, or simply a symmetry operator.

These operators can be used to prove that the eigenvalues of the Laplacian in the representation $\bar{\beta}_k$ on the subspace $V^{p,n,\gamma}$ are symmetric in the entries of γ. That is, we have the following results:

Lemma 5.10. (i) *The subspace $V^{p,n,\gamma}$ is spectrally equivalent to $V^{p,n,(ij)\cdot\gamma}$ for any $i \neq j$.*
(ii) *The subspace $V^{p,n,\gamma}$ is spectrally equivalent to $V^{p,n,\sigma\cdot\gamma}$ for any $\sigma \in S_n$.*

The proof is straightforward, in light of the above discussion.

Before continuing, we note several basic facts about these symmetry operators:

$$\chi_{ij}^2 = \mathrm{Id} \tag{5.1}$$

$$\chi_{ij}\chi_{ik}\chi_{ij} = \chi_{jk} \tag{5.2}$$

$$\chi_{jk}U_{1j} = U_{1k}\chi_{jk} \tag{5.3}$$

for any i, j, k such that $2 \leq i < j < k \leq n$

From (5.1), we deduce that χ_{23} has only +1 and -1 as eigenvalues. From (5.2), we see that given $\chi_{23}, \chi_{24}, \ldots, \chi_{2n}$, we can generate (by composition) any other χ_{ij} (for $2 \leq i < j \leq n$). Equation (5.3) will be useful in the next section.

6. Subspaces and sub-subspaces

We are now able to use Lemma 5.7 (iii) to divide up the eigenvalues of $\triangle_{p,n}(k)$ on $V^{p,\gamma}$ for any γ, using Theorem 6.2, to be proved shortly. However, we first need to define certain maps for convenience.

Definition 6.1. We define a map from $\mathbb{Z}^n$ to $\mathbb{Z}^{n-1}$ which omits the ith index and is denoted p_i:

$$p_i : (a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \mapsto (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n).$$

This then induces a projection p_i^* from $\mathcal{F}_n^{-k}$ onto $\mathcal{F}_{n-1}^{-k}$, defined to be the linear operator with action on the basis elements given by:

$$p_i^* : \psi_\beta(k) \mapsto \psi_{p_i(\beta)}(k).$$

We can now state the theorem.

Theorem 6.2. (i) *For any multi-index γ such that $\gamma_n \geq 2$, $V^{p,n,\gamma}$ is spectrally equivalent to*

$$(\ker U_{1n} \cap V^{p,n,\gamma'}) \oplus (\ker U_{1n} \cap V^{p,n,\gamma''}) \oplus V^{p,n,\gamma'''}$$

where $\gamma' = \gamma + (\gamma_n - 1)e_1 - (\gamma_n - 1)e_n$ (so that $(\gamma')_n = 1$) and $\gamma'' = \gamma' + e_1 - e_n$, $\gamma''' = \gamma' + 2e_1 - 2e_n$.
(ii) *The subspace $V^{p,n,\gamma'''}$ is spectrally equivalent to $V^{p-1,n-1,p_n(\gamma''')}$.*
(iii) *The subspace $\ker U_{1n} \cap V^{p,n,\gamma'}$ is spectrally equivalent to $V^{p-1,n-1,p_n(\gamma')+e_1}$.*

Proof. (i) By Lemma 5.7 (iii), the subspaces $V^{p,n,\gamma}$ and $V^{p,n,\gamma'}$ are spectrally equivalent.

Now as mentioned before,

$$V^{p,n,\gamma'} \cong \mathrm{Im}\, U_{n1} \oplus \ker U_{1n}$$

where each subspace is $\triangle(k)$-invariant (a consequence of Lemma 5.7 (i)). But $\gamma_1 \geq 2$ by assumption; since we are adding a non-negative number to the first index of γ, we also have $(\gamma')_1 \geq 2$, which means that U_{n1} (here a map from $V^{p,n,\gamma''}$ to

$\mathrm{Im}U_{n1}$) is one-to-one, and so $\mathrm{Im}\,U_{n1}$ is spectrally equivalent to $V^{p,n,\gamma''}$. The above decomposition then implies that

$$V^{p,n,\gamma} \text{ is spectrally equivalent to } (\ker U_{1n} \cap V^{p,n,\gamma'}) \oplus V^{p,n,\gamma''}.$$

Similarly we can decompose $V^{p,n,\gamma''}$ into $\mathrm{Im}\,U_{n1} \oplus \ker U_{1n}$; again, U_{n1} is one-to-one and thus an isomorphism from $V^{p,n,\gamma'''}$ to $\mathrm{Im}\,U_{n1}$. Hence $V^{p,n,\gamma''}$ is spectrally equivalent to $(\ker U_{1n} \cap V^{p,n,\gamma''}) \oplus V^{p,n,\gamma'''}$. These two spectral equivalences then imply part (i) of the theorem.

(ii) We are considering $V^{p,n,\gamma'''}$, where $(\gamma''')_n = -1$. Recall that ψ_β is only defined if $\beta_i \geq 0$ for all i from 1 to n, so any element of $V^{p,n,\gamma'''}$ must be of the form $v \wedge \tau^n$, for some v in $V^{p-1,n,\gamma'''+e_n}$ such that $i(Z_n)v = 0 = i(Z_{\bar{n}})v$.

We construct a homomorphism

$$\varphi_1 : V^{p,n,\gamma'''} \to V^{p-1,n-1,p_n(\gamma''')}$$

which takes $v \wedge \tau^n$ to $p_n^*(v)$, where we extend p_n^* by tensoring with the projection from $\Lambda^{p-1}((\mathfrak{h}^{2n+1})^*)$ onto $\Lambda^{p-1}((\mathfrak{h}^{2n-1})^*)$. It can easily be seen that this (linear) homomorphism φ_1 is in fact one-to-one and onto.

For θ_j and θ_j^* the operators defined in Definition 4.2, we have $\theta_n(v \wedge \tau^n) = 0 = \theta_n^*(v \wedge \tau^n)$; further, for any $i, j = 1, 2, \ldots, n-1$, the operators $\theta_j \theta_i^*$ and $\theta_i^* \theta_j$ both commute with the homomorphism φ_1 (that is, φ_1 doesn't affect the action of these operators). Thus $\triangle(k)$ commutes with φ_1, which proves part (ii).

To prove part (iii), construct a linear mapping φ_2 from $\ker U_{1n} \cap V^{p,n,\gamma'}$ to $V^{p-1,n-1,p_n(\gamma')+e_1}$ as follows.

Begin by specifying that φ_2 commutes with $e(\tau^j)a_j^*$ and $e(\tau^{\bar{j}})a_j$ for $2 \leq j \leq n-1$, and with $e(\tau^w)$, and also with the adjoints of these operators. These operators map $\ker U_{1n} \cap V^{p,n,\gamma'}$ to $\ker U_{1n} \cap V^{p+1,n,\gamma'}$ and $V^{p-1,n-1,p_n(\gamma')+e_1}$ to $V^{p,n-1,p_n(\gamma')+e_1}$.

Proceed by defining a 2-form, to be denoted ω_2, in $\ker U_{1n} \cap V^{p,n,\gamma'}$, by

$$\omega_2 := \psi_{\gamma'-e_1-e_n} \tau^{\bar{1}} \wedge \tau^{\bar{n}}$$

and specify that φ_2 maps ω_2 to $\psi_{p_n(\gamma')}\tau^{\bar{1}}$.

Define next forms $\omega_0, \omega_1, \omega_3$ in $\ker U_{1n} \cap V^{*,n,\gamma',\alpha}$ to be the 1-form, 2-form and 3-form respectively given by

$$\begin{aligned} \omega_0 &:= (g+1)^{-1/2}\big(i(Z_{\bar{1}})a_1^* + i(Z_{\bar{n}})a_n^*\big)\omega_2, \\ \omega_3 &:= (g+1)^{-1/2}\big(e(\tau^1)a_1^* + e(\tau^n)a_n^*\big)\omega_2 \qquad \text{and} \\ \omega_1 &:= (g+2)^{-1/2}\big(e(\tau^1)a_1^* + e(\tau^n)a_n^*\big)\omega_0 \end{aligned}$$

for $g := \gamma_1 + \gamma_n - 1$; these forms ω_j are still in $\ker U_{1n}$ since $[e(\tau^1)a_1^* + e(\tau^n)a_n^*, U_{1n}] = 0 = [e(\tau^{\bar{1}})a_1 + e(\tau^{\bar{n}})a_n, U_{1n}]$.

Set φ_2 to also map ω_0 to $\psi_{p_n(\gamma')+e_1}$, map ω_3 to $\psi_{p_n(\gamma')+e_1}\tau^1 \wedge \tau^{\bar{1}}$ and map ω_1 to $\psi_{p_n(\gamma')+2e_1}\tau^1$. The mapping φ_2 can now be seen to be an isomorphism (for example, by counting dimensions of the respective subspaces).

It's necessary to check that ω_0, ω_1 and ω_3 all have length 1 with respect to the inner product that we've chosen on $V^{p,n,\gamma'}$, and also that ω_1 is equal to $-(g+2)^{-1/2}(i(Z_{\bar{1}})a_1^* + i(Z_{\bar{n}})a_n^*)\omega_3$. By considering the actions of adjoints of the operators discussed, it follows (after some calculations) that

$$\begin{aligned} \varphi_2(\theta_1(k) + \theta_n(k)) &= \theta_1(k)\varphi_2, \\ \varphi_2(\theta_1^*(k) + \theta_n^*(k)) &= \theta_1^*(k)\varphi_2 \end{aligned}$$

on $\ker U_{1n} \cap V^{p,n,\gamma'}$, and that φ_2 commutes with the other operators involved in d and d^*.

It follows that $\varphi_2 \triangle_{p,n}(k) = \triangle_{p-1,n-1}(k)\varphi_2$ on all of $\ker U_{1n} \cap V^{p,n,\gamma'}$. Since we've already proved that φ_2 is an isomorphism, this completes the proof. □

This theorem would also be true if we replaced U_{1n} by U_{12} or U_{13} and so on, so that instead of considering $\ker U_{1n} \cap V^{p,\gamma''}$, we need only consider the subspace $\ker U_{12} \cap \ldots \cap \ker U_{1n} \cap V^{p,(|\gamma|,0,\cdots,0)}$ (since all the eigenvalues "missed" here are eigenvalues of $\triangle_{p-1,n-1}(k)$).

This subspace $\ker U_{12} \cap \ldots \cap \ker U_{1n} \cap V^{p,n,(|\gamma|,0,\cdots,0)}$ will be referred to as the reduced subspace, and denoted by $V_{red}^{p,n,|\gamma|}$.

The subspace $V^{p,n,(|\gamma|,0,\cdots,0)}$ is just $V^{p,n,|\gamma|e_1}$, which is how it will be referred to from now on.

We now derive a basis for certain symmetric subspaces and study the action of the Laplacian thereon. As implied previously, the reduced subspace $V_{red}^{p,|\gamma|}$ can be further decomposed, this time with respect to the action of the χ_{2j}'s.

Definition 6.3. We call an element of this subspace which is also in the +1-eigenspace of **all** the χ_{2j}'s (and thus of all χ_{ij}'s, by (5.2)) a **symmetric** element, and an element in the −1-eigenspace of **any** χ_{ij} an **anti-symmetric** element. These two possibilities account for all of the subspace, i.e.

$$V_{red}^{p,|\gamma|} = (E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n}) \oplus (E_{-1}\chi_{23} + \ldots + E_{-1}\chi_{n-1,n})$$

(where $E_\lambda A$ refers to the eigenspace of A corresponding to the eigenvalue λ). The symmetric subspace is defined to be $E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n} \cap \ker U_{12} \cap \ldots \cap \ker U_{1n} \cap V^{p,|\gamma|e_1}$ and will be denoted by $V_{symm}^{p,|\gamma|}$.

In order to understand the eigenvalues of the Laplacian on these subspaces, we begin by characterising explicitly all elements of $E_{-1}\chi_{ij} \cap V_{red}^{p,|\gamma|}$ for $2 \le i < j \le n$. Note that this subspace is preserved by $\triangle_{p,n}(k)$.

