ON THE CROSSING NUMBERS OF CARTESIAN PRODUCTS OF STARS AND GRAPHS OF ORDER SIX

Marián Klešč\textsuperscript{1,2} and Štefan Schrötter\textsuperscript{1}

Faculty of Electrical Engineering and Informatics
Technical University of Košice
Letná 9, 042 00 Košice, Slovak Republic
e-mail: marian.klesc@tuke.sk
stefan.schrotter@tuke.sk

Abstract

The crossing number $cr(G)$ of a graph $G$ is the minimal number of crossings over all drawings of $G$ in the plane. According to their special structure, the class of Cartesian products of two graphs is one of few graph classes for which some exact values of crossing numbers were obtained. The crossing numbers of Cartesian products of paths, cycles or stars with all graphs of order at most four are known. Moreover, except of six graphs, the crossing numbers of Cartesian products $G\Box K_{1,n}$ for all other connected graphs $G$ on five vertices are known. In this paper we are dealing with the Cartesian products of stars with graphs on six vertices. We give the exact values of crossing numbers for some of these graphs and we summarise all known results concerning crossing numbers of these graphs. Moreover, we give the crossing number of $G_1\Box T$ for the special graph $G_1$ on six vertices and for any tree $T$ with no vertex of degree two as well as the crossing number of $K_{1,n}\Box T$ for any tree $T$ with maximum degree five.

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1. Introduction

Let $G$ be a graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of $G$ is a representation of $G$ in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point. The crossing number $\text{cr}(G)$ is the smallest number of edge crossings in any drawing of $G$. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a good drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

The investigation on the crossing number of graphs is a classical and very difficult problem. According to their special structure, the class of Cartesian products of two graphs is one of few graph classes for which some exact values of crossing numbers were obtained. The Cartesian product $G \square H$ of the graphs $G$ and $H$ has vertex set $V(G \square H) = V(G) \times V(H)$ and any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \square H$ if and only if either $u = v$ and $u'$ is adjacent with $v'$ in $H$, or $u' = v'$ and $u$ is adjacent with $v$ in $G$. Let $C_n$ be the cycle of length $n$, $P_n$ be the path of $n$ vertices, and $S_n$ be the star isomorphic to $K_{1,n}$. Harary et al. [7] conjectured that the crossing number of the Cartesian product $C_m \square C_n$ of two cycles is $(m - 2)n$, for all $m, n$ satisfying $3 \leq m \leq n$. It was proved by Glebsky and Salazar [6] that for any fixed $m$, the conjecture holds for all $n \geq m(m + 1)$. The conjecture has also been verified for $m \leq 7$. Beineke and Ringeisen in [2] started to study the crossing numbers of Cartesian products of cycles with all graphs of order at most four. In [11], [13], and [14], the crossing numbers of Cartesian products of cycles, paths and stars with all graphs of order four are given. In the paper, we are dealing with crossing numbers of Cartesian products of stars and small graphs. Some results concerning the crossing numbers of $G \square S_n$ for graphs $G$ on five vertices appear in [15] and [16]. The crossing numbers of Cartesian products of stars with graphs of order five are collected in [18]. The aim of the paper is to establish the crossing numbers of Cartesian products of stars with several graphs of order six. We will use some results concerning the crossing numbers of bipartite and multipartite complete graphs as well as the crossing numbers of join products of special graphs.

The join product of two graphs $G$ and $H$, denoted by $G + H$, is obtained from vertex-disjoint copies of $G$ and $H$ by adding all edges between $V(G)$ and $V(H)$. For $|V(G)| = m$ and $|V(H)| = n$, the edge set of $G + H$ is the union of disjoint edge sets of the graphs $G$, $H$, and the complete bipartite graph $K_{m,n}$. 
Kulli and Muddebihal [19] gave the characterization of all pairs of graphs which join is planar graph. Using Kleitman’s result [12], the crossing numbers for join of two paths, join of two cycles, and for join of path and cycle were studied in [17].

Let $D$ be a good drawing of the graph $G$. We denote the number of crossings in $D$ by $\text{cr}_D(G)$. For a subgraph $H_i$ of the graph $G$, let $D(H_i)$ be the subdrawing of $D$ induced by $H_i$. For edge-disjoint subgraphs $H_i$ and $H_j$ of $G$, we denote by $\text{cr}_D(H_i, H_j)$ the number of crossings of edges in $H_i$ and edges in $H_j$, and by $\text{cr}_D(H_i)$ the number of crossings among edges of $H_i$ in $D$. It is easy to see that for any three edge-disjoint subgraphs $H_i$, $H_j$, and $H_k$ of the graph $G$ the following equations hold:

$$\text{cr}_D(H_i \cup H_j) = \text{cr}_D(H_i) + \text{cr}_D(H_j) + \text{cr}_D(H_i, H_j),$$

\[ (1) \]

$$\text{cr}_D(H_i \cup H_j, H_k) = \text{cr}_D(H_i, H_k) + \text{cr}_D(H_j, H_k).$$

