A SEMI-DISCRETE TAILORED FINITE POINT METHOD FOR A CLASS OF ANISOTROPIC DIFFUSION PROBLEMS

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Abstract. This work proposes a tailored finite point method (TFPM) for the numerical solution of an anisotropic diffusion problem, which has much smaller diffusion coefficient along one direction than the other on a rectangle domain. The paper includes analysis on the differentiability of the solution to the given problem under some compatibility conditions. It has detailed derivation for a semi-discrete TFPM for the given problem. This work also proves a uniform error estimate on the approximate solution. Numerical results show the TFPM is accurate as well as efficient for the strongly anisotropic diffusion problem. Examples include those that do not satisfy compatibility and regularity conditions. For the incompatible problems, numerical experiments indicate that the method proposed can still offer good numerical approximations.

1. Introduction

In this paper, we consider the following anisotropic diffusion problem in two-dimensional space,

\[-\varepsilon^2 \frac{\partial^2 u_\varepsilon}{\partial x^2} - \frac{\partial^2 u_\varepsilon}{\partial y^2} + a(x, y) u_\varepsilon = f(x, y), \quad \forall (x, y) \in \Omega, \tag{1.1}\]

\[u_\varepsilon|_{\partial \Omega} = 0, \tag{1.2}\]

with (cf. Fig. 1)

\[\Omega = \{(x, y) \mid 0 < x < 1, \ 0 < y < 1\} \quad \text{and its boundary} \quad \partial \Omega = \Gamma_e \cup \Gamma_w \cup \Gamma_n \cup \Gamma_n.\]

Suppose that the functions \(a(x, y), \ f(x, y)\) satisfy

\[a, \ f \in C^{(4,\alpha)}(\Omega), \quad (\alpha > 0), \tag{1.3}\]

\[a(x, y) \geq 0, \quad \forall (x, y) \in \Omega, \tag{1.4}\]

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and $0 < \varepsilon \leq 1$. Here, $C^{(k,\alpha)}(\Omega)$ stands for the Hölder Space of index $(k,\alpha)$ on $\Omega$ with $k$ be a non-negative integer and $\alpha$ be a positive real number in the interval $(0,1]$. Let $\{p_i\}_{i=1}^4$ be the four corners of the rectangle domain $\Omega$ (see Fig. 1).

When $0 < \varepsilon \ll 1$, problem (1.1)--(1.2) is an anisotropic diffusion problem and its solution may have boundary layers on a portion of $\partial \Omega$, i.e. at $x = 0$ and $x = 1$. These layers are characterized by rapid transitions in the solution and difficult to capture in a numerical approximation without using a large number of unknowns. Also, such layers tend to cause spurious oscillations in a numerical solution to the problem.

The anisotropic diffusion problem appears in a large variety of applications such as image processing [17, 22], chemical reactions [20] and anisotropic materials [15]. It has been numerically solved with asymptotic preserving method [2], least-square finite element method [16], mimetic finite difference methods [13], etc. A similar anisotropic but first-order ordinary differential system is solved with a finite difference method on Shishkin meshes [14].

This work proposes a tailored finite point method (TFPM) [6, 7, 8, 9, 10, 12] to numerically solve problem (1.1)--(1.2). The tailored finite point method was first proposed by Han, Huang and Kellogg [10] for the numerical solutions of singular perturbation problems of second-order elliptic equations with constant coefficients. Later, Han and Huang [6, 7, 8, 9] and Shih, Kellogg et al. [18, 19] systematically develop this method for the nonhomogeneous reaction-diffusion, convection-diffusion and convection-diffusion-reaction problems. The basic idea of the tailored finite point method is that the numerical scheme is tailor-made at each point/cell based on the local properties of the solution of the given problem. The method is expected to capture boundary layers associated with the problem (1.1)--(1.2) even with coarse grids.

The remainder of this paper is organized as follows. Section 2 makes studies on the differentiability of the solution to problem (1.1)--(1.2) under some compatibility conditions. Section 3 derives a semi-discrete approximation of problem (1.1)--(1.2). The details of the TFPM for the semi-discrete and fully-discrete approximations of the problem are described in Section 4. Section 4 gives a proof for the uniform convergence property of the approximate solution. Finally, numerical examples supporting the theory and demonstrating the efficiency and reliability of the method are presented in Section 5.
2. THE DIFFERENTIABILITY OF THE SOLUTION TO PROBLEM (1.1)–(1.2)

The differentiability of \( u_\varepsilon(x, y) \), the solution to problem (1.1)–(1.2), in the open domain \( \Omega \) only depends on the differentiability of the given functions \( a(x, y) \) and \( f(x, y) \). By condition (1.3) and an interior a priori estimate for the elliptic equation (1.1), it is known that [11]

\[
L_\varepsilon \in C^{6,\alpha}(\Omega). \tag{2.1}
\]

But the differentiability of the solution to problem (1.1)–(1.2) on the closed domain \( \bar{\Omega} \) also depends on some compatibility conditions at the corner points (refer to Grisvard [5] and Han & Kellogg [11]). For problem (1.1)–(1.2), suppose that the source term satisfies the following compatibility conditions:

\[
\begin{aligned}
f(\mathbf{p}_l) &= 0, & l &= 1, \ldots, 4, \\
f_{xx}(\mathbf{p}_l) &= 0, & f_{yy}(\mathbf{p}_l) &= 0, & l &= 1, \ldots, 4.
\end{aligned} \tag{2.2}
\]

Theorem 3.2 in Han & Kellogg [11] implies that \( u_\varepsilon(x, y) \in C^{4,\alpha}(\bar{\Omega}) \).

**Remark 2.1.** If the compatibility conditions (2.2) (or part of them) do not hold, then the partial derivatives of the solution \( u_\varepsilon(x, y) \) may not be continuous at four corners [5, 11].

For any function \( v(x, y) \) that is continuous on \( \bar{\Omega} \), denote by \( \|v\|_\infty = \max_{(x,y)\in\bar{\Omega}} |v(x,y)| \) its maximum norm. Let

\[
M_{k_1,k_2} = \left\| \frac{\partial^{k_1+k_2} u_\varepsilon}{\partial x^{k_1} \partial y^{k_2}} \right\|_\infty, \quad k_1 \geq 0, \ k_2 \geq 0, \ k_1 + k_2 \leq 4,
\]

\[
F_{k_1,k_2} = \left\| \frac{\partial^{k_1+k_2} f}{\partial x^{k_1} \partial y^{k_2}} \right\|_\infty, \quad k_1 \geq 0, \ k_2 \geq 0, \ k_1 + k_2 \leq 4,
\]

\[
A_{k_1,k_2} = \left\| \frac{\partial^{k_1+k_2} a}{\partial x^{k_1} \partial y^{k_2}} \right\|_\infty, \quad k_1 \geq 0, \ k_2 \geq 0, \ k_1 + k_2 \leq 4.
\]

and \( M_0 = M_{0,0}, \ A_0 = A_{0,0}, \ F_0 = F_{0,0} \).