To investigate the eigenvalues of the Laplacian on the anti-symmetric subspace, we first prove a slightly more general lemma than is strictly necessary.

Lemma 6.4. *For any multi-index $\hat{\gamma}$ such that $(\hat{\gamma})_{n-1} = 0 = (\hat{\gamma})_n$, the subspaces $E_{-1}\chi_{n-1,n} \cap \ker U_{1,n-1} \cap \ker U_{1,n} \cap V^{p,n,\hat{\gamma}}$ and $V^{p-2,n-2,p_{n-1}p_n(\hat{\gamma})}$ are spectrally equivalent.*

Proof. We construct a linear mapping $\varphi_3 : E_{-1}\chi_{n-1,n} \cap \ker U_{1,n-1} \cap \ker U_{1,n} \cap V^{p,n,\hat{\gamma}} \to V^{p-2,n-2,p_{n-1}p_n(\hat{\gamma})}$, and show that φ_3 is an isomorphism and commutes with the Laplacian, in a similar manner to the proof of Theorem 6.2iii.

Define a 3-form, $\hat{\omega}_2$, by

$$\hat{\omega}_2 := 2^{-1/2}\psi_{\hat{\gamma}-e_1}\tau^{\bar{1}} \wedge (\tau^n \wedge \tau^{\bar{n}} - \tau^{n-1} \wedge \tau^{\overline{n-1}});$$

note that $\hat{\omega}_2 \in E_{-1}\chi_{n-1,n} \cap \ker U_{1,n-1} \cap \ker U_{1,n} \cap V^{p,n,\hat{\gamma}}$. Set $\varphi_3(\hat{\omega}_2) = \psi_{p_{n-1}p_n(\hat{\gamma})}$.

Again as in the proof of Theorem 6.2iii, define the 2-form $\hat{\omega}_0$, 3-form $\hat{\omega}_1$, and 4-form $\hat{\omega}_3$ as follows, for $\hat{g} := (\hat{\gamma})_1$:

$$\begin{aligned}
\hat{\omega}_0 &:= (\sqrt{\hat{g}+1})^{-1/2}\big(i(Z_{\bar{1}})a_1^* + i(Z_{\overline{n-1}})a_{n-1}^* + i(Z_{\bar{n}})a_n^*\big)\hat{\omega}_2, \\
\hat{\omega}_3 &:= (\sqrt{\hat{g}+1})^{-1/2}\big(e(\tau^1)a_1^* + e(\tau^{n-1})a_{n-1}^* + e(\tau^n)a_n^*\big)\hat{\omega}_2 \quad \text{and} \\
\hat{\omega}_1 &:= (\sqrt{\hat{g}+2})^{-1/2}\big(e(\tau^1)a_1^* + e(\tau^{n-1})a_{n-1}^* + e(\tau^n)a_n^*\big)\hat{\omega}_0
\end{aligned}$$

Note that the operators $e(\tau^1)a_1^*+e(\tau^{n-1})a_{n-1}^*+e(\tau^n)a_n^*$ and $e(\tau^{\bar{1}})a_1+e(\tau^{\overline{n-1}})a_{n-1}+e(\tau^{\bar{n}})a_n$ both commute with $U_{1,n-1}, U_{1,n}$ and $\chi_{n-1,n}$, as do their adjoints, so that $\hat{\omega}_0, \hat{\omega}_1$ and $\hat{\omega}_3$ are all in $E_{-1}\chi_{n-1,n} \cap \ker U_{1,n-1} \cap \ker U_{1,n}$.

We set φ_3 to map $\hat{\omega}_0$ to $\psi_{p_{n-1}p_n(\gamma)+e_1}$, $\hat{\omega}_3$ to $\psi_{p_{n-1}p_n(\hat{\gamma})+e_1}\tau^1 \wedge \tau^{\bar{1}}$ and $\hat{\omega}_1$ to $\psi_{p_{n-1}p_n(\hat{\gamma})+2e_1}\tau^1$.

Again we must check that $\hat{\omega}_0, \hat{\omega}_1$ and $\hat{\omega}_3$ all have length 1 with respect to the inner product that we've chosen on $V^{p,n,\hat{\gamma},\alpha}$, and also that $\hat{\omega}_1$ is equal to $-(g+2)^{-1/2}(i(Z_{\bar{1}})a_1^* + i(Z_{\overline{n-1}})a_{n-1}^* + i(Z_{\bar{n}})a_n^*)\hat{\omega}_3$. If we then set φ_3 to commute with the operators $e(\tau^j)a_j^*, i(Z_j)a_j, e(\tau^{\bar{j}})a_j, i(Z_{\bar{j}})a_j^*$ for $1 < j < n-1$ and $e(\tau^w), i(W)$, it follows as in the proof of Theorem 6.2 (iii) that φ_3 is an isomorphism and commutes with $\triangle(k)$. □

Corollary 6.5. *The subspace $E_{-1}\chi_{ij} \cap V_{red}^{p,n,|\gamma|}$ is spectrally equivalent to a subspace of $V^{p-2,n-2,(|\gamma|+1)e_1}$ for $i \neq j, i, j = 2, \ldots, n$.*

This follows from Lemma 6.4 and appropriate use of the symmetry operators.

Thus the eigenvalues of the Laplacian in the representation $\bar{\beta}_k$ on the anti-symmetric subspace have already been counted, in a sense, since they occur as eigenvalues of the Laplacian on lower-degree forms on a lower-dimensional Heisenberg group. Specifically, they are the same eigenvalues that we would get from applying (say) the homomorphism φ_1 from Theorem 6.2 (ii) twice.

We now begin to analyse the eigenvalues which occur on the symmetric subspace.

Lemma 6.6. *The subspace $V_{symm}^{p,n,|\gamma|}$ is equal to $E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n} \cap \ker U_{12} \cap V^{p,n,|\gamma|e_1}$.*

The proof follows by repeated application of equation (5.3).

We explicitly characterise all symmetric elements, beginning this process by looking at $E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n} \cap V^{p,n,|\gamma|e_1}$ (i.e. dropping the $\ker U_{12}$ condition).

Definition 6.7. We define a p-form, $\varepsilon(p,n)$, which is in $V^{p,n,|\gamma|e_1}$, by:

$$\varepsilon(p,n) \quad := \quad \begin{cases} \psi_{|\gamma|e_1} & \text{if } p{=}0 \\ \sum_{j=2}^n a_j^* e(\tau^j)\varepsilon(p-1,n) & \text{if } p \text{ is odd} \\ (-2/p)\sum_{j=2}^n a_j e(\tau^{\bar{j}})\varepsilon(p-1,n) & \text{if } p \text{ is even and } p \geq 2 \end{cases}$$

so that

$$\varepsilon(1,n) = \textstyle\sum_{j=2}^n \psi_{|\gamma|e_1+e_j}\tau^j; \qquad \varepsilon(2,n) = \displaystyle\sum_{j=2}^n \psi_{|\gamma|e_1}\tau^j \wedge \tau^{\bar{j}};$$

$$\varepsilon(3,n) = \textstyle\sum_{j,l=2}^n \psi_{|\gamma|e_1+e_j}\tau^j \wedge \tau^l \wedge \tau^{\bar{l}}; \qquad \varepsilon(4,n) = \displaystyle\sum_{j,l=2}^n \psi_{|\gamma|e_1}\tau^j \wedge \tau^{\bar{j}} \wedge \tau^l \wedge \tau^{\bar{l}} \ldots$$

Now $\varepsilon(p,n)$ is certainly in $E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n} \cap V^{p,n,|\gamma|e_1}$. In fact, we have the following lemma.

Lemma 6.8. (i) *For $p \geq 3$, a basis for $E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n} \cap V^{p,n,|\gamma|e_1}$ is given by*

$$\big\{\varepsilon(p,n),\ a_1^*\tau^1 \wedge \varepsilon(p-1,n),\ a_1\tau^{\bar{1}} \wedge \varepsilon(p-1,n),\ \tau^1 \wedge \tau^{\bar{1}} \wedge \varepsilon(p-2,n),$$
$$\tau^w \wedge \varepsilon(p-1,n),\ a_1^*\tau^w \wedge \tau^1 \wedge \varepsilon(p-2,n),\ a_1\tau^w \wedge \tau^{\bar{1}} \wedge \varepsilon(p-2,n),$$
$$\tau^w \wedge \tau^1 \wedge \tau^{\bar{1}} \wedge \varepsilon(p-3,n)\big\}$$

(ii) *For p even and $p \geq 4$, $p = 2q$ say, a basis of the symmetric subspace is*

$$\begin{gathered}\big\{-|\gamma|\tau^1 \wedge \tau^{\bar{1}} \wedge \varepsilon(2q-2,n) + a_1\tau^{\bar{1}} \wedge \varepsilon(2q-1,n),\\ \tau^1 \wedge \tau^{\bar{1}} \wedge \varepsilon(2q-2,n) + \varepsilon(2q,n),\\ \tau^w \wedge \tau^1 \wedge \tau^{\bar{1}} \wedge \varepsilon(2q-3,n) + a_1^*\tau^w \wedge \tau^1 \wedge \varepsilon(2q-2,n) + \tau^w \wedge \varepsilon(2q-1,n),\\ a_1\tau^w \wedge \tau^{\bar{1}} \wedge \varepsilon(2q-2,n)\big\}\end{gathered}$$

(iii) *The matrix of $\triangle_{2q,n}(k)$ acting on $V_{symm}^{2q,|\gamma|}$ with respect to the above basis is then*

$$(2k|\gamma| + k^2 + nk)Id + \begin{pmatrix} (q-1)(n-q) & 0 & \sqrt{k} & \sqrt{k} \\ -q|\gamma| & q(n-q-1) & -q\sqrt{k} & 0 \\ \sqrt{k}|\gamma| & -\sqrt{k} & k+q(n-q) & 0 \\ \sqrt{k}(|\gamma|+n-q) & -\sqrt{k} & 0 & -k+q(n-q) \end{pmatrix}$$

and its eigenvalues are

$$\begin{gathered}\big\{2k|\gamma| + nk + k^2 + q(n-q),\\ 2k|\gamma| + nk + k^2 + \tfrac{1}{2}n + q(n-q-1) \pm (\tfrac{1}{4}n^2 + nk + 2k|\gamma| + k^2)^{1/2}\big\}.\end{gathered} \tag{6.1}$$

where the first eigenvalue has multiplicity 2.

Proof. (i) It is easily checked that all these elements are in the required subspace. Also, it can be seen that the subspace $\ker i(Z_1) \cap \ker i(Z_{\bar{1}}) \cap \ker i(W) \cap E_1\chi_{23} \cap \ldots \cap E_1\chi_{2n} \cap V^{p,n,|\gamma|e_1}$ is spanned by $\varepsilon(p,n)$, i.e. $\varepsilon(p,n)$ is the only symmetric element which doesn't contain $\tau^1, \tau^{\bar{1}}$, or τ^w. In this way we see that the given elements do in fact span the subspace in question.

(ii) The proof requires investigating the action of U_{12} (which is all that is necessary by Lemma 6.6) on linear combinations of the basis elements from (i).

(iii) Proving this is a matter of (somewhat tedious) calculations, using firstly the formula (4.2) and secondly calculating the eigenvalues of the matrix, using say Maple. □

We only need to find a basis for even p, due to the following fact: a basis of the symmetric subspace for odd p ($p = 2q+1$ say) can be derived from the bases given in (ii) (more precisely, from the bases corresponding to $p = 2q$ and $p = 2q+2$) by judicious use of $e(\tau^w)$ and $i(W)$, since these operators commute with U_{ij} and χ_{ij} for all i and j.

Further, using the Hodge star operator (which could also give us the basis for p odd from those for even p), we see that the eigenvalues of $\triangle_{2q,n}(k)$ are exactly those of $\triangle_{2n+1-2q,n}(k)$, so that we need not calculate the action of the Laplacian on odd forms separately.