In the paper, some proofs are based on Kleitman’s result on crossing numbers of complete bipartite graphs. More precisely, he proved that

$$\text{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6. \quad (2)$$

Let $D_n$ denote the discrete graph on $n$ vertices and let $G_1$ be the special graph on six vertices which can be seen in Figure 1. In Section 2, we give the crossing number of the join product of $G_1$ with the graph $D_n$. Using this result and properties of the Zip product operation, for any tree $T$ with no vertex of degree two we give the crossing number of $G_1 \square T$ in Section 3. In Section 4, all known results concerning the crossing numbers of Cartesian products of stars with graphs on six vertices are collected. Moreover, we establish the crossing numbers of $G_i \square S_n$ for several other graphs $G_i$ of order six. In the proofs of the paper, we will often use the term “region” also in non-planar drawings. In this case, crossings are considered to be vertices of the “map”.

![Figure 1. The graph $G_1$ on six vertices.](image-url)
2. THE CROSSING NUMBER OF $G_1 + D_n$

The graph $G_1$ in Figure 1 consists of one 6-cycle $abcdef$, denoted by $C_6(G_1)$ in the paper, and of one edge $be$ which, together with the edges of the 6-cycle, form two 4-cycles. The graph $G_1 + D_n$ consists of one copy of the graph $G_1$ and $n$ vertices $t_1, t_2, \ldots, t_n$, where every vertex $t_i$, $i = 1, 2, \ldots, n$, is adjacent to every vertex of $G_1$, see Figure 2. For $i = 1, 2, \ldots, n$, let $T_i$ denote the subgraph induced by six edges incident with the vertex $t_i$ and let $F_i = G_1 \cup T_i$. To simplify the notation, let $G_1(n)$ denote the graph $G_1 + D_n$ in this paper. In Figure 2, one can easily see that

$$G_1 + D_n = G_1(n) = G_1 \cup K_{6,n} = G_1 \cup \left( \bigcup_{i=1}^{n} T_i \right).$$

(3)

Figure 2. The drawing of the graph $G_1 + D_n$.

Lemma 1. $cr(G_1 + D_2) = 2$.

Proof. The graph $G_1 + D_2$ consists of the subgraph $G_1$ and two subgraphs $T^1$ and $T^2$. As $G_1 + D_2$ contains $K_{3,3}$ as a subgraph, $cr(G_1 + D_2) \geq 1$. If there is a drawing of $G_1 + D_2$ with only one crossing, then at least one of the subgraphs $T^1$ and $T^2$ does not cross $G_1$. Without loss of generality, assume that $T^1$ does not cross $G_1$. Then, in the view of the subdrawing of $G_1$, all vertices of $G_1$ are placed on the boundary of one, say outside, region. The subgraph $T^1$ is placed in this region and, as $T^1$ and $G_1$ do not cross each other, the edge $be$ of $G_1$ does not cross $T^1$, too. Hence, if the edges of $C_6(G_1)$ do not cross each other, the subdrawing of $G_1 \cup T^1$ divides the plane such that at most four vertices of $G_1$ are on the boundary of every region as shown in Figure 3(a). If the edges of $C_6(G_1)$ cross each other, then they cross only once and no region of the subdrawing of $C_6(G_1) \cup T^1$ has more than four vertices of $G_1$ on its boundary, see Figure 3(b). This forces, that the subgraph $T^2$ crosses $G_1 \cup T^1$ at least twice in both cases. Thus, $cr(G_1 + D_2) \geq 2$. On the other hand, in Figure 2 it is easy to see that $cr(G_1 + D_2) \leq 2$. This completes the proof.
Theorem 2. \( cr(G_1 + D_n) = 6\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 2\left\lceil \frac{n}{2} \right\rceil \) for \( n \geq 1 \).

Proof. The drawing in Figure 2 shows that \( cr(G_1 + D_n) \leq 6\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 2\left\lceil \frac{n}{2} \right\rceil \) and that the theorem is true if the equality holds. We prove the reverse inequality by induction on \( n \). As the graph \( G_1 + D_1 \) is planar, the case \( n = 1 \) is trivial. Lemma 1 implies that the result is true for the case \( n = 2 \).