**Assumption 2.1.** In the rest of this section, assume that the constants \( F_{k_1,k_2} \) and \( A_{k_1,k_2} \) are bounded uniformly with respect to \( \varepsilon \).

**Lemma 2.1.** There exists a constant \( C \) independent of \( \varepsilon \) such that

\[
M_0 \leq C \equiv \frac{F_0}{8}. \tag{2.3}
\]

**Proof.** This result follows directly from the maximum principle of elliptic problem (1.1)–(1.2). Define the differential operator

\[
L_\varepsilon \equiv -\varepsilon^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + a \tag{2.4}
\]

and introduce an auxiliary function

\[
\varphi(y) = \frac{F_0}{2} y(1 - y),
\]
which depends on variable \( y \) only. It is easy to check that
\[
0 \leq \varphi(y) \leq C = \frac{F_0}{8}, \quad \forall y \in [0, 1],
\]
and
\[
L_\varepsilon \varphi(y) = F_0 + a(x, y) \varphi(y) \geq 0, \quad \forall (x, y) \in \bar{\Omega}.
\]
Let \( v_\pm = u_\varepsilon \pm \varphi \). We have
\[
v_+|_{\partial \Omega} \geq 0, \quad v_-|_{\partial \Omega} \leq 0
\]
and
\[
L_\varepsilon v_\pm(x, y) = f(x, y) \pm L_\varepsilon \varphi(y) = f(x, y) \pm (F_0 + a(x, y) \varphi(y)).
\]
By the definition of \( F_0 \) and the assumption on the coefficient \( a(x, y) \), we get
\[
L_\varepsilon v_+(x, y) \geq 0, \quad L_\varepsilon v_-(x, y) \leq 0, \quad \forall (x, y) \in \Omega.
\]
The maximum principle further implies
\[
v_+(x, y) \geq 0, \quad v_-(x, y) \leq 0, \quad \forall (x, y) \in \bar{\Omega}.
\]
As a result
\[
-\varphi(y) \leq u_\varepsilon(x, y) \leq \varphi(y), \quad \forall (x, y) \in \Omega.
\]
Then inequality (2.3) follows immediately. \( \square \)

**Lemma 2.2.** The following inequalities hold
\[
M_{0,j} \leq \frac{2}{\delta} M_{0,j-1} + \frac{\delta}{2} M_{0,j+1}, \quad \forall \delta \in (0, \frac{1}{2}], \quad j = 1, 2, 3; \tag{2.5}
\]
\[
\varepsilon^j M_{j,0} \leq \frac{2}{\delta} \varepsilon^{j-1} M_{j-1,0} + \frac{\delta}{2} \varepsilon^{j+1} M_{j+1,0}, \quad \forall \delta \in (0, \frac{1}{2}], \quad j = 1, 2, 3. \tag{2.6}
\]

**Proof.** At first, we prove estimate (2.5). By the definition of \( M_{0,1} \), there exists a point \((x_0, y_0) \in \bar{\Omega} \), such that
\[
M_{0,1} = \left| \frac{\partial u_\varepsilon}{\partial y}(x_0, y_0) \right|.
\]
If \( 0 \leq y_0 \leq \frac{1}{2} \) (when \( \frac{1}{2} < y_0 \leq 1 \), the proof is analogous), for any \( \delta \in (0, \frac{1}{2}] \), we have
\[
u_\varepsilon(x_0, y_0 + \delta) = u_\varepsilon(x_0, y_0) + \delta \frac{\partial u_\varepsilon}{\partial y}(x_0, y_0) + \frac{\delta^2}{2} \frac{\partial^2 u_\varepsilon}{\partial y^2}(x_0, y_0 + \xi), \quad \text{with} \ \xi \in [0, \delta]. \tag{2.7}
\]
From (2.7), estimate (2.5) for \( j = 1 \) is obtained. For the case \( j = 2, 3 \), the proof is similar. Furthermore, following the same lines as those for (2.5), we can prove estimate (2.6). \( \square \)

Let
\[
v_j(x, y) = \frac{\partial^j u_\varepsilon(x, y)}{\partial y^j}, \quad j = 1, 2, 3, 4;
\]
\[
w_j(x, y) = \frac{\partial^j u_\varepsilon(x, y)}{\partial x^j}, \quad j = 1, 2, 3, 4.
\]
It can be verified that function $v_2(x, y)$ satisfies the following problem

\[-\varepsilon^2 \frac{\partial^2 v_2}{\partial x^2} - \frac{\partial^2 v_2}{\partial y^2} + av_2 = f_{yy}(x, y) - a_{yy}(x, y)u_z - 2a_y(x, y)v_1, \quad \forall (x, y) \in \Omega, \quad (2.8)\]

\[v_2|_{\Gamma_x} = v_2|_{\Gamma_w} = 0, \quad (2.9)\]

\[v_2|_{\Gamma_n} = -f(x, 1), \quad v_2|_{\Gamma_s} = -f(x, 0). \quad (2.10)\]

The boundary conditions (2.10) are from equation (1.1). Similar to the proof for Lemma 2.1, introducing an auxiliary function

\[
\psi(y) \equiv F_0 + (F_{0,2} + A_{0,2}M_0 + 2A_{0,1}M_{0,1}) \frac{y(1-y)}{2}, \quad \forall y \in [0, 1],
\]

we have

\[0 \leq \psi(y) \leq F_0 + \frac{F_{0,2} + A_{0,2}M_0 + 2A_{0,1}M_{0,1}}{8}, \quad \forall y \in [0, 1] \quad (2.11)\]

and

\[L_\varepsilon \psi(y) = F_{0,2} + A_{0,2}M_0 + 2A_{0,1}M_{0,1} + a(x, y)\psi(y).\]

Then it is easy to check that

\[L_\varepsilon (v_2 + \psi) \geq 0, \text{ and } L_\varepsilon (v_2 - \psi) \leq 0.\]

The maximum principle implies

\[M_{0,2} = \max_{(x,y) \in \Omega} |v_2(x, y)| \leq \max_{y \in [0,1]} |\psi(y)|.\]