We note specifically that all the eigenvalues given in equation (6.1) are greater than $(n-1)k + k^2/c^2$.

Remark 6.9. Through tedious but straightforward calculations, one can verify that the eigenvalues of the Laplacian on symmetric 1-forms and symmetric 2-forms are also given by equation (6.1), though for 1-forms the first eigenvalue has multiplicity one. Similarly, it can be shown that the eigenvalues of the Laplacian on $V_{symm}^{p,n,0}$ are just $\{k^2 + nk + q(n-q), k^2 + (n+1)k + q(n-q+1)\}$. For more details, see [33].

7. The spectrum of the Laplacian in a representation

In this section, we will prove the following result:

Theorem 7.1. *For any positive integers p, n, let $\triangle_{p,n}$ be the Laplacian on p-forms on H^{2n+1} and $\triangle_{p,n}(k)$ the corresponding operator in the representation ρ_k which corresponds to Fourier transform in k over the centre variable. Then the eigenvalues of $\triangle_{p,n}(k)$ are*

$$\begin{aligned}\{ \quad & 2k(g-1) + k^2 + (n-p)k, \\ & k^2 + (n-p+r+1)k + \left\lfloor \frac{r+1}{2} \right\rfloor \left(n - p + \left\lfloor \frac{r}{2} \right\rfloor + 1\right), \\ & 2k(g-1) + k^2 + (n-p+r)k + \left\lfloor \frac{r}{2} \right\rfloor \left(n - p + \left\lfloor \frac{r+1}{2} \right\rfloor\right), \\ & 2kg + k^2 + (n-p+r)k + \frac{1}{2}(n-p+r) + \left\lfloor \frac{r-1}{2} \right\rfloor \left(n - p + \left\lfloor \frac{r}{2} \right\rfloor\right) \\ & \pm(1/4(n-p+r)^2 + (n-p+r)k + 2kg + k^2)^{1/2} \\ & : g \in \mathbb{Z}, g \geq 1 \text{ and } r = 1, \dots, p\}.\end{aligned}$$

For any k, the lowest eigenvalue of $\triangle_{p,n}(k)$ is $k^2 + (n-p)k$, and its multiplicity is $\binom{n}{p}$.

Here $\lfloor n \rfloor$ is defined to be the greatest integer smaller than n.

The bulk of the work to prove Theorem 7.1 has already been done (in particular, see equation (6.1)). To complete the proof, note that the spectrum of $\triangle_{p,n}(k)$ contains all the eigenvalues of $\triangle_{p-1,n-1}(k)$. But this latter set in turn includes all the eigenvalues of $\triangle_{p-2,n-2}(k)$ and so on, so that the spectrum of $\triangle_{p,n}(k)$ contains the spectrum of $\triangle_{p-r,n-r}(k)$ for any r between 1 and p.

The only new (as yet "unlisted") eigenvalues at each stage (working from lower degree forms to higher degree) are those which occur on the symmetric subspaces $V_{symm}^{p-r,n-r,|\gamma|}$, for $|\gamma| = 0, 1, \dots$. So these, together with the eigenvalues of the Laplacian acting on functions, are all of the eigenvalues of the Laplacian on p-forms, $\triangle_{p,n}(k)$.

It remains to prove that the given eigenvalue is indeed lowest, and that its multiplicity is as specified.

Lemma 7.2. *The lowest eigenvalue of $\triangle_{p,n}(k)$ is $(n-p)k + k^2$.*

Proof. Note that this eigenvalue occurs; for example, $\psi_0(k)\tau^1 \wedge \ldots \wedge \tau^p$ is an eigenvector with this eigenvalue.

For most of the eigenvalues in the list given in Theorem 7.1, it is clear that they are greater than this eigenvalue (since we are considering the case $k > 0$). We need only consider the last eigenvalue which contains the negative square root. We have that

$$2kg + k^2 + nk + \frac{1}{2}n - (\sqrt{1/4n^2 + nk + 2kg + k^2}) > k^2 + (n-1)k,$$

for $n, g \geq 1$. So this eigenvalue will always be greater than $k^2 + (n-p+r-1)k$, which is greater than or equal to $k^2 + (n-p)k$ since $r \geq 1$.

So all other eigenvalues are greater than the eigenvalue in question. □

Lemma 7.3. *The multiplicity of the lowest eigenvalue of $\triangle_{p,n}(k)$ is $\binom{n}{p}$.*

Proof. We note from the proof of the preceding lemma that the eigenvalues of $\triangle_{p-r,n-p+r}(k)$ (for $r \leq p$) on the symmetric subspace $V_{symm}^{p-r,n-p+r,|\gamma|}$ are strictly greater than the value under consideration, $(n-p)k + k^2$, for any $|\gamma|$.

This eigenvalue, which occurs as an eigenvalue of $\triangle_{0,n-p}(k)$ acting on $\psi_0(k)$ in $\mathcal{F}_{n-p}^{-k}$, is only found in the spectrum of $\triangle_{p,n}(k)$ due to repeated applications of Theorem 6.2ii, together with isometries $\chi_{j,n-p+r}$ (for certain values of j between 1 and $n-p+r$).

That is, we have: $V^{0,n-p,0}$ is spectrally equivalent to $V^{1,n-p+1,\sigma_1\cdot(-1,0,\dots,0)}$ (for some transposition σ_1 in S_{n-p+1}), which in turn is spectrally equivalent to $V^{2,n-p+2,\sigma_2\cdot(-1,-1,0,\dots,0)}$ (for σ_2 some permutation in S_{n-p+2}, actually the product of σ_1 and a disjoint transposition) and so on; by induction, we infer that $V^{0,n-p,0}$ is spectrally equivalent to $V^{p,n,\sigma_p\cdot(-1,\dots,-1,0,\dots,0)}$ for some $\sigma_p \in S_n$ with the multi-index consisting of -1 repeated p times and 0 repeated $n-p$ times. (Here S_q stands for the permutation group on q symbols.)

There are $\binom{n}{p}$ different ways of choosing σ_p which give different multi-indices, which proves this lemma, since all of these subspaces $V^{p,n,\sigma_p\cdot -e_1-\dots-e_p}$ are of (complex) dimension 1. □

Note that the direct sum of these subspaces, the eigenspace of the lowest eigenvalue, is the subspace denoted by Lott in [22] by $\mathcal{S}^p$.

This concludes the proof of the lemma and thus proves Theorem 7.1.

8. Calculation of Novikov-Shubin invariants

In this section, we prove the following corollary.

Corollary 8.1. *The pth Novikov-Shubin invariant of H^{2n+1} is given by*

$$\alpha_p(H^{2n+1}) = \begin{cases} n+1, & p \neq n, n+1, \\ \frac{1}{2}(n+1), & p = n, n+1. \end{cases}$$

Furthermore, for any discrete subgroup Γ of H^{2n+1} such that $M = H/\Gamma$ is a compact manifold, $\alpha_p(M) = \alpha_p(H^{2n+1})$; any manifold which is homotopy equivalent to such a manifold M also has the same Novikov-Shubin invariants, and any manifold M' whose fundamental group $\pi_1(M')$ is a discrete subgroup of H^{2n+1} has its first Novikov-Shubin invariant given by $\alpha_1(M') = \alpha_1(H^{2n+1})$.

Proof. This corollary follows from Theorem 7.1 by analysis of the form of $\mathrm{Tr}_\Gamma e^{-t\triangle_p}$ and by showing that the lowest eigenvalue of $\triangle_p(k)$ does indeed determine the decay, as follows.

We need to first establish how $\mathrm{Tr}_\Gamma e^{-T\triangle_p}$ can be found given only the eigenvalues of $\triangle_p(k)$ for all k.

Let Γ be a discrete subgroup of H, and let $\mathcal{A}$ be the von Neumann algebra defined as in section 1. Suppose we have an operator A in $\mathcal{A}$ which is also Γ-trace class, is positive, self-adjoint, and has smooth kernel $k_A(x,y)$. Suppose also that A is not only Γ-invariant, but also H-invariant.

Then $L_g A u = A L_g u$ for any $g \in H$, $u \in L^2(H)$. This implies that the kernel of A satisfies $k_A(g^{-1}x, y) = k_A(x, gy)$ for all $x, y \in H$, i.e. that $k_A(x,y) = k_A(e, x^{-1}y)$ and k_A is a convolution kernel.

In particular, if $A = \pi_R(f)$ for some $f \in C_0^\infty(H)$, then

$$\mathrm{Tr}_\Gamma(A) = \mathrm{vol}(H/\Gamma)t(A),$$

where t is the trace defined in Theorem 3.1. Since $\pi_R(C_0^\infty(H))$ is dense in $\mathcal{V}_H$, we have that $\mathrm{Tr}_\Gamma(A) = \mathrm{vol}(H/\Gamma)t(A)$ for any $A \in \mathcal{V}_H$. (This argument is taken from [2]; see that article for more details. The assumption there is that the Lie group in question is semi-simple, but the same arguments hold for nilpotent Lie groups, for example H.)

The Plancherel Theorem for H (see [9] or [14]) implies that this trace also decomposes under equation (3.2),

$$t(A) = \int_{k\in\mathbb{R}} \mathrm{tr}_k(A(k))|k|^n dk,$$

where $A = \int_{k\in\mathbb{R}}^{\oplus} A(k)|k|^n dk$. We take $A(k)$ to be the operator A in the representation $\bar{\beta}_k$ (since A is left-invariant), so that tr_k is just the Hilbert-Schmidt trace on trace-class operators on $\bar{\mathcal{F}}_n^k$.

In particular, if $\{\lambda_j(k)\}_{j=1}^\infty$ are the eigenvalues of $A(k)$, with $\lambda_1(k) \geq \lambda_2(k) \geq \dots$, then $\mathrm{tr}_k(A(k)) = \sum_{j=1}^\infty \lambda_j(k)$.

Now to apply this theory to our situation. We have to extend all the traces above by tensoring with the trace on $End(\Lambda^p(\mathfrak{h}^*))$, but this carries through all the above discussion. The Laplacian $\triangle_{p,n}$ is (left) H-invariant, positive, self-adjoint, and has smooth kernel; therefore, so does the heat operator $e^{-t\triangle_{p,n}}$. Further, the heat operator is bounded, and thus is in $\mathcal{A}$.

We still need the eigenvalues of the heat operator; however, it is a corollary of the spectral theorem for self-adjoint operators that if an operator B has an eigenvalue λ, then e^{-tB} will have an eigenvalue $e^{-t\lambda}$ (in fact, the eigenvector will be the same).

So given all the eigenvalues of the Laplacian on p-forms on H^{2n+1} in the representations $\bar{\beta}_k$, we can determine the Novikov-Shubin invariants.

However, not all of this information is needed just to calculate the pth Novikov-Shubin invariant; only the value of the lowest eigenvalue of $\triangle_p(k)$ and its multiplicity (for all k) is strictly necessary, as shown below.

Here we consider k to be fixed. Suppose we have ordered the eigenvalues of $\triangle_p(k)$; that is, (indexing by positive integers) so that $\lambda_1(k) = \dots = \lambda_m(k) < \lambda_{m+1}(k) \leq \lambda_{m+2}(k)\dots$ (where m is the multiplicity of the lowest eigenvalue). Then

$$\sum_{j=1}^\infty e^{-T\lambda_j(k)} = me^{-T\lambda_1(k)} + \sum_{j=m+1}^\infty e^{-T\lambda_j(k)}$$

As $\triangle_{1,n}(k)$ is an unbounded operator, we have $\lambda_j(k) \to \infty$ as $j \to \infty$; thus the last term will always be less than $e^{-T\lambda_1(k)}$, and will not affect the decay as $T \to \infty$.

A further complication occurs if the multiplicity of the lowest eigenvalue of $\triangle_{p,n}(k)$ varies with k; however, this is not the case here. As proved above, the multiplicity of the lowest eigenvalue of the Laplacian on p-forms on the Heisenberg group depends only on n and p.