Suppose now that for \( n \geq 3 \)

\[
(4) \quad cr(G_1(n-2)) \geq 6\left\lceil \frac{n-2}{2} \right\rceil \left\lceil \frac{n-3}{2} \right\rceil + 2\left\lceil \frac{n-2}{2} \right\rceil
\]

and consider such a drawing \( D \) of \( G_1(n) \) that

\[
(5) \quad cr_D(G_1(n)) < 6\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 2\left\lceil \frac{n}{2} \right\rceil.
\]

The drawing \( D \) has the following properties:

Property 1. \( cr_D(T^i, T^j) \neq 0 \) for all \( i, j = 1, 2, \ldots, n, i \neq j \).

Assume that for some \( i \neq j, \ cr_D(T^i, T^j) = 0 \). The subgraph \( G_1 \cup T^i \cup T^j \) is isomorphic to the graph \( G_1 + D_2 \). If the edges of \( G_1 \) do not cross each other, then, by Lemma 1, \( cr_D(G_1, T^i \cup T^j) \geq 2 \). If both \( T^i \) and \( T^j \) cross \( G_1 \), then \( cr_D(G_1, T^i \cup T^j) \geq 2 \) again. The last possibility is that one of \( T^i \) and \( T^j \), say \( T^i \), does not cross \( G_1 \), and the edges of \( G_1 \) cross each other. This forces that the edges of \( C_6(G_1) \) cross each other, otherwise the edge be crosses \( T^i \), see Figure 3(a). Hence, aside from the number of internal crossings in \( G_1 \), the subdrawing of \( C_6(G_1) \cup T^i \) divides the plane in such a way that there is no region with more than four vertices of \( C_6(G_1) \) on its boundary. This implies that the edges of \( T^j \) joining \( t_j \) with the vertices of \( G_1 \) cross \( C_6(G_1) \cup T^i \) at least twice. As \( T^j \) does not cross \( T^i \), \( cr_D(G_1, T^i \cup T^j) \geq 2 \) again. Moreover, as \( cr(K_{6,3}) = 6 \), in \( D \), every subgraph \( T^k, k = 1, 2, \ldots, n, k \neq i, j \), crosses \( T^i \cup T^j \) at least six times. Since \( G_1(n) = G_1 + D_n = G_1(n-2) \cup (T^i \cup T^j) \) and \( G_1(n-2) = K_{6,n-2} \cup G_1 \), using (1) and (4) we have

\[
(1) \quad cr(K_{6,3}) = 6,
\]

\[
(4) \quad cr(G_1(n-2)) \geq 6\left\lceil \frac{n-2}{2} \right\rceil \left\lceil \frac{n-3}{2} \right\rceil + 2\left\lceil \frac{n-2}{2} \right\rceil
\]

and

\[
(5) \quad cr_D(G_1(n)) < 6\left\lceil \frac{n}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil + 2\left\lceil \frac{n}{2} \right\rceil.
\]
\[\text{cr}_D(G_1(n)) = \text{cr}_D(G_1(n - 2)) + \text{cr}_D(T^i \cup T^j) + \text{cr}_D(K_{6,n-2}, T^i \cup T^j) + \text{cr}_D(G_1, T^i \cup T^j) + \text{cr}_D(K_{6}, T^i \cup T^j) + \text{cr}_D(G_1, T^i \cup T^j) + \text{cr}_D(K_{6}, T^i \cup T^j)
\]

This contradicts (5), and therefore \(\text{cr}_D(T^i, T^j) \neq 0\) for all \(i, j = 1, 2, \ldots, n, \ i \neq j\).

**Property 2.** The edges of \(G_1\) are crossed less than \(2\left\lfloor \frac{n}{2} \right\rfloor\) times in \(D\).

Using (1) and (3) together with \(\text{cr}(K_{6,n}) = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor\) we have

\[
\text{cr}_D(G_1(n)) = \text{cr}_D(K_{6,n}) + \text{cr}_D(G_1) + \text{cr}_D(K_{6,n}, G_1)
\]

\[
\geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \text{cr}_D(G_1) + \text{cr}_D(K_{6,n}, G_1).
\]

This, together with the assumption (5), implies that

(6) \[\text{cr}_D(G_1) + \text{cr}_D(K_{6,n}, G_1) < 2\left\lfloor \frac{n}{2} \right\rfloor\]

and hence, the edges of \(G_1\) are crossed less than \(2\left\lfloor \frac{n}{2} \right\rfloor\) times in \(D\).

The inequality (6) immediately implies the next property.

**Property 3.** In \(D\), there is at least one subgraph \(T^i\) which does not cross \(G_1\).

Assume, without loss of generality, that \(\text{cr}_D(G_1, T^i) = 0\). Then for the subgraph \(F^n = G_1 \cup T^n\) of the graph \(G_1(n)\) we have the next property.

**Property 4.** In \(D\), there is at least one subgraph \(T^i, i \in \{1, 2, \ldots, n - 1\}\), for which \(\text{cr}_D(F^n, T^i) \leq 3\).