By (2.11) and (2.5) in Lemma 2.2, we obtain

\[M_{0,2} \leq F_0 + \frac{F_{0,2} + A_{0,2}M_0 + 2A_{0,1}M_{0,1}}{8} \leq F_0 + \frac{F_{0,2} + A_{0,2}M_0}{8} + \frac{A_{0,1}}{2\delta}M_0 + \frac{\delta A_{0,1}}{8}M_{0,2}, \quad \forall \delta \in (0, \frac{1}{2}]. \quad (2.12)\]

Let

\[\delta = \min \left\{ \frac{1}{2}, \frac{4}{A_{0,1}} \right\}, \quad (2.13)\]

we arrive at

\[M_{0,2} \leq 2F_0 + \frac{F_{0,2}}{4} + \left(\frac{A_{0,2}}{4} + \frac{A_{0,1}}{\delta}\right)M_0 \equiv C_2. \quad (2.14)\]

As we have (let $\delta = \frac{1}{2}$ in equation (2.5))

\[M_{0,1} \leq 4M_0 + \frac{M_{0,2}}{4}, \]

we get the following Lemma.

**Lemma 2.3.** There exists a constant $C$ independent of $\varepsilon$ such that

\[M_{0,1} \leq C, \quad M_{0,2} \leq C. \quad (2.15)\]

Furthermore, from equation (1.1) and inequality (2.3), we have
Lemma 2.4. There exists a constant $C$ independent of $\varepsilon$ such that
\[ \varepsilon^{j} M_{j,0} \leq C, \quad j = 1, 2. \] (2.16)

Now we try to get a prior estimate of $v_4$. First, we discuss the boundary condition of $v_4$ at $\Gamma_s$ and $\Gamma_n$. From equation (2.8), we have
\[ v_4 = \frac{\partial^2 v_2}{\partial y^2} = -\varepsilon^2 \frac{\partial^4 u_\varepsilon}{\partial y^2 \partial x^2} + a(x, y)v_2 - f_{yy}(x, y) + a_{yy}(x, y)u_\varepsilon + 2a_y(x, y)v_1. \] (2.17)

Differentiating equation (1.1) with respect to $x$ twice, we obtain
\[ \frac{\partial^4 u_\varepsilon}{\partial y^2 \partial x^2} = -\varepsilon^2 w_4 + a(x, y)w_2 + a_{xx}(x, y)u_\varepsilon + 2a_x(x, y)w_1 - f_{xx}(x, y). \] (2.18)

Substituting (2.18) into (2.17) and using boundary conditions $w_j|_{\Gamma_n} = w_j|_{\Gamma_s} = 0$ ($j = 1, 2, 3, 4$) yield
\[ v_4|_{\Gamma_n} = \left( \varepsilon^2 f_{xx} + av_2 - f_{yy} + 2a_y v_1 \right)|_{\Gamma_n} \equiv g_4^i(x), \] (2.19)
\[ v_4|_{\Gamma_s} = \left( \varepsilon^2 f_{xx} + av_2 - f_{yy} + 2a_y v_1 \right)|_{\Gamma_s} \equiv g_4^s(x). \] (2.20)

This means, by Lemma 2.3, we can find a constant $C$ independent of $\varepsilon$ such that
\[ \|g_4^i\|_{^r,\infty} \leq C, \quad \|g_4^s\|_{^s,\infty} \leq C. \] (2.21)

The function $v_4$ satisfies
\[ -\varepsilon^2 \frac{\partial^2 v_4}{\partial x^2} - \frac{\partial^2 v_4}{\partial y^2} + av_4 = f_{yyyy} - a_{yyyy}u_\varepsilon - 4a_{yyyy}v_1 - 6a_{yy}v_2 - 4a_y v_3, \forall (x, y) \in \Omega, \] (2.22)
\[ v_2|_{\Gamma} = v_2|_{\Gamma_w} = 0, \] (2.23)
\[ v_2|_{\Gamma_n} = g_4^i(x), \quad v_2|_{\Gamma_s} = g_4^s(x). \] (2.24)

As before, applying the maximum principle to problem (2.22)–(2.24), we get

Lemma 2.5. There exists a constant $C$ independent of $\varepsilon$ such that
\[ \left\| \frac{\partial^4 u_\varepsilon}{\partial y^4} \right\|_\infty \leq C. \] (2.25)

Combining Lemma 2.1 – Lemma 2.5, we obtain

Theorem 2.1. There exists a constant $C$ independent of $\varepsilon$ such that
\[ \left\| \frac{\partial^j u_\varepsilon}{\partial y^j} \right\|_\infty \leq C, \quad j = 0, 1, \ldots, 4. \] (2.26)

Similarly, we have

Theorem 2.2. There exists a constant $C$ independent of $\varepsilon$ such that
\[ \left\| \varepsilon^j \frac{\partial^j u_\varepsilon}{\partial x^j} \right\|_\infty \leq C, \quad j = 0, 1, \ldots, 4. \] (2.27)

Finally, we get the theorem
The solution of problem (1.1)–(1.2) is smooth in \( y \)-direction. It is expected the gradient of the solution in \( y \)-direction will in general have small variations provided that the coefficients \( a_j(x) \) and the source term \( f_j(x) \) are nice functions. We can discretize the problem in \( y \)-direction by the standard finite difference method, which allows us to transform the second-order PDE into an ODE system. It is much like the method of lines.