Finally we need to consider the relevant integrals. For $a > 0$ and $b > 0$ constants, it can be shown that

$$\int_0^\infty k^m e^{-T(ak+k^2.f(k))} dk = \left(\frac{1}{aT}\right)^{m+1} + O(T^{-m-2}),$$

$$\int_0^\infty k^m e^{-T(bk^2+k^3.f(k))} dk = O(T^{-1/2(m+1)})$$

for m a positive integer, and $f(k)$ a power series in k (positive for all $k > 0$).

We see that the Heisenberg group H^{2n+1} is L^2 acyclic, that is has all L^2 Betti numbers zero, since $\lim_{T\to\infty} \mathrm{Tr}_\Gamma(e^{-T\triangle_{p,n}}) = 0$. In fact, this was known previously; it can be shown using techniques in [25].

So if $p < n$, we have that the decay of

$$\mathrm{Tr}_\Gamma e^{-T\triangle_p} = \int_{\mathbb{R}} \mathrm{tr}_k e^{-T\triangle_p(n,k)} |k|^n dk$$

is determined by the lowest eigenvalue, so that the integral is of the first kind, with $m = n$ and $a = n - p$ (with $f(k) = 1$). Thus $\alpha_p(H^{2n+1}) = n + 1$ if $p < n$, and using the Hodge star operator, if $p > n + 1$.

If $p = n$, the decay of the trace of the heat kernel is again determined by the lowest eigenvalue k^2, and thus by an integral of the second kind, with $m = n$ and $b = 1$ (with $f(k) = 0$). Thus $\alpha_n(H^{2n+1}) = 1/2(n + 1) = \alpha_{n+1}(H^{2n+1})$, again using the Hodge star operator.

The remainder of Corollary 8.1 follows from the definition of the Novikov-Shubin invariants and the fact that they are homotopy-invariant (see [3, 17]), as well as the fact that the first Novikov-Shubin invariant $\alpha_1(M)$ of a manifold M a function only of the fundamental group of M, which was proved in [23] □

9. General nilpotent Lie groups

For all of the following sections, the universal reference for background and definitions is [6], which covers these topics in detail; this reference will be assumed even if not specifically mentioned.

This section contains few new results; in particular, the results given here agree with the calculations in [29] on the spectrum of the Laplacian on functions in a representation, and are mostly an elaboration of Appendix A of that article. However, the definitions of this section are necessary for later sections.

We first outline the notation that will be used frequently from now on.

Let $\mathfrak{n}$ be a general nilpotent Lie algebra, with N the corresponding connected and simply connected Lie group. (Again, N is unique up to isomorphism; see [6].) Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$, and $\mathfrak{v}$ the complement of $\mathfrak{z}$. Let l be the dimension of $\mathfrak{z}$, and m the dimension of $\mathfrak{v}$.

Explicit formulae for d, d^*. From now on, we'll consider N to be a step 2 nilpotent Lie group.

Let $\{X_1, \dots, X_{m+l}\}$ be an orthonormal basis for $\mathfrak{n}$, and let X_j also denote the left-invariant vector field derived from X_j. Let $\{\tau^1, \dots, \tau^{m+l}\}$ be the corresponding basis of 1-forms. Then with respect to these bases, we can find explicit formulae for d, d^* and the Laplacian on functions and 1-forms at least.

If we select a basis $X_1, \dots, X_{m+l}$ for $\mathfrak{n}$, and let the structure constants be C_{ij}^k, that is $[X_i, X_j] = \sum_k C_{ij}^k X_k$, then we have that

$$\begin{aligned}
d &= \sum_{i=1}^{m+l} e(\tau^i)X_i + \sum_{i,j=1}^{m}\sum_{k=1}^{l} C_{ij}^k e(\tau^i)e(\tau^j)i(X_k),\\
d^* &= -\sum_{i=1}^{m+l} i(X_i)X_i + \sum_{i,j=1}^{m}\sum_{k=1}^{l} C_{ij}^k e(\tau^k)i(X_j)i(X_i),\\
\triangle_0 &= -\sum_{i=1}^{m+l} X_i^2,\\
\triangle_1 &= -\left(\sum_{i,j=1}^{m+l}[X_i,X_j]e(\tau^i)i(X_j) + X_j^2 - \sum_k X_j C_{ji}^k e(\tau^i)i(X_k)\right.\\
&\qquad \left. + \sum_{k>j} C_{jk}^i X_j e(\tau^i)i(X_k) - C_{jk}^i \sum_q C_{jk}^m e(\tau^i)i(X_m)\right).
\end{aligned}$$

If instead we allow $\{X_1, \dots, X_{m+l}\}$ to be complex vector fields (or to be an algebraic basis for $u(\mathfrak{n})$), which are orthonormal with respect to our chosen basis, then we have slightly different formulae for $d^*, \triangle_0$ and $\triangle_1$:

$$\begin{aligned}
d^* &= -\sum_{i=1}^{m+l} i(X_i)\bar{X}_i + \sum_{i,j=1}^{m}\sum_{k=1}^{l} \bar{C}_{ij}^k e(\tau^k)i(X_j)i(X_i),\\
\triangle_0 &= -\sum_{i=1}^{m+l} \bar{X}_i X_i, \qquad (9.1)\\
\triangle_1 &= -\left(\sum_{i,j=1}^{m+l}[X_i,\bar{X}_j]e(\tau^i)i(X_j) + \bar{X}_j X_j - \sum_k \bar{X}_j C_{ji}^k e(\tau^i)i(X_k)\right.\\
&\qquad \left. + \sum_{k>j} \bar{C}_{jk}^i X_j e(\tau^i)i(X_k) - \bar{C}_{jk}^i \sum_q C_{jk}^m e(\tau^i)i(X_m)\right). \qquad (9.2)
\end{aligned}$$

Kirillov theory. Take any element $\lambda \in \mathfrak{n}^*$. Then we can define a character ζ_λ on Z (the centre of N, and the image of $\mathfrak{z}$ under exp) to be $\zeta_\lambda(\exp z) = e^{i\lambda(z)}$ for $z \in Z$.

Let π_λ be the representation of N induced (in the sense of Mackey) from this representation ζ_λ of Z. We write $\mathcal{H}_\lambda$ for its representation space. We also denote the corresponding representation of $\mathfrak{n}$ by π_λ.

Kirillov theory (see for example [6, 20]) tells us that every unitary representation π of N is unitarily equivalent to π_λ for some λ; furthermore, two representations $\pi_\lambda, \pi_{\lambda'}$ are unitarily equivalent iff they are in the same $\mathrm{Ad}^*(N)$ orbit, i.e. if there exists an element g of N such that $\lambda = \mathrm{Ad}^* g(\lambda')$. That is, $\hat{N}$ is the set of coadjoint orbits of $\mathfrak{n}^*$.

In particular, for any elements W of $\mathfrak{z}$ and λ of $\mathfrak{n}^*$, we have that

$$\pi_\lambda(W) = \sqrt{-1}\,\lambda(W)\mathrm{Id}$$

since π_λ is a unitary representation.

We'll actually consider the conjugate representation $\bar{\pi}_\lambda$, since results from the Heisenberg group (where $\bar{\beta}_k$ was the relevant representation) will then be more easily comparable; it also corresponds to left-invariant operators. Again, this representation is canonically isomorphic to $\pi_{-\lambda}$, and so we'll write its representation space as $\mathcal{H}_{-\lambda}$; Note that for $W \in \mathfrak{z}$,

$$\bar{\pi}_\lambda(W) = -\sqrt{-1}\,\lambda(W)\mathrm{Id}.$$

The Plancherel theorem for nilpotent Lie groups. We define the bilinear form b_λ on $\mathfrak{n}$ associated to any $\lambda \in \mathfrak{n}^*$ as follows:

$$b_\lambda(X, Y) := \lambda([X, Y]).$$

We also define the radical of this bilinear form, r_λ, as $r_\lambda := \{X \in \mathfrak{n} : b_\lambda(X, Y) = 0\ \forall Y \in \mathfrak{n}\}$. Then b_λ is non-degenerate on $\mathfrak{n}/r_\lambda$; from the theory of linear algebra, we know that this space is even-dimensional, of dimension $2n$ say.

Let $\{X_1, \ldots, X_{2n}\}$ be a basis for $\mathfrak{n}/r_\lambda$. Then the Pfaffian $\mathrm{Pf}(\lambda)$ is defined, up to sign, by

$$\mathrm{Pf}(\lambda)^2 = \det \mathrm{B}_\lambda,$$

where B_λ is the matrix with (i, j)th entry $b_\lambda(X_i, X_j)$. Once a choice of sign is made, $\mathrm{Pf}(\lambda)$ is a polynomial function of λ; specifically, a polynomial in $\lambda_1, \ldots, \lambda_{m+l}$ (where $\lambda_i = \lambda(X_i)$) of degree n (see [31]).

Then it is well-known (see for example [6]) that the Plancherel measure on π_λ is Lebesgue measure on $\hat{N}$ multiplied by the Pfaffian $\mathrm{Pf}(\lambda)$; that is,

$$L^2(N) \cong \int_{\hat{N}} \mathcal{H}_\lambda \otimes \mathcal{H}_{-\lambda} |\mathrm{Pf}(\lambda)| \mathrm{d}\lambda. \tag{9.3}$$

Again, we take the Laplacian on p-forms on N, $\triangle_p$, to be left-invariant, and write $\triangle_p(\lambda)$ for its decomposition in the representation $\bar{\pi}_\lambda$. That is, $\triangle_p(\lambda)$ is an operator on $\mathcal{H}_{-\lambda} \otimes \Lambda^p(\mathfrak{n}^*)$.

Lower bound on spectrum. For general nilpotent groups, we can in fact use a similar method to that of Lemma 4.3 to find a lower bound on the spectrum of the Laplacian in a representation for any nilpotent Lie group, not just a step 2 nilpotent Lie group.

Theorem 9.1. *For any $\lambda \in \mathfrak{n}^*$, $\triangle_p(\lambda) \geq |(\lambda|_{\mathfrak{z}})|^2 Id$.*

Proof. For N and $\mathfrak{n}$ as above, let $\{X_1, \ldots, X_m\}$ be a basis for $\mathfrak{v}$ and $\{W_1, \ldots, W_l\}$ a basis for $\mathfrak{z}$, with $\{\tau^{W_1}, \ldots, \tau^{W_l}\}$ the dual basis. Identify these elements with left-invariant vector fields and 1-forms as before.

Define the operators $d_z := \sum_{q=1}^l e(\tau^{W_q})W_q$ and $d_v := d - d_z$, which both take L^2 p-forms on N to L^2 $(p+1)$-forms on N. Now d_v can be written

$$d_v = (\sum_{j=1}^m e(\tau^j)X_j) + \sum_{i,j,k} C_{ij}^k e(\tau^i)e(\tau^j)i(X_k)$$

(where the last term implicitly includes the case of $X_k = W_q$, i.e. that X_k is in the centre), but importantly, there is no term $e(\tau^{W_q})$ in d_v (for any $q = 1, \ldots, l$).

This implies that $i(W_q)d_v + d_v i(W_q) = 0$, which means that $d_z^* d_v + d_v d_z^* = 0$, since $d_z^* = -\sum_{q=1}^l i(W_q)W_q$. Similarly $d_v^* d_z + d_z d_v^* = 0$.

So

$$\begin{aligned}\triangle_p &= d_v^* d_v + d_v d_v^* + d_z^* d_z + d_z^* d_z \\ &= d_v^* d_v + d_v d_v^* - \sum_{q=1}^{l} W_q^2 \\ &\geq -\sum_{q=1}^{l} W_q^2\end{aligned}$$

where the inequality follows since $d_v^* d_v + d_v d_v^*$ is a positive operator. But this means that $\triangle_p(\lambda) \geq \sum_{q=1}^{l} \lambda(W_q)^2$. □

10. Heisenberg-type groups

The main reference for this section is [6].