Otherwise, as \(G_1(n) = K_{6,n-1} \cup F^n\) and \(\text{cr}_D(F^n) = \text{cr}_D(G_1 \cup T^n) = 0\), we have

\[
\text{cr}_D(G_1(n)) = \text{cr}_D(K_{6,n-1}) + \text{cr}_D(F^n) + \text{cr}_D(K_{6,n-1}, F^n)
\]

\[
\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 4(n - 1) \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lceil \frac{n}{2} \right\rceil,
\]

which contradicts (5).

Consider now the subdrawing \(D^*\) of \(D\) induced by \(F^n\). As we assumed above, no edge of \(T^n\) crosses \(G_1\). Our next analysis depends on whether or not the edges of the 6-cycle \(C_6(G_1)\) cross each other in \(D^*\). Assume first, that the edges of \(C_6(G_1)\) do not cross each other. Since \(\text{cr}_D(G_1, T^n) = 0\), in \(D^*\), all edges of \(T^n\) are placed in one of two regions, say outside, in the view of the subdrawing of \(C_6(G_1)\) and the edge be of \(G_1\), not belonging to \(C_6(G_1)\), is placed inside the 6-cycle \(C_6(G_1)\). The unique such drawing \(D^*\) is shown in Figure 3(a). If, in \(D\), some vertex \(t_i, i \in \{1, 2, \ldots, n - 1\}\), is placed inside \(C_6(G_1)\), then \(G_1\) is crossed by at least two edges joining \(t_i\) with the vertices of \(G_1\). Moreover, by Property 1, \(T^n\) crosses \(T^n\) and therefore, \(\text{cr}_D(F^n, T^i) \geq 3\). Outside \(C_6(G_1)\) there are two vertices on the boundary of every region. Hence, for all other vertices \(t_i\) not placed inside \(C_6(G_1)\), the edges of \(T^i\) cross the edges of \(F^n\) at least five times.
Let \( r \) be the number of vertices \( t_i, i \in \{1, 2, \ldots, n - 1\} \), which are placed in \( D \) inside the cycle \( C_b(G_1) \). Thus, the corresponding subgraphs \( T^i \) cross the edges of \( G_1 \cup T^n \) at least three times. The calculating of the necessary crossings in \( D \) gives
\[
\text{cr}_D(G_1(n)) = \text{cr}_D(K_{6,n-1}) + \text{cr}_D(F^n) + \text{cr}_D(K_{6,n-1}, F^n) \\
\geq 6 \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 0 + 3r + 5(n - r - 1) \\
= 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor + (5n - 2r - 5 - 6 \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \left\lfloor \frac{n}{2} \right\rfloor).
\]
This, together with the assumption (5), gives
\[
2r > 5n - 5 - 6 \left\lfloor \frac{n-1}{2} \right\rfloor - 2 \left\lfloor \frac{n}{2} \right\rfloor - 2 \left\lfloor \frac{n}{2} \right\rfloor.
\]
This contradicts Property 2, because every of \( r \) subgraphs \( T^i \) crosses \( G_1 \) at least twice.

Let us turn to the case \( \text{cr}_D(C_b(G_1)) = 0 \). As \( \text{cr}_D(G_1, T^n) = 0 \), the subdrawing of \( C_b(G_1) \) induced by \( D^* \) divides the plane in such a way that all vertices of \( C_b(G_1) \) are placed on the boundary of one region, say outside. The whole subgraph \( T^n \) is placed in this region and on the boundary of every subregion formed by the edges of \( T^n \) and \( G_1 \) there are two vertices of \( G_1 \). The edges of \( C_b(G_1) \) cross each other at least once and, by Property 1 and Property 4, on the boundary of at least one region inside \( C_b(G_1) \) there are at least four vertices of \( G_1 \). Because of good drawing, this requirement forces that \( C_b(G_1) \) has only one internal crossing and the unique subdrawing of \( C_b(G_1) \cup T^n \) is shown in Figure 3(b). The edge \( be \) of \( G_1 \) not belonging to \( C_b(G_1) \) must be adjacent to at least one vertex of \( C_b(G_1) \) on the boundary of the region \( \alpha \). Assume first that both vertices of \( G_1 \) incident with the edge \( be \) are placed on the boundary of the region \( \alpha \). These vertices are in distance three. As two edges incident with the same vertex do not cross each other in the good drawing \( D^* \) and the edge \( be \) does not cross \( T^n \), the unique possibility for placing the edge \( be \) is shown in Figure 3(c). It is easy to verify that, in this case, every subgraph \( T^i, i = 1, 2, \ldots, n - 1 \), crosses the edges of \( F^n \) at least four times in \( D \). This contradicts Property 4. Hence, the edge \( be \) is adjacent to only one vertex on the boundary of the region \( \alpha \) in such a way that it splits the region \( \alpha \) into two subregions. So, in \( D^* \), there is no region with at least four vertices on its boundary. In this case, in \( D \), every subgraph \( T^i, i = 1, 2, \ldots, n - 1 \), crosses \( F^n \) at least four times. This contradiction with Property 4 completes the proof.