Taking a positive integer \( M \), let \( \Delta y = \frac{1}{M} \) and
\[
y_j = j\Delta y, \quad j = 0, 1, \cdots, M.
\]
We discretize the second-order partial derivative with respect to the \( y \) variable in problem (1.1)-(1.2) with the centered three-point finite difference. This gives us the following semi-discrete system
\[
-\varepsilon^2 \frac{d^2 u^\varepsilon_j(x)}{dx^2} - \frac{u^\varepsilon_{j+1}(x) - 2u^\varepsilon_j(x) + u^\varepsilon_{j-1}(x)}{(\Delta y)^2} + a_j(x)u^\varepsilon_j(x) = f_j(x), \quad 1 \leq j \leq M - 1,
\]
\[
u^\varepsilon_j(0) = \frac{\varepsilon}{M} \quad u^\varepsilon_j(1) = 0, \quad 1 \leq j \leq M - 1,
\]
\[
u^\varepsilon_j(x) = u^\varepsilon_0(x) = 0, \quad \forall x \in [0, 1],
\]
with
\[
a_j(x) = a(x, y_j), \quad f_j(x) = f(x, y_j), \quad j = 1, \cdots, M - 1.
\]
Here, \( u^\varepsilon_j(x) \) is the finite difference approximation of \( u^\varepsilon(x, y_j) \) for each \( j \in \{0, 1, \cdots, M\} \). Let
\[
U^\varepsilon_j(x) = u^\varepsilon(x, y_j), \quad j = 0, 1, \cdots, M
\]
be the exact solution at the discrete points \( \{y_j\}_{j=0}^M \). The differentiability of functions \( \{U^\varepsilon_j(x), j = 0, 1, \cdots, M\} \) that we derived in Section 2 allows us to make local Taylor expansions for them. As a result, the functions satisfy the modified semi-discrete system
\[
-\varepsilon^2 \frac{d^2 U^\varepsilon_j(x)}{dx^2} - \frac{U^\varepsilon_{j+1}(x) - 2U^\varepsilon_j(x) + U^\varepsilon_{j-1}(x)}{(\Delta y)^2} + a_j(x)U^\varepsilon_j(x) = f_j(x) + R^\varepsilon_j(x),
\]
\[
1 \leq j \leq M - 1,
\]
\[
U^\varepsilon_j(0) = \frac{\varepsilon}{M} \quad U^\varepsilon_j(1) = 0, \quad 1 \leq j \leq M - 1,
\]
\[
U^\varepsilon_0(x) = u^\varepsilon_0(x) = 0, \quad \forall x \in [0, 1],
\]
with the remainder term \( R^\varepsilon_j(x) \) on the order of \( \Delta y^2 \) in that there is a constant \( C \) independent of \( \varepsilon \) such that
\[
|R^\varepsilon_j(x)| \leq C(\Delta y)^2, \quad 1 \leq j \leq M - 1.
\]

The estimate (3.8) for the remainder term results from the application of Theorems 2.1–2.3.
Lemma 3.1. Let

\[ \mathbf{v}(x) = (v_1^\varepsilon(x), v_2^\varepsilon(x), \ldots, v_{M-1}^\varepsilon(x))^T \]

be a vector-valued function. If for any \(0 < x < 1\) and \(1 \leq j \leq M - 1\),

\[ -\varepsilon^2 \frac{d^2 v_j^\varepsilon(x)}{dx^2} - \frac{v_{j-1}^\varepsilon(x) - 2v_j^\varepsilon(x) + v_{j+1}^\varepsilon(x)}{(\Delta y)^2} + a_j(x)v_j^\varepsilon(x) \leq 0 \]

then we have

\[ v_j^\varepsilon(x) \leq \max \left\{ 0, \max_{1 \leq j \leq M-1} \{ v_j^\varepsilon(0), v_j^\varepsilon(1) \}, \max_{0 \leq x \leq 1} \{ v_0^\varepsilon(x), v_M^\varepsilon(x) \} \right\}. \quad (3.9) \]

Proof. First, we prove that if

\[ -\varepsilon^2 \frac{d^2 v_j^\varepsilon(x)}{dx^2} - \frac{v_{j-1}^\varepsilon(x) - 2v_j^\varepsilon(x) + v_{j+1}^\varepsilon(x)}{(\Delta y)^2} + a_j(x)v_j^\varepsilon(x) < 0, \quad (3.10) \]

then \(v_j^\varepsilon(x)\) can not attain its maximum non-negative value in the interior of interval \(0 < x < 1\) and for \(1 \leq j \leq M - 1\). Assume that \(v_j^\varepsilon(x)\) attains its maximum non-negative value at point \(x = x^*\) for some \(1 \leq j = j^* \leq M - 1\). This indicates

\[ \varepsilon^2 \frac{d^2 v_{j^*}^\varepsilon(x^*)}{dx^2} \leq 0 \quad \text{and} \quad \frac{v_{j^*-1}^\varepsilon(x^*) - 2v_{j^*}^\varepsilon(x^*) + v_{j^*+1}^\varepsilon(x^*)}{(\Delta y)^2} \leq 0. \]

The condition (3.10) further implies

\[ 0 \leq a_{j^*}(x^*)v_{j^*}^\varepsilon(x^*) < \varepsilon^2 \frac{d^2 v_{j^*}^\varepsilon(x^*)}{dx^2} + \frac{v_{j^*-1}^\varepsilon(x^*) - 2v_{j^*}^\varepsilon(x^*) + v_{j^*+1}^\varepsilon(x^*)}{(\Delta y)^2} \leq 0. \]

This is a contradiction. As a result, the inequality (3.9) holds.

In the next, we prove the lemma for the case

\[ -\varepsilon^2 \frac{d^2 v_j^\varepsilon(x)}{dx^2} - \frac{v_{j-1}^\varepsilon(x) - 2v_j^\varepsilon(x) + v_{j+1}^\varepsilon(x)}{(\Delta y)^2} + a_j(x)v_j^\varepsilon(x) \leq 0. \quad (3.11) \]

Let

\[ \xi_j = \frac{1}{8} - \frac{1}{2} \left( y_j - \frac{1}{2} \right)^2 \quad \text{and} \quad w_j^\eta(x) = v_j^\varepsilon(x) - \eta \xi_j \]

with \(\eta > 0\) be a small positive parameter. It obvious that

\[ \xi_j \geq 0 \quad \text{and} \quad \frac{\xi_{j-1} - 2\xi_j + \xi_{j+1}}{(\Delta y)^2} = -1. \quad (3.12) \]
We see that \( w_j^n(x) \) satisfies the condition (3.10) since
\[
-\varepsilon^2 \frac{d^2 w_j^n(x)}{dx^2} - \frac{w_j^n(x) - 2w_j^n(x) + w_{j+1}^n(x)}{(\Delta y)^2} + a_j(x)w_j^n(x) = 0.
\]

\[
= -\varepsilon^2 \frac{d^2 v_j^e(x)}{dx^2} - \frac{v_{j-1}^e(x) - 2v_j^e(x) + v_{j+1}^e(x)}{(\Delta y)^2} + a_j(x)v_j^e(x) + \eta \frac{\xi_{j-1} - 2\xi_j + \xi_{j+1}}{(\Delta y)^2}
\]

\[
\leq -\varepsilon^2 \frac{d^2 v_j^e(x)}{dx^2} - \frac{v_{j-1}^e(x) - 2v_j^e(x) + v_{j+1}^e(x)}{(\Delta y)^2} + a_j(x)v_j^e(x) + \eta \frac{\xi_{j-1} - 2\xi_j + \xi_{j+1}}{(\Delta y)^2}
\]

Here, we used the condition (3.11). The previous conclusion implies
\[
w_j^n(x) \leq \max \left\{ 0, \max_{1 \leq j \leq M-1} \{ w_j^n(0), w_j^n(1) \}, \max_{0 \leq \varepsilon \leq 1} \{ w_0^n(x), w_M^n(x) \} \right\}. \tag{3.13}
\]

Since the parameter \( \eta \) can be arbitrarily small, letting \( \eta \to 0 \), we get the inequality (3.9) for \( v_j^e(x) \).