Let $\mathfrak{n}$ be a step 2 nilpotent Lie algebra with positive definite inner product $\langle .,. \rangle$. Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$, and let $\mathfrak{v}$ be the complement of $\mathfrak{z}$ in $\mathfrak{n}$. For each element $W \in \mathfrak{z}$, define a skew-symmetric linear transformation $J(W)$ from $\mathfrak{v}$ to $\mathfrak{v}$ by:

$$\langle J(W)X, Y \rangle = \langle W, [X, Y] \rangle$$

for all $X, Y \in \mathfrak{v}$.

Definition 10.1. A step 2 nilpotent Lie algebra $\mathfrak{n}$ with metric $\langle .,. \rangle$ is of Heisenberg type (or H-type) if $J(W)^2 = -|W|^2 \mathrm{Id}$ on $\mathfrak{v}$ for all $W \in \mathfrak{z}$.

We can then derive the following formula:

$$\langle J(W)X, J(W')X \rangle = \langle W, W' \rangle |X|^2 \tag{10.1}$$

which is true for all W, W' in $\mathfrak{z}$, and for all X in $\mathfrak{v}$; this and other formulae concerning $J(W)$ can be found in, for example, [7].

There is a connection between $J(W)$ and representations π_λ; to see it more clearly, we'll need the following notation.

Definition 10.2. Let $\{W_1, \ldots, W_l\}$ be an orthonormal basis for $\mathfrak{z}$, and $\{\tau^{W_1}, \ldots, \tau^{W_l}\}$ the dual basis for $\mathfrak{z}^*$.

For any element $W = \sum_{q=1}^{l} A_q W_q$ of $\mathfrak{z}$ (with $A_q \in \mathbb{R}$), define

$$\lambda_W := \sum_{q=1}^{l} A_q \tau^{W_q},$$

the corresponding element of $\mathfrak{z}^*$.

Similarly, for any element $\lambda = \sum_{q=1}^{l} B_q \tau^{W_q}$ of $\mathfrak{z}^*$, define the corresponding element W_λ of $\mathfrak{z}$ by

$$W_\lambda := \sum_{q=1}^{l} B_q W_q.$$

Trivially, $\lambda_{W_\lambda} = \lambda$ and $W_{\lambda_W} = W$.

Now by definition,

$$\langle J(W_\lambda)U, V \rangle = \lambda([U, V]), \tag{10.2}$$

for all $\lambda \in \mathfrak{z}^*$, U, V in $\mathfrak{v}$. Equivalently,

$$\langle J(W)U, V \rangle = \lambda_W([U, V])$$

for all $W \in \mathfrak{z}, U, V \in \mathfrak{v}$, and we use these two equations interchangeably.

Useful for our purposes will be the following lemma, which has a straightforward proof, but is not (as far as I know) found in the literature.

Lemma 10.3. *Let $\mathfrak{n}$ be any step 2 nilpotent Lie algebra with positive definite inner product $\langle .,. \rangle$. If $\mathfrak{n}$ is H-type, then for any nonzero $\lambda \in \mathfrak{n}^*$, there is a basis $\{X_{j\lambda}, Y_{j\lambda}\}_{j=1}^n$ of $\mathfrak{v}$ such that*

$$\lambda([X_{j\lambda}, X_{k\lambda}]) = 0 = \lambda([Y_{j\lambda}, Y_{k\lambda}])$$

for $j, k = 1, \ldots, n$, and

$$\lambda([X_{j\lambda}, Y_{k\lambda}]) = \delta_{jk}|\lambda|.$$

We take the inner product on $\mathfrak{n}^*$ to be that induced by the inner product on $\mathfrak{n}$.

Proof. Choose any non-zero $\lambda \in \mathfrak{n}^*$; in fact, we will assume without loss of generality that $\lambda \in \mathfrak{z}^*$ (replacing λ by another element in its $\mathrm{Ad}^*(N)$ orbit if necessary).

Now $\{\mathfrak{v}, b_\lambda\}$ is a symplectic vector space (because b_λ is an anti-symmetric bilinear form on $\mathfrak{v}$, which is non-degenerate since $\mathfrak{n}$ is H-type). So we can find a basis for $\mathfrak{v}$ (which depends on λ) $u_{1\lambda}, \ldots, u_{n\lambda}, v_{1\lambda}, \ldots, v_{n\lambda}$ such that

$$b_\lambda(u_{i\lambda}, u_{j\lambda}) = 0 = b_\lambda(v_{i\lambda}, v_{j\lambda}) \text{ and } b_\lambda(u_{i\lambda}, v_{j\lambda}) = \delta_{ij}.$$

(For the proof, and more on symplectic vector spaces, see [18].) However, these elements $u_{i\lambda}, v_{j\lambda}$ are not necessarily normalized. We define $X_{j\lambda} := u_{j\lambda}/\|u_{j\lambda}\|$, and $Y_{j\lambda} := v_{j\lambda}/\|v_{j\lambda}\|$, so that $X_{1\lambda}, \ldots, X_{n\lambda}, Y_{1\lambda}, \ldots, Y_{n\lambda}$ are an orthonormal basis for $\mathfrak{v}$.

Then since $\mathfrak{n}$ is H-type, we have that

$$\begin{aligned} \langle J(W_\lambda)^2 X_{i\lambda}, X_{i\lambda} \rangle &= -|\lambda|^2 \\ \Longrightarrow -\langle J(W_\lambda) X_{i\lambda}, J(W_\lambda) X_{i\lambda} \rangle &= -|\lambda|^2 \\ \Longrightarrow J(W_\lambda) X_{i\lambda} &= |\lambda| Y_{i\lambda}, \end{aligned}$$

where the last implication follows because $X_{i\lambda}$ is a scalar multiple of $u_{i\lambda}$ (and because of the equation (10.2) which connects b_λ and $J(W_\lambda)$). □

In fact, the converse of this lemma is also true; if such a basis of $\mathfrak{z}$ exists for any nonzero $\lambda \in \mathfrak{n}^*$, then $\mathfrak{n}$ is H-type (see [33]).

Corollary 10.4. *For any $\mathfrak{n}, \lambda$ as above, the basis $X_{j\lambda}, Y_{j\lambda}$ of $\mathfrak{v}$ satisfies*

$$[X_{j\lambda}, Y_{j\lambda}] = \frac{1}{|\lambda|} \sum_{q=1}^{l} \lambda_q W_q.$$

Proof. Define W_λ as before. As noted in the proof of the preceding lemma, we have that $J(W_\lambda) X_{j\lambda} = |\lambda| Y_{j\lambda}$. But from equation (10.1), we have that for any $p = 1, \ldots, l$:

$$\begin{aligned} \langle J(W_p) X_{j\lambda}, J(W_\lambda) X_{j\lambda} \rangle &= \langle W_p, W_\lambda \rangle \\ \Longrightarrow \langle J(W_p) X_{j\lambda}, |\lambda| Y_{j\lambda} \rangle &= \lambda_p \\ \Longrightarrow |\lambda| \langle W_p, [X_{j\lambda}, Y_{j\lambda}] \rangle &= \lambda_p \end{aligned}$$

But this is true for all p, so the result follows. □

Remark 10.5. This corollary says nothing about other commutation relations, such as $[X_{j\lambda}, X_{k\lambda}]$; indeed, the only groups for which all other commutation relations vanish are the Heisenberg groups.

Definition 10.6. For any $\lambda \in \mathfrak{n}^*/\{0\}$, we define $Z_{j\lambda}$ and $Z_{\bar{j}\lambda}$ to be the elements of $u(\mathfrak{n})$ given by:

$$Z_{j\lambda} := 2^{-1/2}(X_{j\lambda} - iY_{j\lambda}), \quad Z_{\bar{j}\lambda} := 2^{-1/2}(X_{j\lambda} - iY_{j\lambda}).$$

The commutation relations of these elements are

$$[Z_{j\lambda}, Z_{\bar{j}\lambda}] = i|\lambda|^{-1} \sum_{q=1}^{l} \lambda_q W_q,$$

from Corollary 10.4. Thus $\bar{\pi}_\lambda([Z_{j\lambda}, Z_{\bar{j}\lambda}]) = |\lambda|$.

We can also think of $Z_{j\lambda}, Z_{\bar{j}\lambda}$ as complex left-invariant vector fields acting on N. With respect to them, we can write

$$\triangle_0(\lambda) = -\sum_{j=1}^{n} \left(Z_{j\lambda} Z_{\bar{j}\lambda} + Z_{\bar{j}\lambda} Z_{j\lambda} \right) - \sum_{q=1}^{l} W_q^2.$$

In particular,

$$[\triangle_0(\lambda), \bar{\pi}_\lambda(Z_{j\lambda})] = 2|\lambda| \bar{\pi}_\lambda(Z_{j\lambda}), \quad [\triangle_0(\lambda), \bar{\pi}_\lambda(Z_{\bar{j}\lambda})] = -2|\lambda| \bar{\pi}_\lambda(Z_{\bar{j}\lambda})$$

so that $\bar{\pi}_\lambda(Z_{j\lambda}), \bar{\pi}_\lambda(Z_{\bar{j}\lambda})$ act as raising and lowering operators with respect to $\triangle_0(\lambda)$.

Creation and annihilation operators. Creation and annihilation operators and a complete basis for $\mathcal{H}_\lambda$ can now be defined, analogously to their definition for $\mathcal{F}_n^k$.

For any j between 1 and n, let a_j, a_j^* be the operators on $\mathcal{H}_{-\lambda}$ defined by:

$$a_j := \sqrt{-1} |\lambda|^{-1/2} \bar{\pi}_\lambda(Z_{\bar{j}\lambda}), \quad a_j^* := \sqrt{-1} |\lambda|^{-1/2} \bar{\pi}_\lambda(Z_{j\lambda}).$$

Then $[a_j, a_j^*] = \mathrm{Id}$. We call a_j an annhilation operator and a_j^* a creation operator.

For any λ, choose an element $v \in \mathcal{H}_{-\lambda}$ which is in the kernel of a_j for all $j = 1, \ldots, n$. (This element is unique up to scalar multiples, otherwise the following construction would give a subspace of $\mathcal{H}_\lambda$ which was $\bar{\pi}_\lambda$-invariant; but this is impossible since $\bar{\pi}_\lambda$ is an irreducible representation.) Define $\psi_0(\lambda)$ to be $v/\|v\|$.

For any multi-index $\beta \in \mathbb{Z}_+^n$, we define

$$\psi_\beta(\lambda) := \frac{1}{\sqrt{\beta!}} (a^*)^\beta \psi_0(\lambda).$$

Then $\{\psi_\beta(\lambda)\}_{\beta \in \mathbb{Z}_+^n}$ is a complete basis of $\mathcal{H}_\lambda$ - otherwise, again, it would be the basis for a closed, $\bar{\pi}_\lambda$-invariant subspace of $\mathcal{H}_\lambda$.

This leads to an explicit realisation of the representation $\bar{\pi}_\lambda$, with representation space $\mathcal{F}^{-\lambda}$, the generalized anti-Fock space (i.e. the conjugate to the generalized Fock space [30]) - see [33].

An explicit formula for the Laplacian. For H-type groups, the formulae for the Laplacian in particular simplifies, so that we have

$$
\begin{aligned}
(10.3)\, \triangle_1(\lambda) \quad = \quad & |\lambda|^2 + n|\lambda| + \sum_{j=1}^{n} \Big(2|\lambda| a_j^* a_j + |\lambda| (i(Z_j) e(\tau^j) + e(\tau^{\bar{j}}) i(Z_{\bar{j}})\Big) \\
& + \sum_{i,j=1}^{n} \sum_{q=1}^{l} \Big(\bar{\pi}_\lambda(Z_{\bar{j}\lambda}) C^q_{i,j} e(\tau^i) + \bar{\pi}_\lambda(Z_{\bar{j}\lambda}) C^q_{i+n,j} e(\tau^{\bar{i}}) \\
& + \bar{\pi}_\lambda(Z_{j\lambda}) C^q_{i,j+n} e(\tau^i) + \bar{\pi}_\lambda(Z_{j\lambda}) C^q_{i+n,j+n} e(\tau^{\bar{i}}) \Big) \, i(W_q) \\
& + \big(\bar{\pi}_\lambda(Z_{j\lambda}) \bar{C}^q_{i,j} i(Z_i) + \bar{\pi}_\lambda(Z_{j\lambda}) \bar{C}^q_{i+n,j} i(Z_{\bar{i}}) \\
& + \bar{\pi}_\lambda(Z_{\bar{j}\lambda}) C^q_{i,j+n} i(Z_i) + \bar{\pi}_\lambda(Z_{\bar{j}\lambda}) C^q_{i+n,j+n} i(Z_{\bar{i}})) e(\tau^{w_q}) \Big)
\end{aligned}
$$

There are similarities with the formula for the Laplacian on p-forms on the Heisenberg group (4.2), but the middle terms (involving $C^q_{i,j}$ and so on) are rather different. Nevertheless, we can list some of the eigenvalues of this Laplacian in a representation, using these similarities.