### 3. The Crossing Number of \( G_1 \square T \)

In this section, using Zip product operation introduced in [3], we establish the crossing number of the Cartesian product of the graph \( G_1 \) with any tree without vertices of degree two. For better reading, we repeat the related terms and results
introduced by Bokal in [3] and [4].

For \( i = 1, 2 \), let \( H_i \) be a graph and let \( v_i \in V(H_i) \) be its vertex of degree \( d \).
Let \( N_i = N_{H_i}(v_i) \) be the set of neighbouring vertices of \( v_i \) and let \( \sigma : N_1 \rightarrow N_2 \) be a bijection. We call \( \sigma \) a zip function of the graphs \( H_1 \) and \( H_2 \) at vertices \( v_1 \) and \( v_2 \).

The zip product of the graphs \( H_1 \) and \( H_2 \) according to \( \sigma \) is defined to be the graph \( H_1 \circ_\sigma H_2 \), obtained from the disjoint union of \( H_1 - v_1 \) and \( H_2 - v_2 \) after adding the edges \( u\sigma(u), u \in N_1 \).

Let \( H_1 \circ_\sigma H_2 \) denote the set of all graphs obtained as a zip product \( H_1 \circ_\sigma H_2 \) for some zip function \( \sigma : N_1 \rightarrow N_2 \).

Let \( D_i \) be a drawing of a graph \( H_i \). The drawing imposes a cyclic ordering of the edges incident with \( v_i \), which can be extended to its neighbourhood \( N_i \).
Let the bijection \( \pi_i : N_i \rightarrow \{1, \ldots, d\} \) be one of the corresponding labellings. We define \( \sigma : N_1 \rightarrow N_2, \sigma = \pi^{-1}_2 \pi_1 \), to be the zip function of the drawings \( D_1 \) and \( D_2 \) at vertices \( v_1 \) and \( v_2 \). The zip product of \( D_1 \) and \( D_2 \) according to \( \sigma \) is the drawing \( D_1 \circ_\sigma D_2 \), obtained from \( D_1 \) by adding a mirrored copy of \( D_2 \) that has \( v_2 \) on the infinite face disjointly into some face of \( D_1 \) that contains the vertex \( v_1 \), removing the vertices \( v_1 \) and \( v_2 \) together with small disks around them from the drawings, and then joining the edges according to the function \( \sigma \). As \( \sigma \) respects the ordering of the edges around \( v_1 \) and \( v_2 \), the edges between \( D_1 \) and \( D_2 \) may be drawn without introducing any new crossings. Clearly \( D_1 \circ_\sigma D_2 \) is a drawing of \( H_1 \circ_\sigma H_2 \), which implies the following lemma.

**Lemma 3** [3]. For \( i = 1, 2 \), let \( D_i \) be an optimal drawing of \( H_i \), let \( v_i \in V(H_i) \) be a vertex of degree \( d \), and let \( \sigma \) be a zip function of \( D_1 \) and \( D_2 \) according to \( v_1 \) and \( v_2 \). Then \( \text{cr}(H_1 \circ_\sigma H_2) \leq \text{cr}(H_1) + \text{cr}(H_2) \).

Let \( G(i) = G + D_i \) be the suspension of order \( i \) of a graph \( G \). The vertices of \( D_i \) are called apices of \( G(i) \). For a multiset \( L \subseteq V(H_2) \), we denote with \( H_1 \sqcup_L H_2 \) the capped Cartesian product of graphs \( H_1 \) and \( H_2 \), that is, the graph obtained by adding a distinct vertex \( v' \) to \( H_1 \sqcup H_2 \) for each copy of a vertex \( v \in L \) and joining \( v' \) to all the vertices of \( H_1 \sqcup \{v\} \). We call each \( v' \) a cap of \( v \). Let \( \chi_L(v) \) denote the multiplicity of \( v \) in \( L \) and \( \ell(v) := \deg_{H_2}(v) + \chi_L(v) \). An edge \( uv \in E(H_2) \) is unbalanced if \( \ell(u) \neq \ell(v) \). Let \( \beta(H_2) \) be the number of unbalanced edges of \( H_2 \).