Now we are ready to prove the following theorem.

**Theorem 3.1.** Let
\[
\mathcal{U}^\varepsilon(x) = (u_1^\varepsilon(x), \ldots, u_{M-1}^\varepsilon(x))^T,
\]
\[
\mathcal{U}^e(x) = (U_1^e(x), \ldots, U_{M-1}^e(x))^T.
\]

The following error estimate holds
\[
\|\mathcal{U}^\varepsilon - \mathcal{U}^e\|_\infty \leq C(\Delta y)^2, \tag{3.14}
\]
where \( C \) is a constant independent of \( \varepsilon \).

**Proof.** Subtracting the semi-discrete system (3.1)-(3.3) by the semi-discrete system (3.5)-(3.7) yields a new semi-discrete system
\[
-\varepsilon^2 \frac{d^2 e_j^\varepsilon(x)}{dx^2} - \frac{e_{j-1}^\varepsilon(x) - 2e_j^\varepsilon(x) + e_{j+1}^\varepsilon(x)}{(\Delta y)^2} + a_j(x)e_j^\varepsilon(x) = R_j^\varepsilon(x), \quad 1 \leq j \leq M - 1,
\]
\[
e_j^\varepsilon(0) = e_j^\varepsilon(1) = 0, \quad 1 \leq j \leq M - 1,
\]
\[
e_0^\varepsilon(x) = e_M^\varepsilon(x) = 0, \quad \forall x \in [0,1],
\]
for the vector-valued error function
\[
e^\varepsilon(x) = (e_1^\varepsilon(x), e_2^\varepsilon(x), \ldots, e_{M-1}^\varepsilon(x))^T = \mathcal{U}^e - \mathcal{U}^\varepsilon
\]
with \( e_j^\varepsilon(x) = U_j^e(x) - u_j^\varepsilon(x) \). Let
\[
\varphi_j = \left[ \frac{1}{8} - \frac{1}{2} \left( y_j - \frac{1}{2} \right)^2 \right] \| R_j^\varepsilon(x) \|_\infty
\]
with \( \| R_j^\varepsilon(x) \|_\infty = \max_{1 \leq j \leq M-1, 0 \leq x \leq 1} |R_j^\varepsilon(x)| \). Let \( v_j^\varepsilon(x) = e_j^\varepsilon(x) - \varphi_j \). We have
\[
-\varepsilon^2 \frac{d^2 v_j^\varepsilon(x)}{dx^2} - \frac{v_{j-1}^\varepsilon(x) - 2v_j^\varepsilon(x) + v_{j+1}^\varepsilon(x)}{(\Delta y)^2} + a_j(x)v_j^\varepsilon(x) \leq R_j^\varepsilon(x) - \| R_j^\varepsilon(x) \|_\infty \leq 0.
\]
Lemma 3.1 implies \( v_j^\varepsilon(x) \leq 0 \) for all \( j \)'s and \( x \in [0, 1] \). Thus we obtain
\[
e_j^\varepsilon(x) \leq \varphi_j \leq \max_{0 \leq j \leq M} \varphi_j = \frac{1}{8} \| R_j^\varepsilon(x) \|_\infty.
\]
Let \( w_j^\varepsilon(x) = -e_j^\varepsilon(x) - \varphi_j \). We get
\[
-\varepsilon^2 \frac{d^2 w_j^\varepsilon(x)}{dx^2} - \frac{w_{j-1}^\varepsilon(x) - 2w_j^\varepsilon(x) + w_{j+1}^\varepsilon(x)}{(\Delta y)^2} + a_j(x)w_j^\varepsilon(x) \leq -R_j^\varepsilon(x) - \| R_j^\varepsilon(x) \|_\infty \leq 0.
\]
Similarly, Lemma 3.1 implies \( w_j^\varepsilon(x) \leq 0 \) for all \( j \)'s and \( x \in [0, 1] \). We arrive at
\[
-e_j^\varepsilon(x) \leq \varphi_j \leq \max_{0 \leq j \leq M} \varphi_j = \frac{1}{8} \| R_j^\varepsilon(x) \|_\infty.
\]
Combining (3.15) and (3.16), we get
\[
\| e_j^\varepsilon(x) \|_\infty \leq \frac{1}{8} \| R_j^\varepsilon(x) \|_\infty.
\]
Together with (3.8), we finally obtain the error estimate (3.14).

4. A TAILORED FINITE POINT METHOD FOR PROBLEM (3.1)-(3.3)

The problem (3.1)-(3.3) can be rewritten as
\[
-\varepsilon^2 \frac{d^2 U^\varepsilon}{dx^2} + A(x) U^\varepsilon = F(x), \quad \forall x \in (0, 1),
\]
\[
U^\varepsilon(0) = U^\varepsilon(1) = 0,
\]
where \( A(x) \) is an \((M-1) \times (M-1)\) matrix function, \( F(x) \) is an \((M-1)\) dimensional vector function,
\[
A(x) = \begin{pmatrix}
\frac{2}{(\Delta y)^2} + a_1(x) & -\frac{1}{(\Delta y)^2} & 0 \\
-\frac{1}{(\Delta y)^2} & \frac{2}{(\Delta y)^2} + a_1(x) & \ddots \\
0 & \ddots & \ddots & -\frac{1}{(\Delta y)^2} \\
\end{pmatrix},
\]
\[
F(x) = \begin{pmatrix}
f_1(x) \\
f_2(x) \\
\vdots \\
f_{M-1}(x)
\end{pmatrix}.
\]
Taking a positive integer $N$, let 

$$\Delta x = \frac{1}{N} \quad \text{and} \quad x_i = i \Delta x, \quad i = 0, 1, \ldots, N.$$ 

We will introduce piecewise constant matrix functions $A_h(x)$ and $F_h(x)$ for $0 \leq x \leq 1$, which are approximations of $A(x)$ and $F(x)$ respectively. On each interval $[x_i, x_{i+1}]$ (i.e., $x_i^* = \frac{x_i + x_{i+1}}{2}$), define the matrix functions $A_h(x)$ and $F_h(x)$ by 

$$A_h(x) = A(x_i^*), \quad \text{for } x \in (x_i, x_{i+1}),$$ 

$$F_h(x) = F(x_i^*), \quad \text{for } x \in (x_i, x_{i+1}).$$ 

This means the matrix functions $A_h(x)$ and $F_h(x)$ are constants on each subinterval $(x_i, x_{i+1})$ ($i = 0, 1, \ldots, N - 1$) and have jump discontinuities at points $x = x_i$ ($i = 1, 2, \ldots, N - 1$).