Lemma 10.7. *For any multi-index $\beta \in \mathbb{Z}^n_+$, we define the following elements of $\mathcal{H}_{-\lambda} \otimes \mathfrak{n}^*$:*

$$
v_1 := \sum_{j=1}^{n} \sqrt{\beta_j + 1}\, \psi_{\beta + e_j}(\lambda) \tau^j, v_2 := \sum_{j=1}^{n} \sqrt{\beta_j}\, \psi_{\beta - e_j}(\lambda) \tau^{\bar{j}}, v_3 := \sum_{q=1}^{l} \lambda_p \psi_\beta(\lambda) \tau^{w_p}.
$$

Then $\{v_1, v_2, v_3\}$ span a $\triangle_1(\lambda)$-invariant subspace of $\mathcal{H}_{-\lambda} \otimes \mathfrak{n}^*$. Further, with respect to these elements, $\triangle_1(\lambda)$ has matrix

$$
\triangle_1(\lambda) = (|\lambda|(2|\beta| + n) + |\lambda|^2) Id + \begin{pmatrix} |\lambda| & 0 & -|\lambda|^{3/2} \\ 0 & -|\lambda| & |\lambda|^{3/2} \\ -|\lambda|^{-1/2}(|\beta| + n) & |\lambda|^{-1/2}|\beta| & n \end{pmatrix}
$$

and eigenvalues

$$
\left\{ |\lambda|(2|\beta| + n) + |\lambda|^2, |\lambda|(2|\beta| + n) + |\lambda|^2 + \frac{n}{2} \pm \sqrt{\frac{n^2}{4} + |\lambda|(2|\beta| + n) + |\lambda|^2} \right\}
$$

The proof is by computation, using the formula 10.3 for $\triangle_1(\lambda)$. The matrix described in this lemma would be self-adjoint if the 1-forms v_1, v_2, v_3 were correctly normalized. Further, the first of the above eigenvalues comes from the action of d on functions (i.e. the corresponding eigenvector is in Imd), but the other two do not.

Note the similarities between this lemma and Lemma 6.8 (ii) for $q = n$; in fact, if we set $c = 1$, and identify k with $|\lambda|$ and $|\gamma|$ with $|\beta|$, the eigenvalues agree exactly.

Symmetry operators on H-type groups. In fact, H-type groups are easily classified. The following result was noted by Kaplan in [19].

Theorem 10.8. *The map $J : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ extends to a representation of the Clifford algebra of $\mathfrak{z}$.*

That is, $\mathfrak{v}$ is a Clifford module over $Cl(\mathfrak{z})$.

Recall that the Clifford algebra associated to a vector space V and quadratic form q, denoted $Cl(V, q)$, is generated by elements of V. For $v \in V$, we write $C(v)$ for Clifford multiplication by v (i.e. the corresponding element in $Cl(V, q)$). Then $C(v)C(w)+C(w)C(v) = -2q(v, w)$. For more on Clifford algebras, see for example [21].

An explicit example of the structure of H-type groups, as related to Clifford modules, is found in [7], where the cases $\dim \mathfrak{z} = 1, 3$ and 7 are discussed.

Now it is also well-known (see for example [21]) that for any finite-dimensional vector space V, if $\dim V \cong 3 \pmod 4$, then $Cl(V)$ has two non-isomorphic irreducible representations, and that otherwise it only has one. (All of these representations are finite-dimensional.)

We consider first the case that dim $\mathfrak{z}$ is not congruent to 3, mod 4. Let M be the unique (up to isomorphism) irreducible module for $Cl(\mathfrak{z})$, and let m be its dimension. Then

$$v \cong M_1 \oplus \cdots \oplus M_r$$

for some r, where each M_i is a copy of M. In particular, the action of $Cl(\mathfrak{z})$ is the same on each M_i. That is, we can find a basis $\{X_j\}_{j=1}^{mr}$ of $\mathfrak{v}$ (where $X_j \in M_i$ iff $m(i-1) < j \leq mi$), such that

$$[X_{mi+j}, X_{mi+l}] = [X_j, X_l] \tag{10.4}$$

for all $j, l = 1, \ldots, m$, and for all $i = 1, \ldots, r-1$. The structure constants are similarly related: for all i, j, l, q in the appropriate sets, we have that $C^q_{mi+j,mi+l} = C^q_{j,l}$.

So whenever dim $\mathfrak{z}$ is not congruent to 3 mod 4, we have the following definition.

Definition 10.9. The (i, j) symmetry operator on $\mathfrak{n}$, χ'_{ij}, is defined for all $1 \leq i < j \leq r$ by the following rules: χ'_{ij} is linear, χ'_{ij} maps $X_{m(i-1)+l}$ to $X_{m(j-1)+l}$ and $X_{m(j-1)+l}$ to $X_{m(i-1)+l}$ for $l = 1, \ldots, m$, and χ'_{ij} is the identity on the complement of $M_i \oplus M_j$.

(For example, if we think of $\mathfrak{v}$ as M^r, then $\chi'_{12}(v_1, v_2, \ldots, v_r)$ would just be $(v_2, v_1, \ldots, v_r)$.)

If $\dim \mathfrak{z} \cong 3 \pmod 4$, then let U and V be the non-isomorphic irreducible modules for $Cl(\mathfrak{z})$. The complement $\mathfrak{v}$ must be isomorphic to $U_1 \oplus \cdots \oplus U_r \oplus V_{r+1} \oplus \cdots \oplus V_{r+s}$, for some r, s, where $\mathfrak{z}$ acts on each U_i as on U and on each V_j as on V.

In this case, we define χ'_{ij} only for $1 \leq i < j \leq r$ or for $r+1 \leq i < j \leq r+s$, but the rest of the definition is the same.

Lemma 10.10. *With notation as above, whenever χ'_{ij} is defined, it is a Lie algebra isomorphism.*

The proof is trivial, given (10.4).

11. The "Double" Heisenberg Group

In this last section, we investigate a particular class of H-type groups: those with 2-dimensional centre. We know from the above classification that there will be at most one such group of a given dimension (up to isomorphism); in fact, since the irreducible modules of $Cl(\mathbb{R}^2)$ are 4-dimensional, a H-type group with 2-dimensional centre must have dimension $4n + 2$, for some positive integer n.

Definition 11.1. Let D^{4n+2} denote the "double" Heisenberg group of dimension $4n+2$. That is, the Lie algebra $\mathfrak{d}^{4n+2}$ of D^{4n+2} has basis $\{X_1, \ldots, X_{4n}, W_1, W_2\}$ and non-zero commutation relations defined by

$$\begin{aligned}
[X_{4j+1}, X_{4j+3}] &= W_1, \qquad [X_{4j+1}, X_{4j+4}] = W_2, \\
[X_{4j+2}, X_{4j+3}] &= W_2, \qquad [X_{4j+2}, X_{4j+4}] = -W_1
\end{aligned}$$

for $j = 0, \ldots, n-1$.

(We also write D instead of D^{4n+2} and $\mathfrak{d}$ instead of $\mathfrak{d}^{4n+2}$ when the dimension is understood.)

Raising and lowering operators. We begin by defining the raising and lowering operators for D^6, then indicate how to generalise to D^{4n+2}. We fix a non-zero linear functional $\lambda \in \mathfrak{z}^*$ throughout.

We find $Z_{1\lambda}, \ldots, Z_{\bar{2}\lambda}$ as indicated in Lemma 10.3; they are given by

$$\begin{aligned}
Z_{1\lambda} &:= (\sqrt{2}|\lambda|)^{-1}(i\lambda_1 X_1 + i\lambda_2 X_2 + |\lambda| X_3), \\
Z_{\bar{1}\lambda} &:= (\sqrt{2}|\lambda|)^{-1}(-i\lambda_1 X_1 - i\lambda_2 X_2 + |\lambda| X_3), \\
Z_{2\lambda} &:= (\sqrt{2}|\lambda|)^{-1}(i\lambda_2 X_1 - i\lambda_1 X_2 + |\lambda| X_4), \\
Z_{\bar{2}\lambda} &:= (\sqrt{2}|\lambda|)^{-1}(-i\lambda_1 X_1 + i\lambda_2 X_2 + |\lambda| X_4).
\end{aligned}$$

These elements of $u(\mathfrak{d})$ have the following non-zero commutation relations:

$$\begin{aligned}
[Z_{1\lambda}, Z_{\bar{1}\lambda}] &= i(|\lambda|)^{-1}(\lambda_1 W_1 + \lambda_2 W_2) = [Z_{2\lambda}, Z_{\bar{2}\lambda}], \\
[Z_{1\lambda}, Z_{2\lambda}] &= i(|\lambda|)^{-1}(-\lambda_2 W_1 + \lambda_1 W_2) = -[Z_{\bar{1}\lambda}, Z_{\bar{2}\lambda}].
\end{aligned}$$

But again, since D is a H-type group, in the representation $\bar{\pi}_\lambda$ we have:

$$\begin{aligned}
\bar{\pi}_\lambda([Z_{1\lambda}, Z_{\bar{1}\lambda}]) &= |\lambda| = \bar{\pi}_\lambda([Z_{2\lambda}, Z_{\bar{2}\lambda}]), \\
\bar{\pi}_\lambda([Z_{1\lambda}, Z_{2\lambda}]) &= 0 = \bar{\pi}_\lambda([Z_{\bar{1}\lambda}, Z_{\bar{2}\lambda}]).
\end{aligned}$$

For $n \geq 2$, the remaining Z_j's and $Z_{\bar{j}}$'s are defined analogously; for example,

$$Z_{3\lambda} := (\sqrt{2}|\lambda|)^{-1}(i\lambda_1 X_5 + i\lambda_2 X_6 + |\lambda| X_7).$$

As we know, the commutation relations also carry over unchanged.

We again write $\{\tau^1, \tau^{\bar{1}}, \ldots, \tau^{2n}, \tau^{\overline{2n}}\}$ for the dual basis corresponding to $\{Z_{1\lambda}, \ldots, Z_{\overline{2n}\lambda}\}$.

We define the creation and annihilation operators on $\mathcal{H}_{-\lambda}$ as we did for all H-type groups:

$$a_j = \sqrt{-1}|\lambda|^{-1/2}\pi_\lambda(Z_{j\lambda}), \quad a_j^* = \sqrt{-1}|\lambda|^{-1/2}\pi_\lambda(Z_{\bar{j}\lambda}).$$

Commuting operators. For D^{4n+2}, the symmetry operators χ'_{ij} are easily defined; for example, χ'_{12} interchanges X_1 and X_5, X_2 and X_6, and so on, or equivalently, $Z_{1\lambda}$ and $Z_{3\lambda}$, $Z_{2\lambda}$ and $Z_{4\lambda}$, $Z_{\bar{1}\lambda}$ and $Z_{\bar{3}\lambda}$, and $Z_{\bar{2}\lambda}$ and $Z_{\bar{4}\lambda}$ are interchanged.