A drawing \( D \) of \( G(i) \) is apex-homogeneous if there exists a permutation \( \rho \) of the vertices of \( G \) such that the vertex rotation around every apex in \( D \) is \( \rho \) or \( \rho^{-1} \).
Two drawings \( D^{(i)} \) of \( G(i) \) and \( D^{(j)} \) of \( G(j) \) are pairwise apex-homogeneous, if they are apex-homogeneous with respect to the same permutation \( \rho \). A graph \( G \) has all apex-homogeneous drawings if there exist drawings \( D^{(i)} \) of \( G(i) \), \( i \geq 1 \), such that every two of them are pairwise apex-homogeneous. The next result given by Bokal enables us to establish the crossing numbers of the Cartesian products of our graph \( G_1 \) with all trees which do not contain vertices of degree two.

**Theorem 4** [4]. Let \( G \) be a graph of order \( n \), let \( T \) be a tree, and let \( L \subseteq V(T) \) be a multiset with either \( \ell(v) \geq 3 \) or, if \( G \) has a dominating vertex, \( \ell(v) \geq 2 \) for
every \( v \in V(T) \). Define

\[
B = \sum_{v \in V(T)} \text{cr}(G(\ell(v))).
\]

Then, \( B \leq \text{cr}(G \square L_T) \leq B + \frac{\beta(T)}{2} \binom{n}{2} \). Also, \( \text{cr}(G \square L_T) = B \) whenever \( G \) has all apex-homogeneous drawings.

Consider now a graph \( H_{G_1} \) obtained by joining all vertices of \( G_1 \) to six vertices of a connected graph \( H \) such that every vertex of \( G_1 \) be adjacent to exactly one vertex of \( H \). Let \( H_{G_1}^* \) be the graph obtained from \( H_{G_1} \) by contracting the edges of \( G_1 \).

**Lemma 5.** \( \text{cr}(H_{G_1}^*) \leq \text{cr}(H_{G_1}) \).

**Proof.** Assume an optimal drawing of \( H_{G_1} \). We remark that the edges of \( G_1 \) can cross each other in this drawing, as well as they can be crossed by some other edges of \( H_{G_1} \). Let us denote by \( x_1 \) and \( x_2 \) the number of crossings on the edges \( ab \) and \( bc \) of \( G_1 \), respectively. Similarly, let \( x_3 \) and \( x_4 \) denote the number of crossings on the edges \( ed \) and \( ef \), see one of the possible cases in Figure 4(a).

Let us contract the graph \( G_1 \) into the vertex \( b \) and connect this vertex with six vertices of \( H \) in such a way that the segments along the edges \( ab \) and \( bc \) in \( G_1 \) are used twice and the segments along the edges \( af, be, \) and \( cd \) only once. The segments along the edges \( ef \) and \( ed \) are not used as shown in Figure 4(b). In the worst case, at most \( x_1 + x_2 \) new crossings can appear only on the new edges in the segments along the edges \( ab \) and \( bc \). But \( x_3 + x_4 \) crossings on the edges \( ed \) and \( ef \) do not appear in the new drawing. Hence, if \( x_1 + x_2 \leq x_3 + x_4 \), the resulting drawing of \( H_{G_1}^* \) does not have more crossings than the original drawing of \( H_{G_1} \). If the edges \( ed \) and \( ef \) cross some other edges of \( G_1 \) in the original drawing, these crossings do not appear in the drawing of \( H_{G_1}^* \), and the number of crossings is less than in the drawing of \( H_{G_1} \). Moreover, possible crossings among the edges of \( G_1 \) not appeared on the edges \( ed \) and \( ef \) cannot be crossings in a good drawing of \( H_{G_1}^* \). Thus, if \( x_1 + x_2 \leq x_3 + x_4 \) our drawing after contracting the graph \( G_1 \) into vertex \( b \) does not have more crossings than the original drawing of \( H_{G_1} \). Due to symmetry of the graph \( G_1 \), the same holds if \( x_3 + x_4 \leq x_1 + x_2 \).

Assume that the statement of Lemma 5 is not true. Then there is a good drawing of the graph \( H_{G_1} \) in which \( x_1 + x_2 > x_3 + x_4 \) and \( x_3 + x_4 > x_1 + x_2 \). This contradiction completes the proof.