Now we consider the following approximation of problem (4.1)–(4.2),

$$-\varepsilon^2 \frac{d^2 U_h^e}{dx^2} + A_h(x) U_h^e = F_h(x), \quad \forall x \in (x_i, x_{i+1}), \quad i = 0, \ldots, N - 1 \quad (4.3)$$

$$U_h^e(0) = U_h^e(1) = 0, \quad (4.4)$$

$$[U_h^e]_{x=x_i} = \left[ \frac{dU_h^e}{dx} \right]_{x=x_i} = 0, \quad (4.5)$$

with the jumps defined by

$$[U_h^e]_{x=x_i} = U_h^e(x_i + 0) - U_h^e(x_i - 0), \quad (4.6)$$

$$\left[ \frac{dU_h^e}{dx} \right]_{x=x_i} = \frac{dU_h^e}{dx}(x_i + 0) - \frac{dU_h^e}{dx}(x_i - 0). \quad (4.7)$$

Here, the semi-discrete solution

$$U_h^e(x) = (u_1^{e,h}(x), \ldots, u_{M-1}^{e,h}(x))^T$$

is an approximation of $U^e(x)$. Correspondingly, each component $u_j^{e,h}(x)$ of the vector-valued function is also an approximation of $u_j^e(x)$.

**Lemma 4.1.** The problem (4.3)–(4.5) has a unique solution $U_h^e(x)$ and the solution satisfies the following estimate

$$\|U_h^e\|_\infty \leq C, \quad (4.8)$$

where $C$ is a constant independent of $\varepsilon$.

**Lemma 4.2.** The following uniform error estimate holds

$$\|U^e - U_h^e\|_\infty \leq C \Delta x, \quad (4.9)$$

where $C$ is a constant independent of $\varepsilon$ and $\Delta x$. 
For $x \in [0, 1]$, we define
\[
\begin{aligned}
u_{\varepsilon,h}(x, y) &= \frac{y_{j+1} - y}{\Delta y} u_{j}^{\varepsilon,h}(y) + \frac{y - y_{j}}{\Delta y} u_{j+1}^{\varepsilon,h}(y), \quad \text{for} \quad y \in [y_{j}, y_{j+1}], \quad (j = 0, \cdots, M - 1)
\end{aligned}
\]
where
\[
u_{0}^{\varepsilon,h}(x) = u_{M}^{\varepsilon,h}(x) \equiv 0.
\]

Combining with Theorem 3.1, we get the following error estimate:

**Theorem 4.1.** Suppose that $u^\varepsilon$ is the solution of problem (1.1)–(1.2), $u_{\varepsilon,h}$ is defined above. The following error estimate holds
\[
\|u_{\varepsilon,h} - u_{\varepsilon}\|_\infty \leq C \left\{ \Delta x + (\Delta y)^2 \right\},
\]
where $C$ is a constant independent of $\varepsilon$, $\Delta x$ and $\Delta y$.

Now we discuss how to get the solution $U_{\varepsilon}^i(x)$ exactly. We will propose a tailored finite point method to solve it [6, 7, 8, 9, 10, 12].

On each subinterval $[x_i, x_{i+1}]$, $U_{\varepsilon}^i(x)$ satisfies
\[
-\varepsilon^2 \frac{d^2 U_{\varepsilon}^i(x)}{dx^2} + A_i U_{\varepsilon}^i(x) = F_i, \quad \text{for} \quad x \in (x_i, x_{i+1}),
\]
with
\[
A_i = A(x_i^*), \quad F_i = F(x_i^*).
\]

Let
\[
U_{h}^{f;i}(x) = A_i^{-1} F_i,
\]
which is a particular solution to the nonhomogeneous equation (4.11). Let
\[
U_{h}^{i}(x) = \xi^i e^{\frac{\lambda x}{\varepsilon}} \in \mathbb{R}^{M-1},
\]
be a general solution of the homogeneous equation
\[
-\varepsilon^2 \frac{d^2 U_{h}^{i}(x)}{dx^2} + A_i U_{h}^{i}(x) = 0, \quad \text{for} \quad x \in (x_i, x_{i+1}).
\]

Substituting (4.13) into (4.14), we arrive at
\[
A_i \xi^i = \lambda^2 \xi^i.
\]

Since $A_i$ is a symmetric definite real matrix, it has $M - 1$ positive real eigenvalues
\[
0 < \mu_1^i \leq \mu_2^i \leq \cdots \leq \mu_{M-1}^i,
\]
and corresponding eigenvectors
\[
\xi_1^i, \xi_2^i, \cdots, \xi_{M-1}^i.
\]

Let $\lambda_k^i = \sqrt{\mu_k^i}$. The general solution of equation (4.11) on the interval $[x_i, x_{i+1}]$ is given by
\[
U_{h}^{i}(x) = U_{h}^{f;i}(x) + \sum_{k=1}^{M-1} \left( \alpha_k^i \exp \left( \frac{\lambda_k^i(x - x_{i+1})}{\varepsilon} \right) + \alpha\overline{k}^i \exp \left( -\frac{\lambda_k^i(x - x_i)}{\varepsilon} \right) \right) \xi_k^i.
\]
with some constants $\alpha^\pm_k$ $(k = 1, \ldots, M - 1)$. Let

$$\mathbb{K}^i = (\xi_1^i, \ldots, \xi_{M-1}^i),$$

$$\Lambda^i_+(x) = \text{diag}\left\{\exp\left(\frac{\lambda^i_1(x-x_{i+1})}{\varepsilon}\right), \ldots, \exp\left(\frac{\lambda^i_{M-1}(x-x_{i+1})}{\varepsilon}\right)\right\},$$

$$\Lambda^i_-(x) = \text{diag}\left\{\exp\left(-\frac{\lambda^i_1(x-x_i)}{\varepsilon}\right), \ldots, \exp\left(-\frac{\lambda^i_{M-1}(x-x_i)}{\varepsilon}\right)\right\},$$

$$\alpha^+_i = (\alpha^+_{1i}, \alpha^+_{2i}, \ldots, \alpha^+_{M-1i})^T,$$

$$\alpha^-_i = (\alpha^-_{1i}, \alpha^-_{2i}, \ldots, \alpha^-_{M-1i})^T.$$  

Then equality (4.16) can be rewritten by

$$\mathcal{U}^E_h(x) = \mathcal{U}^{E,i}_h(x) + \mathbb{K}^i \Lambda^i_+(x) \alpha^+_i + \mathbb{K}^i \Lambda^i_-(x) \alpha^-_i, \quad \forall x \in [x_i, x_{i+1}]. \tag{4.17}$$