We can define transposition operators U_{ij} as for the Heisenberg group, in terms of a_j and $e(\tau^j)$, but they do not commute with the Laplacian on 1-forms, $\triangle_1(\lambda)$. Instead, if $n \geq 2$, $[\triangle_1(\lambda), U_{13} - U_{42}] = 0 = [\triangle_1(\lambda), U_{31} - U_{24}]$; and I conjecture that $\triangle_1(\lambda)$ also commutes with $U_{23} - U_{41}$ and thus with $U_{32} - U_{14}$.

The Laplacian on 1-forms does not even commute with U_{11} or U_{22}, but instead with $U_{11} - U_{22}$, so that there is no corresponding subspace $V^{1,n,\gamma}$, but instead two disjoint subspaces, as we'll see shortly.

Use could be made of these operators in some way, but the situation is somewhat more complicated than for the Heisenberg group - due primarily to the extra non-zero commutation relations.

Some eigenvalues of the Laplacian on 1-forms. For any multi-index $\beta \in \mathbb{Z}_+^n$ with all indices positive, define the following 1-forms for $j = 1, \ldots, n$:

$$u_j := (a_{2j} e(\tau^{2j-1}) - a_{2j-1} e(\tau^{2j}))\psi_\beta(\lambda), v_j := (a_{2j}^* e(\tau^{\overline{2j-1}}) - a_{2j-1}^* e(\tau^{\overline{2j}}))\psi_\beta(\lambda),$$
$$w_j := (a_{2j-1}^* e(\tau^{2j-1}) + a_{2j}^* e(\tau^{2j}))\psi_\beta(\lambda), w_j' := (a_{2j-1} e(\tau^{\overline{2j-1}}) + a_{2j} e(\tau^{\overline{2j}}))\psi_\beta(\lambda).$$

Define also the number $\mu' := |\lambda|(2n + 2|\beta|) + |\lambda|^2$.

Theorem 11.2. *If $n \geq 2$, then for any multi-index $\beta \in \mathbb{Z}_+^n$, the Laplacian $\triangle_1(\lambda)$ in the representation $\bar{\pi}_\lambda$ acting on 1-forms on D^{4n+2} has eigenvalues including $\{\mu' - 3|\lambda|, \mu' + 3|\lambda|, \mu' + |\lambda|, \mu' - |\lambda|\}$, each with multiplicity $n-1$. The corresponding eigenvectors are, respectively, $\{(\beta_{2j+1} + \beta_{2j+2})u_j - (\beta_{2j-1} + \beta_{2j})u_{j+1}, (\beta_{2j+1} + \beta_{2j+2})v_j - (\beta_{2j-1} + \beta_{2j})v_{j+1}, (\beta_{2j+1} + \beta_{2j+2})w_j - (\beta_{2j-1} + \beta_{2j})w_{j+1}, (\beta_{2j+1} + \beta_{2j+2})w_j' - (\beta_{2j-1} + \beta_{2j})w_{j+1}'\}$, for $j = 1, \ldots, n-1$. There are also two $\triangle_1(\lambda)$-invariant subspaces with bases*

$$\left\{ \sum_{j=1}^n u_j,\ \sum_{j=1}^n v_j,\ \lambda_2 \tau^{w_1} - \lambda_1 \tau^{w_2} \right\}, \left\{ \sum_{j=1}^n w_j, \sum_{j=1}^n w_j',\ \lambda_1 \tau^{w_1} + \lambda_2 \tau^{w_2} \right\}.$$

With respect to these bases, $\triangle_1(\lambda)$ has matrices

$$(2|\lambda|(n + |\beta|) + |\lambda|^2) Id + \begin{pmatrix} -3|\lambda| & 0 & -|\lambda|^{3/2} \\ 0 & 3|\lambda| & |\lambda|^{3/2} \\ -|\lambda|^{-1/2}|\beta| & -|\lambda|^{-1/2}(|\beta| + 2n) & 2n \end{pmatrix}$$

and

$$(2|\lambda|(n + |\beta|) + |\lambda|^2) Id + \begin{pmatrix} |\lambda| & 0 & -|\lambda|^{3/2} \\ 0 & -|\lambda| & |\lambda|^{3/2} \\ -|\lambda|^{-1/2}(|\beta| + 2n) & -|\lambda|^{-1/2}|\beta| & 2n \end{pmatrix}.$$

On the second subspace, $\triangle_1(\lambda)$ has eigenvalues

$$\left\{ 2|\lambda|(n + |\beta|) + |\lambda|^2, 2|\lambda|(n + |\beta|) + |\lambda|^2 + n \pm \sqrt{n^2 + 2|\lambda|(n + |\beta|) + |\lambda|^2)} \right\}.$$

Define functions $\mu_{low}(b, n)$ and $\mu_{high}(b, n)$ for positive integers b, n as follows:

$$\mu_{low}(b, n) := -\left(\frac{b + n + \sqrt{(b + n)^2 + 24n^2}}{2n} \right) |\lambda|, \quad \mu_{high}(b, n) := -3|\lambda|$$

Then the lowest eigenvalue μ_0 of the first of the above matrices is bounded by

$$\mu_{low}(|\beta|, n) + 2(|\beta| + n)|\lambda| + |\lambda|^2 < \mu_0 < \mu_{high}(|\beta|, n) + 2(|\beta| + n)|\lambda| + |\lambda|^2,$$

for $|\lambda| > 0$, $|\beta|, n \geq 1$, while the other two eigenvalues are greater than $2(|\beta| + n)|\lambda| + |\lambda|^2$.

Proof. Most of the proof consists of tedious calculations, using either of the formulae (9.2) or (10.3).

The eigenvalues of the first matrix in the theorem are worth discussing in some detail, since they come from a cubic which is decidedly non-trivial to solve.

Let $p(\mu)$ be the characteristic polynomial of this matrix (minus the constant term $2(|\beta|+n)|\lambda|+|\lambda|^2$). That is,

$$p(\mu)=\mu^3-2n\mu^2-|\lambda|(2|\beta|+9|\lambda|+2n)\mu+12n|\lambda|^2.$$

Then we can approximate its zeros (i.e. the eigenvalues of the matrix) if we know where it is positive and negative. Calculations (for example, using a computer package such as Maple) give that $p(\mu_{low}(|\beta|,n))=\mu_{low}(|\beta|,n)^3-9\mu_{low}(|\beta|,n)|\lambda|^2$ is negative (since $|\mu_{low}(|\beta|,n)|>3|\lambda|$)), while $p(\mu_{high}(|\beta|,n))=6|\beta||\lambda|^2$ is positive. Further, $p(0)$ is positive, while $p'(\mu)$ has a positive zero, indicating (by standard results in calculus) that the other two zeros of $p(\mu)$ are both positive. □

We briefly discuss special cases, i.e. what happens when some or all of the indices β_i are zero.

If β_{2j-1}, say, is zero, then u_j and w'_j both simplify; but if $\beta_{2j-1}=0=\beta_{2j}$, then u_j and w'_j are also zero. In particular, if $\beta=0$, then every u_j and every w'_j are zero; also, $|\beta|$ must be greater than or equal to 2 in order to have an eigenvector of the form $(\beta_{2l-1}+\beta_{2l})u_j-(\beta_{2j-1}+\beta_{2j})u_l$ (since one of β_{2l-1},β_{2l} must be non-zero, and one of β_{2j-1},β_{2j} must be non-zero).

The case $\beta=0$ has to be considered separately, but it can be shown that in this case, all eigenvalues are greater than $2n|\lambda|+|\lambda|^2$.

This motivates the following result.

Corollary 11.3. *The lowest eigenvalue of the Laplacian on 1-forms on D^{4n+2} in the representation π_λ, $\triangle_{1,n}(\lambda)$, has multiplicity 1 for all n,λ, and lies between $\left(2(n+1)-\frac{n+1+\sqrt{(n+1)^2+24n^2}}{2n}\right)|\lambda|+|\lambda|^2$ and $(2n-1)|\lambda|+|\lambda|^2$. Further, the coefficient of $|\lambda|$ in the lower bound is positive.*

Proof. For fixed β, the lowest eigenvalue on the first subspace in Theorem 11.2 is between

$$2(|\beta|+n)|\lambda|+|\lambda|^2+\mu_{low}(|\beta|,n)\text{ and }2(|\beta|+n)|\lambda|+|\lambda|^2+\mu_{high}(|\beta|,n).$$

Both $\mu_{low}(|\beta|,n)$ and $\mu_{high}(|\beta|,n)$ are increasing as $|\beta|$ increases; in particular, $\mu_{low}(2,n)>\mu_{high}(1,n)\quad\forall n\geq 1$.

All other eigenvalues are also greater than $\mu_{high}(1,n)+2(n+1)|\lambda|+|\lambda|^2$; in particular, the lowest of the other eigenvalues (coming from $(\beta_{2l-1}+\beta_{2l})u_j-(\beta_{2j-1}+\beta_{2j})u_l$) is $(2n+1)|\lambda|+|\lambda|^2$, and the lowest eigenvalue on the second subspace is $2n|\lambda|+|\lambda|^2$. So the lowest eigenvalue of $\triangle_1(\lambda)$ is between $\mu_{low}(1,n)+2(n+1)|\lambda|+|\lambda|^2$, and $\mu_{high}(1,n)+2(n+1)|\lambda|+|\lambda|^2$.

That the coefficient of $|\lambda|$ is positive follows from more calculations. For $n=1$, the value of $\mu_{low}(1,1)+4|\lambda|$ is exactly $(3-\sqrt{7})|\lambda|$; for $n>1$, we use the fact that $\sqrt{(n+1)^2+24n^2}$ is less than $5n+1$ to derive the estimate: $\mu_{low}(1,n)+(2n+2)|\lambda|$ is greater than $(2n-1-\frac{1}{n})|\lambda|$, which is positive. □

Corollary 11.4. *For any $n\geq 1$, the first Novikov-Shubin invariant of D^{4n+2} is given by*

$$\alpha_1(D^{4n+2})=2n+2=\alpha_0(D^{4n+2}).$$

Proof. From Corollary 11.3, we have an estimate for the lowest eigenvalue of $\triangle_{1,n}(\lambda)$, which has multiplicity of one for all n and λ. As in section 4.4, we can now calculate the eigenvalues; most of the procedure of that section still holds here. The result depends on the decay of the following integral:

$$\int_{\mathbb{R}^2} e^{-T(a|\lambda|+f(|\lambda|).|\lambda|^2)}|\lambda|^{2n}d\lambda_1 d\lambda_2$$

for a positive and $f(x)$ a positive power series. We can rewrite this integral in polar coordinates; it becomes

$$\int_0^{2\pi}\int_0^{\infty} e^{-T(ar+f(r).r^2)}r^{2n+1}drd\theta,$$

which (again using an equation from) evaluates to $2\pi\left(\frac{1}{aT}\right)^{2n+2}+O(T^{-2n-3})$. □

Appendix A. An explicit formula for the Laplacian

The formula for the Laplacian in section 3.1 was given without proof - though it was indicated how the explicit formulae for d and d^* could be proved. Here we derive the formula for the Laplacian, given those for d and d^*.

First, we need to review some properties of the operators $e(*), i(*)$.

Let U, V be vectors selected from the basis $\{Z_1,\dots,Z_n,Z_{\bar{1}},\dots,Z_{\bar{n}},W\}$. Let τ^U,τ^V be the corresponding elements of the dual basis. Then we have the following properties:

$$\{e(\tau^U), i(V)\} = \langle U,V\rangle \tag{A.1}$$

$$\{e(\tau^U), e(\tau^V)\} = 0 = \{i(U), i(V)\} \tag{A.2}$$

$$e(\tau^V) = [i(V)]^* \tag{A.3}$$

where $\{.,.\}$ is the anti-commutator, $\{A,B\} := AB+BA$.

We also note that vector fields such as Z_j, which operate only on functions, commute with the operators $e(\tau)$ and $i(V)$ for all τ, V in the above orthonormal bases.

Finally, it can be shown that the adjoint of Z_j is $-Z_{\bar{j}}$ and the adjoint of W is $-W$.