In the rest of the section, we give the crossing numbers of the Cartesian products of the graph \( G_1 \) with trees \( T \) not containing vertices of degree two. For the special case \( T = S_n \), the crossing number of the graph \( G_1 \square S_n \) is obtained.
Theorem 6. Let $T$ be any tree of order $n$ with no vertices of degree two and let $d_i$ be the number of vertices of degree $i$ in $T$. Then

$$\text{cr}(T \Box G_1) = \sum_{i=1}^{\Delta(T)} (d_i \cdot \text{cr}(G_1 + D_i)).$$

Proof. Let $T'$ be the tree obtained from $T$ by removing all vertices of degree one in $T$. For a vertex $v$ of $T'$, let $r_v$ be the number of $T$-leaves adjacent to $v$ in $T$, and let $L$ be the set of vertices in $T'$, each with multiplicity $r_v$. As $T$ has no vertices of degree two, $\ell(v) = d_{T'}(v) + r(v) = d_T(v) \geq 3$ for all $v \in V(T')$. The drawing in Figure 2 shows that $G_1$ has all apex-homogeneous drawings. Thus, by Theorem 4, $\text{cr}(G_1 \Box L T') = \sum_{i=1}^{\Delta(T)} (d_i \cdot \text{cr}(G_1 + D_i)).$

The graph $G_1 \Box L T'$ is obtained from $G_1 \Box T$ by contracting all the $G_1$ edges corresponding to $G_1\{u\}$, where $u$ is a leaf of $T$. Then, the iterative applications of Lemma 5 show that $\text{cr}(G_1 \Box L T') \leq \text{cr}(G_1 \Box T')$. On the other hand, the graph $G_1 \Box L T'$ is obtained from $G_1 \Box L T'$ by zipping a copy of $G' = G_1 + \{v\}$ at each cap of $G_1 \Box L T'$. As $G' = G_1 + \{v\}$ is planar, Lemma 3 implies that $\text{cr}(G_1 \Box L T') \geq \text{cr}(G_1 \Box T)$, and the proof is done.

By Theorem 2, $\text{cr}(G_1 + D_n) = 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor$. As $\text{cr}(G_1 + D_1) = 0$, for the special tree $S_n$ we have the next result.

Figure 5. The drawing of $G_1 \Box S_n$ with $6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor$ crossings.
Corollary 7. \( \text{cr}(G_1 \Box S_n) = 6 \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor \) for \( n \geq 1 \).

The drawing of the Cartesian product \( G_1 \Box S_n \) with \( 6 \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor \) crossings is shown in Figure 5. In the next section, we will use parts of similar drawings for some special graphs \( G_i \) of order six.

4. The Crossing Number of \( G_i \Box S_n \) for Some Graphs of Order Six

There are 112 connected graphs of order six. In the rest of the paper, we collect old and new results concerning the crossing numbers of Cartesian products of eighteen graphs on six vertices with stars. In Section 3, the crossing number of \( G_1 \Box S_n \) is established. In Figure 6, the other 16 graphs \( G_i, i = 2, 3, \ldots, 17 \), on six vertices are presented. The last graph, not presented in Figure 6, is the complete tripartite graph \( K_{2,2,2} \).

As corollary of Theorem 4, in [4] Bokal estimated the following results:

Corollary 8 [4]. Let \( T \) be a tree and \( n \geq 1 \). Then, for \( d_v = \text{deg}_T(v) \),

\[
\text{cr}(S_n \Box T) = \sum_{v \in V(T), d_v \geq 2} \text{cr}(K_{d_v, n}) .
\]

Corollary 9 [4]. Let \( n \geq 1 \) be any integer and \( T \) a subcubic tree with \( n_2 \) vertices of degree two and \( n_3 \) vertices of degree three. Then,

\[
\text{cr}(S_n \Box T) = \left\lfloor \frac{n}{2} \right\rfloor (n_2 + 2n_3) \left\lfloor \frac{n-1}{2} \right\rfloor + n_3 .
\]

In [10], Huang and Zhao proved that \( \text{cr}(K_{1,4,n}) = 4 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2 \left\lfloor \frac{n}{2} \right\rfloor \). Mei and Huang proved in [21] that the crossing number of the complete tripartite graph \( K_{1,5,n} \) is \( 6 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 4 \left\lfloor \frac{n}{2} \right\rfloor \). This, together with Asano’s [1] result \( \text{cr}(K_{1,3,n}) = 2 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \) and the known fact that \( \text{cr}(K_{1,2,n}) = \text{cr}(K_{3,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \) enables us to state:
Corollary 10. Let \( n \geq 1 \) be any integer and \( T \) a tree with maximum degree five. Let \( T \) have \( n_2 \) vertices of degree two, \( n_3 \) vertices of degree three, \( n_4 \) vertices of degree four, and \( n_5 \) vertices of degree five. Then,

\[
\text{cr}(S_n \Box T) = \left\lfloor \frac{n}{2} \right\rfloor \left( (n_2 + 2n_3 + 4n_4 + 6n_5) \left\lfloor \frac{n-1}{2} \right\rfloor + n_3 + 2n_4 + 4n_5 \right).
\]