At points $x = x_i, x_{i+1}$, we obtain

$$\mathcal{U}^E_h(x_i) = \mathcal{U}^{E,i}_h(x_i) + \mathbb{K}^i \Lambda^i_+(x_i) \alpha^+_i + \mathbb{K}^i \Lambda^i_-(x_i) \alpha^-_i, \tag{4.18}$$

$$\mathcal{U}^E_h(x_{i+1}) = \mathcal{U}^{E,i}_h(x_{i+1}) + \mathbb{K}^i \Lambda^i_+(x_{i+1}) \alpha^+_i + \mathbb{K}^i \Lambda^i_-(x_{i+1}) \alpha^-_i. \tag{4.19}$$

Let

$$\mathbb{A}^i = \begin{pmatrix} \mathbb{K}^i \Lambda^i_+(x_i) & \mathbb{K}^i \Lambda^i_-(x_i) \\ \mathbb{K}^i \Lambda^i_+(x_{i+1}) & \mathbb{K}^i \Lambda^i_-(x_{i+1}) \end{pmatrix}.$$  

Then the coefficients $\alpha^i_{\pm}$ are given by

$$\begin{pmatrix} \alpha^+_i \\ \alpha^-_i \end{pmatrix} = (\mathbb{A}^i)^{-1} \begin{pmatrix} \mathcal{U}^E_h(x_i) - \mathcal{U}^{E,i}_h(x_i) \\ \mathcal{U}^E_h(x_{i+1}) - \mathcal{U}^{E,i}_h(x_{i+1}) \end{pmatrix}. $$

Assume that

$$(\mathbb{A}^i)^{-1} \equiv \begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix}. $$

We have

$$\begin{pmatrix} \alpha^+_i \\ \alpha^-_i \end{pmatrix} = \begin{pmatrix} A_{11}^i \mathcal{U}^E_h(x_i) + A_{12}^i \mathcal{U}^E_h(x_{i+1}) - (A_{11}^i + A_{12}^i) \mathcal{U}^{E,i}_h(x_i) \\ A_{21}^i \mathcal{U}^E_h(x_i) + A_{22}^i \mathcal{U}^E_h(x_{i+1}) - (A_{21}^i + A_{22}^i) \mathcal{U}^{E,i}_h(x_{i+1}) \end{pmatrix}. \tag{4.20}$$

From (4.16), we get

$$\frac{d\mathcal{U}^E_h(x_i + 0)}{dx} = \mathbb{K}^i \frac{d}{dx} \Lambda^i_+(x_i) \alpha^+_i + \mathbb{K}^i \frac{d}{dx} \Lambda^i_-(x_i) \alpha^-_i, \tag{4.21}$$

$$\frac{d\mathcal{U}^E_h(x_i - 0)}{dx} = \mathbb{K}^{-1} \frac{d}{dx} \Lambda^{-1}_+(x_i) \alpha^{i-1}_+ + \mathbb{K}^{-1} \frac{d}{dx} \Lambda^{-1}_-(x_i) \alpha^{i-1}_-. \tag{4.22}$$

Continuity condition (4.7) on the first-order derivative implies

$$\mathbb{K}^i \frac{d}{dx} \Lambda^i_+(x_i) \alpha^+_i + \mathbb{K}^i \frac{d}{dx} \Lambda^i_-(x_i) \alpha^-_i = \mathbb{K}^{-1} \frac{d}{dx} \Lambda^{-1}_+(x_i) \alpha^{i-1}_+ + \mathbb{K}^{-1} \frac{d}{dx} \Lambda^{-1}_-(x_i) \alpha^{i-1}_-. \tag{4.23}$$
Substituting (4.20) into (4.23), we finally get a numerical scheme
\[ A_i U_h^i(x_{i-1}) + B_i U_h^i(x_i) + C_i U_h^i(x_{i+1}) = F_i, \quad i = 1, 2, \ldots, N - 1, \] (4.24)
\[ U_h^i(0) = U_h^i(1) = 0, \] (4.25)
where
\[ A_i = -K^{-1} \frac{d}{dx} \Lambda_{11}^i(x_i) A_{11}^i - K^{-1} \frac{d}{dx} \Lambda_{12}^i(x_i) A_{12}^i, \]
\[ B_i = K^{-1} \frac{d}{dx} \Lambda_{11}^i(x_i) A_{11}^i_1 + K^{-1} \frac{d}{dx} \Lambda_{12}^i(x_i) A_{12}^i - K^{-1} \frac{d}{dx} \Lambda_{11}^i(x_i) A_{12}^i - K^{-1} \frac{d}{dx} \Lambda_{12}^i(x_i) A_{22}^i, \]
\[ C_i = K^{-1} \left( \frac{d}{dx} \Lambda_{11}^i(x_i) (A_{11}^i + A_{12}^i) + \frac{d}{dx} \Lambda_{12}^i(x_i) (A_{21}^i + A_{22}^i) \right) U_h^{i+1}, \]
\[ F_i = K^{-1} \left( \frac{d}{dx} \Lambda_{11}^i(x_i) (A_{11}^i + A_{12}^i) + \frac{d}{dx} \Lambda_{12}^i(x_i) (A_{21}^i + A_{22}^i) \right) U_h^{i-1}. \]

5. Numerical examples

This section presents numerical examples to demonstrate the efficiency and reliability of the tailored finite point method. In the examples, the error \( e_h(x, y) \) of a numerical solution is computed in the discrete \( L^2 \)-norm defined by
\[ \|e_h\|_{L^2}^2 = \Delta x \Delta y \sum_{i=1}^{N} \sum_{j=1}^{M} \left| e_h \left( x_i + \frac{1}{2} \Delta x, y_j + \frac{1}{2} \Delta y \right) \right|^2. \]

**Example 5.1.** First, let \( a(x, y) = 0, \ f(x, y) = 100 \sin 2\pi y \) and
\[ u_\varepsilon(x, y) = \frac{25}{\pi^2} \left( 1 - \frac{e^{-\frac{2\pi x}{\varepsilon}} + e^{-\frac{2\pi (y-1)}{\varepsilon}}}{1 - \exp(-\frac{2\pi}{\varepsilon})} \right) \sin 2\pi y. \] (5.1)

The function \( u_\varepsilon \) given in (5.1) is the exact solution to problem (1.1)–(1.2) with \( a \) and \( f \) given above. In this example, functions \( a \) and \( f \) satisfy the conditions given in Assumption 2.1. The numerical results are shown in Figures 2, 3 and Table 1. Second-order convergence rates for the numerical solutions are observed in the discrete \( L^2 \) norm.