Recall from section 3.1 that

$$\begin{aligned} d &= \sum_{j=1}^{n}\left(e(\tau^j)Z_j + e(\tau^{\bar{j}})Z_{\bar{j}}\right) + e(\tau^w)W - i\sum_{j=1}^{n} e(\tau^j)e(\tau^{\bar{j}})i(W) \\ d^* &= -\sum_{j=1}^{n}\left(i(Z_{\bar{j}})Z_j + i(Z_j)Z_{\bar{j}}\right) - i(W) + i\sum_{j=1}^{n} e(\tau^w)i(Z_{\bar{j}})i(Z_j) \end{aligned}$$

We define, as before, the operators θ_j for $j=1,\dots,n$, by

$$\theta_j = e(\tau^j)Z_j + e(\tau^{\bar{j}})Z_{\bar{j}} - ie(\tau^j)e(\tau^{\bar{j}})i(W)$$

so that $d = e(\tau^w)W + \sum_{j=1}^{n}\theta_j$. We can then define operators $\eta_{j,l}$ and A_j for $j \neq l$ and $j,l = 1,\dots,n$:

$$\eta_{j,l} := \theta_j\theta_l^* + \theta_l^*\theta_j, A_j := \theta_j\theta_j^* + \theta_j^*\theta_j;$$

with these definitions, we can write

$$\triangle_{p,n} = \sum_{j=1}^{n} A_j + \sum_{j\neq k}\eta_{j,k} - W^2.$$

We now calculate A_j and $\eta_{j,l}$.

Firstly,

$$\begin{aligned}
A_j &= \theta_j\theta_j^* + \theta_j^*\theta_j \\
&= \{e(\tau^j)Z_j + e(\tau^{\bar{j}})Z_{\bar{j}} - ie(\tau^j)e(\tau^{\bar{j}})i(W), -i(Z_{\bar{j}})Z_j - i(Z_j)Z_{\bar{j}} \\
&\quad +ie(\tau^w)i(Z_{\bar{j}})i(Z_j)\} \\
&= \{e(\tau^j)Z_j, -i(Z_j)Z_{\bar{j}}\} + ie(\tau^w)i(Z_{\bar{j}})Z_j \\
&\quad +\{e(\tau^{\bar{j}})Z_{\bar{j}}, -i(Z_{\bar{j}})Z_j\} - ie(\tau^w)i(Z_j)Z_{\bar{j}} \\
&\quad -ie(\tau^j)i(W)Z_j + ice(\tau^{\bar{j}})i(W)Z_{\bar{j}} \\
&\quad +e(\tau^j)i(Z_j)e(\tau^{\bar{j}})i(Z_{\bar{j}})i(W)e(\tau^w) + i(Z_j)e(\tau^j)i(Z_{\bar{j}})e(\tau^{\bar{j}})e(\tau^w)i(W)) \\
&= -2Z_jZ_{\bar{j}} + iW\left(i(Z_j)e(\tau^j) + e(\tau^{\bar{j}})i(Z_{\bar{j}})\right) \\
&\quad +ie(\tau^w)\left(i(Z_{\bar{j}})Z_j - i(Z_j)Z_{\bar{j}}\right) + i\left(e(\tau^{\bar{j}})Z_{\bar{j}} - e(\tau^j)Z_j\right)i(W) \\
&\quad +e(\tau^j)i(Z_j)e(\tau^{\bar{j}})i(Z_{\bar{j}})i(W)e(\tau^w) + i(Z_j)e(\tau^j)i(Z_{\bar{j}})e(\tau^{\bar{j}})e(\tau^w)i(W))
\end{aligned}$$

More simply,

$$\begin{aligned}
\eta_{j,l} &= \theta_j\theta_l^* + \theta_l^*\theta_j \\
&= \{-ice(\tau^j)e(\tau^{\bar{j}})i(W), ice(\tau^w)i(Z_{\bar{l}})i(Z_l)\} \\
&= c^2e(\tau^j)e(\tau^{\bar{j}})i(Z_{\bar{l}})i(Z_l).
\end{aligned}$$

Summing these expressions gives the required formula for the Laplacian.

In fact, the equations (A.1)—(A.3), together with the commutation relations of the Lie group in question, can be used to define a Lie superalgebra. This theme is developed somewhat in [33]; for more on Lie superalgebras and their connection with d and the Laplacian, see also [32, 36].

Appendix B. Proof of the Kernel Lemma

To prove: if $v \in \ker U_{12} \cap V^{p,n,\gamma}$, then $\gamma_2 \leq 1$.

Proof. Recall that $U_{12} = a_1^*a_2 - e(\tau^2)i(Z_1) + e(\tau^{\bar{1}})i(Z_{\bar{2}})$. Suppose $v \in \ker U_{12}$. Then in particular $e(\tau^2)e(\tau^{\bar{1}})U_{12}v = 0$ which implies that $e(\tau^2)e(\tau^{\bar{1}})a_1^*a_2v = 0$.

Write v in the form

$$v = \tau^{\bar{1}} \wedge v_1 + \tau^2 \wedge v_2 + \tau^{\bar{1}} \wedge \tau^2 \wedge v_3 + v_4,$$

for v_i forms such that $i(Z_{\bar{1}})v_i = 0 = i(Z_2)v_i$ for $i = 1, \ldots, 4$. Then we've just shown above that $v_4 \in \ker a_2$.

If we apply U_{12} to v and equate coefficients of terms with τ^2 and so on, we get the following equations (since $v \in \ker U_{12}$):

$$a_1^*a_2v_3 + i(Z_1)v_1 - i(Z_{\bar{2}})v_2 = 0, \tag{B.1}$$

$$a_1^*a_2v_2 - i(Z_1)v_4 = 0, \tag{B.2}$$

$$a_1^*a_2v_1 + i(Z_{\bar{2}})v_4 = 0. \tag{B.3}$$

But we know that $v_4 \in \ker a_2$. Equations (B.2) and (B.3) then imply that v_2 and v_1 respectively are in $\ker a_2^2$ (even if $v_4 = 0$). From equation (B.1), we see that v_3 is in $\ker a_2^3$. Actually, equation (B.3) also implies that v_1 is in $\ker(i(Z_{\bar{2}})a_2)$, which together with equation (B.1) implies that $v_3 \in \ker(i(Z_{\bar{2}})a_2^2)$.

If we now require that $v \in V^{p,n,\gamma}$ (and recall that for functions, if $\psi_\beta(k) \in \ker a_2^3$, then $\beta_2 \leq 2$) then the conditions that $\tau^{\bar{1}} \wedge \tau^2 \wedge v_3 \in V^{p,n,\gamma}$ and $v_3 \in \ker a_2^3 \cap \ker(i(Z_{\bar{2}})a_2^2)$ together imply that $\gamma_2 \leq 1$, if $v_3 \neq 0$. Similarly the conditions on v_1, v_2 and v_4 imply that $\gamma_2 \leq 1$, so that the result holds even if one or more of the v_i's is 0. □

References

1. M. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Astérisque **32** (1976), 43–72.
2. M. Atiyah and W. Schmid, *A geometric construction of the discrete series for semisimple Lie groups*, Invent. math. **42** (1977), 1–62.
3. J. Block, V. Mathai, and S. Weinberger, *Homotopy invariance of Novikov-Shubin invariants and L^2 Betti numbers*, To appear in Proc. Amer. Math. Soc., Nov 1997.
4. A. Carey, M. Farber, and V. Mathai, *Determinant lines, von Neumann algebras and L^2 torsion*, J. reine angew. Math. **484** (1997), 153–181.
5. A.L. Carey, T. Coulhon, V. Mathai, and J. Phillips, *Von Neumann spectra near the spectral gap*, To appear in Bull. Sci. Math. (France).
6. L.J. Corwin and F.P Greenleaf, *Representations of nilpotent Lie groups and their applications (part 1)*, Cambridge University Press, Cambridge, England, 1990.
7. M. Cowling, A. H. Dooley, A. Korányi, and F. Ricci, *H-type groups and Iwasawa decompositions*, Advances in Math. **87** (1991), 1–41.
8. J. Dixmier, *C*-algebras*, North-Holland, Amsterdam, 1981, Revised edition; translation of C*-algebres et leurs representations.
9. ———, *Von neumann algebras*, North-Holland, Amsterdam, 1981, Translation of Algebres d'operateurs dans l'espace hilbertien (algebres de Von Neumann).
10. J. Dodziuk, *De Rham-Hodge theory for L^2-cohomology of infinite coverings*, Topology **16** (1977), 157–165.
11. A.V. Efremov, *Cellular decompositions and Novikov-Shubin invariants*, Russ. Math. Surveys **46** (1991), 219–220.
12. M. Farber, *Homological algebra of Novikov-Shubin invariants and Morse inequalities*, Geom. Funct. Anal. **6** (1996), no. 4, 628–665.
13. G.B. Folland, *Harmonic analysis in phase space*, Princeton University Press, Princeton, N.J., 1989.
14. ———, *A course in abstract harmonic analysis*, CRC Press, Boca Raton, c1995.
15. L. Gårding, *Notes on continuous representations of lie groups*, Proc. Nat. Acad. Sci. USA **33** (1947), 331–332.
16. C.S. Gordon and E.N. Wilson, *The spectrum of the Laplacian on Riemannian Heisenberg manifolds*, Mich. Math. J **33** (1986), 253–271.
17. M. Gromov and M.A. Shubin, *Von Neumann spectra near zero*, Geom. Anal. and Funct. Anal. **1** (1991), 375–404.
18. V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge; New York, 1984.
19. A. Kaplan, *On the geometry of Lie groups of Heisenberg type*, Bull. London Math. Soc. **15** (1983), 35–42.
20. A.A. Kirillov, *Elements of the theory of representations*, Springer-Verlag, Berlin; New York, 1976, Translation of Elementy teorii predstavlenii.
21. H.B. Lawson, Jr. and M. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, N.J., 1989.
22. J. Lott, *Heat kernels on covering spaces and topological invariants*, J. Diff. Geom. **35** (1992), 471–510.
23. J. Lott and W. Lück, *L^2-topological invariants of 3-manifolds*, Invent. math. **120** (1995), 15–60.
24. V. Mathai, *Von Neumann algebra invariants of Dirac operators*, To appear in J. of Funct. Anal., 1997.
25. V. Mathai and A. Carey, *L^2-Acyclicity and L^2-torsion invariants*, Contemp. Math. **105** (1990), 91–118.

26. F.I. Mautner, *Unitary representations of locally compact groups*, Annals of Math. **52** (1950), no. 3, 528–556.
27. S. Novikov and M.A. Shubin, *Morse inequalities and von Neumann invariants of non-simply-connected manifolds*, Uspekhi Mat. Nauk **41** (1986), no. 5, 222–223, (Russian).
28. ______, *Morse theory and von Neumann II_1-factors*, Doklady Akad. Nauk SSSR **289** (1986), 289–292.
29. H. Pesce, *Calcul du spectre d'une nilvarièté de rang deux et applications*, Trans. Amer. Math. Soc. **339** (1993), no. 1, 433–461.
30. F. Ricci, *Harmonic analysis on generalized Heisenberg groups*, Preprint.
31. I. Satake, *Linear algebra*, M. Dekker, New York, 1975, Translation of Senkei daisugaku.
32. M. Scheunert, *The theory of Lie superalgebras: an introduction*, Springer-Verlag, Berlin; New York, 1979.
33. L. Schubert, *Spectral properties of the Laplacian on p-forms on the Heisenberg group*, Ph.D. thesis, The University of Adelaide, 1997.
34. I.M. Segal, *A non-commutative extension of abstract integration*, Ann. Math. **57** (1953), 401–457, Corrections in 58 (1954), pp595–596.
35. M. Spivak, *A comprehensive introduction to differential geometry (volume 1)*, Publish or Perish Inc., Berkeley, 1979.
36. S. Sternberg, *Some recent results on the metaplectic representation*, Group theoretical methods in physics (New York) (P. Kramer and A. Rieckers, eds.), Lecture Notes in Physics, no. 79, Springer-Verlag, 1978.
37. N. Varopoulos, *Random walks and Brownian motion on manifolds*, Sympos. Math. **29** (1988), 97–109.

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