In [3], Bokal proved the conjecture given by Jendrol’ and Ščerbová [11] that \( \text{cr}(P_m \Box S_n) = (m-2)\left\lfloor \frac{n-1}{2} \right\rfloor \) for the path \( P_m \) of length \( m - 1 \). Hence, \( \text{cr}(G_2 \Box S_n) = \text{cr}(P_5 \Box S_n) = 4\left\lfloor \frac{n-1}{2} \right\rfloor \) for the graphs \( G_3, G_4, \) and \( G_5 \) in Figure 6. Corollary 9 implies that \( \text{cr}(G_3 \Box S_n) = \text{cr}(G_5 \Box S_n) = 4\left\lfloor \frac{n-1}{2} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor \) and \( \text{cr}(G_4 \Box S_n) = 4\left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \). Moreover, by Corollary 10, \( \text{cr}(G_6 \Box S_n) = 5\left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \). This results enables us to give the crossing numbers of Cartesian products of stars with the graphs \( G_7, G_8, G_9, \) and \( G_{10} \). The graph \( G_7 \) contains \( G_6 \) as a subgraph, so \( \text{cr}(G_7 \Box S_n) \geq 5\left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \). On the other hand, in Figure 7(a) there is a left-side drawing of the graph \( G_7 \Box S_n \) with \( \left\lceil \frac{n}{2} \right\rceil \) non-central copies of \( G_7 \) and the right \( \left\lfloor \frac{n}{2} \right\rfloor \) non-central copies of \( G_7 \) are omitted (compare with the drawing of the graph \( G_1 \Box S_n \) in Figure 5). The drawing in Figure 7(a) implies that there is a drawing of the graph \( G_7 \Box S_n \) with \( 5\left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \) crossings and therefore, \( \text{cr}(G_7 \Box S_n) = 5\left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor \).

(a) (b)
(c) (d)

Figure 7. The half-drawings of the graphs \( G_7 \Box S_n, G_8 \Box S_n, G_{10} \Box S_n, \) and \( G_{17} \Box S_n \).
The graph $G_8$ contains $G_5$ as a subgraph, so $\text{cr}(G_8 \Box S_n) \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. Figure 7(b) shows that there is a drawing of the graph $G_8 \Box S_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ crossings. This states that $\text{cr}(G_8 \Box S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. Both graphs $G_9$ and $G_{10}$ contain $G_4$. This implies that $\text{cr}(G_9 \Box S_n) \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ and $\text{cr}(G_{10} \Box S_n) \geq 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$. In the half-drawing of the graph $G_{10} \Box S_n$ in Figure 7(c) we can see that the crossing number of the graph $G_{10} \Box S_n$ is at most $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$. Using the fact that the graph $G_9 \Box S_n$ is a subgraph of $G_{10} \Box S_n$, we have that $\text{cr}(G_9 \Box S_n) = \text{cr}(G_{10} \Box S_n) = 4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$.

In [21], Mei and Huang proved that $\text{cr}(K_{1,5,n}) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$. Since the graph $S_5 \Box S_n = G_{11} \Box S_n$ is a subdivision of the graph $K_{1,5,n}$, $\text{cr}(G_{11} \Box S_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$. All graphs $G_i$, $i = 12, 13, \ldots, 17$, in Figure 6 contain the graph $G_{11}$ as a subgraph and therefore, $\text{cr}(G_i \Box S_n) \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ for $i = 12, 13, \ldots, 17$. The left-side drawing of the graph $G_{17} \Box S_n$ in Figure 7(d) implies that there is a drawing of the graph $G_{17} \Box S_n$ with at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ crossings. Thus, we have the same lower and upper bound for the crossing number of all six considered graphs, which implies that $\text{cr}(G_i \Box S_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 4\lfloor \frac{n}{2} \rfloor$ for all $i = 12, 13, \ldots, 17$.

In [9], Ho proved that the crossing number of the complete 4-partite graph $K_{2,2,2,2,n}$ is $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 3n$. Using this result, in [5] it is shown that the crossing number of the Cartesian product $K_{2,2,2,2} \Box S_n$ is $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 6n$.

5. Comments

Except the graph $K_5 \Box S_n$, the known results concerning crossing numbers of Cartesian products of stars with graphs of order five are collected in [18]. The crossing number of $K_5 \Box S_n$ was presented in [20]. For six remaining graphs $G_i$ on five vertices, the problem of estimating $\text{cr}(G_i \Box S_n)$ is still open, even though some incorrect result were published. For example, in [8] the incorrect proof states that for the tree $T$ on five vertices with one vertex of degree two and one vertex of degree three, the crossing number of $T \Box S_n$ is $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. This contradicts Corollary 9.

We suppose that the application of Zip product operation can be used to estimate the unknown values of the crossing number for Cartesian products of some graphs on five vertices with trees, and also with stars. The same we expect for larger graphs, namely for some of 94 remaining graphs of order six.

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