**Table 1.** \( L^2 \) errors of the numerical solutions for Example 5.1.

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( \Delta x ) and ( \Delta y )</th>
<th>( \frac{1}{8} )</th>
<th>( \frac{1}{16} )</th>
<th>( \frac{1}{32} )</th>
<th>( \frac{1}{64} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>( u_\varepsilon - u_{\varepsilon,h} ) |_{L^2}</td>
<td>8.88E-2</td>
<td>2.23E-2</td>
<td>5.56E-3</td>
<td>1.39E-3</td>
</tr>
<tr>
<td>0.001</td>
<td>( u_\varepsilon - u_{\varepsilon,h} ) |_{L^2}</td>
<td>8.88E-2</td>
<td>2.25E-2</td>
<td>5.67E-3</td>
<td>1.43E-3</td>
</tr>
</tbody>
</table>
Example 5.2. Now let \( a(x, y) = 0, f(x, y) = 1 \) and
\[
 u_\varepsilon(x, y) = \frac{y(1 - y)}{2} + \sum_{k=1}^{\infty} 2((-1)^k - 1) \frac{e^{-\frac{k\pi x}{\varepsilon}} + e^{\frac{k\pi(x-1)}{\varepsilon}}}{(k\pi)^3} \left( 1 + \exp\left(-\frac{k\pi y}{\varepsilon}\right) \right) \sin k\pi y. \tag{5.2}
\]
The function \( u_\varepsilon \) given in (5.2) is the exact solution to problem (1.1)–(1.2).
In this example, function $f$ does not satisfy all the compatibility conditions (2.2). The numerical results are shown in Figure 4 ($\varepsilon = 0.1$) and Figure 5 ($\varepsilon = 0.001$) with mesh size $\Delta x = \Delta y = 0.05$. The method still yields good approximations on coarse grids ($h \gg \varepsilon$).

**Figure 4.** $u_{\varepsilon,h}$ (left) and $|u_{\varepsilon} - u_{\varepsilon,h}|$ (right), $\varepsilon = 0.1$.

**Figure 5.** $u_{\varepsilon,h}$ (left) and $|u_{\varepsilon} - u_{\varepsilon,h}|$ for example 5.2, $\varepsilon = 0.001$.

**Example 5.3.** Let $a(x, y) = 1$ and $f(x, y) = (64x(1-x)y(1-y))^{0.1}$.

In this example, function $f$ also does not satisfy all the compatibility conditions (2.2). The numerical results are shown in Figure 6 ($\varepsilon = 0.1$) and Figure 7 ($\varepsilon = 0.001$). The method also yields good numerical solutions when $h \gg \varepsilon$.

**Example 5.4.** Let

$$a(x, y) = xy, \quad u_{\varepsilon}(x, y) = 100x(1-x)y(1-y) \left( \exp \left( -\frac{x}{\varepsilon} \right) + \exp \left( \frac{x-1}{\varepsilon} \right) \right),$$

and $f(x, y) = L_{\varepsilon} u_{\varepsilon}$.

In this example, the function $f$ does not satisfy the conditions given in Assumption 2.1. The numerical results are shown in Figures 8, 9 and Table 2. The second-order convergence rates for the numerical solutions are also observed in the discrete $l^2$ norm.
6. Conclusion

This work proposes a tailored finite point method (TFPM) for the strongly anisotropic diffusion problem (1.1)–(1.2), which has much slower diffusion rate along one direction than the other in a rectangle domain. It presents analysis on the differentiability of the solution to problem (1.1)–(1.2) under some compatibility conditions and proves a uniform error estimate for the approximate solution to the discrete problem. Numerical experiments show that, for the problem that satisfies the compatibility conditions, the TFPM yields second-order convergence rates in the discrete $l^2$ norm. For the problem that does not satisfy the compatibility and regularity conditions, the method proposed may still offer good approximate solutions.

Table 2. $l^2$ errors of the numerical solutions for Example 5.4 with $\Delta y = \frac{1}{16}$.

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>$\frac{1}{16}$</th>
<th>$\frac{1}{32}$</th>
<th>$\frac{1}{64}$</th>
<th>$\frac{1}{128}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon = 0.1$ $|u_\varepsilon - u_{\varepsilon,h}|_{l^2}$</td>
<td>5.98E-3</td>
<td>1.45E-3</td>
<td>3.48E-4</td>
<td>8.58E-5</td>
</tr>
<tr>
<td>$\varepsilon = 0.01$ $|u_\varepsilon - u_{\varepsilon,h}|_{l^2}$</td>
<td>8.08E-3</td>
<td>2.12E-3</td>
<td>5.30E-4</td>
<td>1.27E-4</td>
</tr>
</tbody>
</table>
In this work, for simplicity, the TFPM is proposed only for the problem defined in a rectangle domain. However, the method is by no means limited to the simple domain. Instead, the method can be generalized for problems on complex domains together with a structured grid method such as the kernel-free boundary integral method[21]. This extension will make the TFPM applicable to a larger class of problems.

Another potential extension of the TFPM is its application for those problems where an advection or convection term is added into the PDE.

\[ \text{Figure 8. Graphs of } |u_\varepsilon - u_{\varepsilon,h}| \text{ for example 5.4, } \Delta y = \frac{1}{16}, \varepsilon = 0.1: \ (a) \Delta x = \frac{1}{16}; \ (b) \Delta x = \frac{1}{32}; \ (c) \Delta x = \frac{1}{64}; \ (d) \Delta x = \frac{1}{128}. \]

\[ \text{Figure 9. Convergence rates of } \|u_\varepsilon - u_{\varepsilon,h}\|_2 \text{ for Example 5.4.} \]
References
