Edge-superconnectivity
of semiregular cages with odd girth

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Abstract

A graph is said to be edge-superconnected if each minimum edge-cut consists of all the edges incident with some vertex of minimum degree. A graph \( G \) is said to be a \( \{d, d+1\} \)-semiregular graph if all its vertices have degree either \( d \) or \( d+1 \). A smallest \( \{d, d+1\} \)-semiregular graph \( G \) with girth \( g \) is said to be a \( (\{d, d+1\}; g) \)-cage. We show that every \( (\{d, d+1\}; g) \)-cage with odd girth \( g \) is edge-superconnected.

1 Introduction

We only consider undirected simple graphs without loops or multiple edges. Unless otherwise stated, we follow [9] for basic terminology and definitions. Let \( G \) stand for a graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The distance \( d_G(u, v) = d(u, v) \) between two vertices of the graph \( G \) is the length of a shortest path between \( u \) and \( v \), and the diameter of \( G \) denoted by \( diam(G) \) is the maximum distance between any pair of vertices; when \( G \) is not connected, then \( diam(G) = +\infty \). For \( w \in V \) and \( S \subset V \), \( d(w, S) = d_G(w, S) = \min\{d(w, s) : s \in S\} \) denotes the distance between \( w \) and \( S \). For every \( S \subset V \) and every nonnegative integer \( r \geq 0 \), \( N_r(S) = \{w \in V : d(w, S) = r\} \) denotes the neighborhood of \( S \) at distance \( r \). Thus the set of vertices adjacent to a vertex \( v \) is \( N(v) = N_1(\{v\}) \), and
the degree of a vertex \( v \) in \( G \) is \( \text{deg}_G(v) = \text{deg}(v) = |N(v)| \), whereas the minimum degree \( \delta = \delta(G) \) is the minimum degree over all vertices of \( G \). A graph is called \( r \)-regular if every vertex of the graph has degree \( r \).

A graph \( G \) is called connected if every pair of vertices is joined by a path. An edge-cut in a graph \( G \) is a set \( W \) of edges of \( G \) such that \( G - W \) is disconnected. A graph is \( k \)-edge-connected if every edge-cut contains at least \( k \) edges. If \( W \) is a minimal edge-cut of a connected graph \( G \), then necessarily, \( G - W \) contains exactly two components. The edge-connectivity \( \lambda = \lambda(G) \) of a graph \( G \) is the minimum cardinality of an edge-cut of \( G \). A classic result is \( \lambda \leq \delta \) for every graph \( G \). A graph is maximally edge-connected if \( \lambda = \delta \).

One might be interested in more refined indices of reliability. Even two graphs with the same edge-connectivity \( \lambda \) may be considered to have different reliabilities. As a more refined index than the edge-connectivity, edge-superconnectivity is proposed in [6, 7]. A subset of edges \( W \) is called trivial if it contains the set of edges incident with some vertex of the graph. Clearly, if \( |W| \leq \delta - 1 \), then \( W \) is nontrivial. A graph is said to be edge-superconnected if \( \lambda = \delta \) and every minimum edge-cut is trivial.

The degree set \( D \) of a graph \( G \) is the set of distinct degrees of the vertices of \( G \). The girth \( g(G) \) is the length of a shortest cycle in \( G \). A \((D; g)\)-graph is a graph having degree set \( D \) and girth \( g \). Let \( n(D; g) \) denote the least order of a \((D; g)\)-graph. Then a \((D; g)\)-graph with order \( n(D; g) \) is called a \((D; g)\)-cage. If \( D = \{r\} \) then a \((D; g)\)-cage is a \((r; g)\)-cage. When \( D = \{r, r + 1\} \), we refer to \((D; g)\)-cages as semiregular cages.

The existence of \((r; g)\)-cages was proved by Erdös and Sachs [10] in the decade of the 60’s, and using this result Chartrand et al. [8] proved the existence of \((D; g)\)-cages. Some of the structural properties of \((r; g)\)-cages that have been studied are the vertex and the edge connectivit; concerning this problem Fu, Huang and Rodger [11] conjectured that every \((r; g)\)-cage is \( r \)-connected, and they proved the statement for \( r = 3 \). Other contributions supporting this conjecture can be seen in [15, 16, 17, 20]. Moreover, some structural properties of \((r; g)\)-cages have been extended for \((D; g)\)-cages, for example the monotonicity of the order with respect to the girth (see Theorem 1) and the upper bound for the diameter (see Theorem 2). The edge-superconnectivity of cages was established in [18, 19]. For semiregular cages, it has been proved in [3] that they are maximally edge connected. The main objective of this work is to prove that every \((\{d, d +
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1); g)-cage with odd girth $g \geq 5$ is edge-superconnected. With this aim we need the following two results.

**Theorem 1** [4] Let $g_1, g_2$ be two integers such that $3 \leq g_1 < g_2$. Then $n(\{d, d+1\}; g_1) < n(\{d, d+1\}; g_2)$.

**Theorem 2** [5] The diameter of a $(\{d, d+1\}; g)$-cage is at most $g$.

## 2 Main theorem

In order to study the edge-superconnectivity of a graph in terms of its diameter and its girth, the following results were established [1, 2, 13].

**Proposition 3** Let $G = (V, E)$ be a connected graph with minimum degree $\delta \geq 2$ and girth $g$. Let $W \subset E$ be a minimum nontrivial edge-cut, let $H_i$ be a component of $G - W$, and let $W_i \subset V(H_i)$ be the set of vertices of $H_i$ which are incident with some edge in $W$, $i = 0, 1$. Then there exists some vertex $x_i \in V(H_i)$ such that

(a) [1, 13] $d(x_i, W_i) \geq \lfloor (g - 1)/2 \rfloor$, if $|W_i| \leq \delta - 1$.

(b) [2] $d(x_i, W_i) \geq \lceil (g - 3)/2 \rceil$, if $|W| \leq \xi - 1$, where $\xi = \min \{\deg(u) + \deg(v) - 2 : uv \in E\}$ is the minimum edge-degree of $G$.

For every minimum edge-cut $W$ of $G$ such that $H_0, H_1$ are the two components of $G - W$, we will write henceforth $W = [W_0, W_1]$ with $W_0 \subset V(H_0)$ and $W_1 \subset V(H_1)$ containing all endvertices of the edges in $W$. Note that $|W_i| \leq |W|$, $i = 0, 1$. From now on, let

$$\mu_i = \max \{d(x, W_i) : x \in V(H_i)\}, \quad i = 0, 1.$$ 

When $W$ is nontrivial and $|W| \leq \xi - 1$, it follows from Proposition 3 that $\mu_i \geq \lceil (g - 3)/2 \rceil$. Likewise, $\mu_0$ and $\mu_1$ satisfy some other basic properties shown in next lemma.

**Lemma 4** Let $G = (V, E)$ be a connected graph with minimum degree $\delta \geq 3$ and odd girth $g \geq 5$. Let $W = [W_0, W_1] \subset E$ be a minimum nontrivial edge-cut with cardinality $|W| \leq \delta$. Let $G - W = H_0 \cup H_1$, where $W_i \subset V(H_i)$. If $\mu_i = \lfloor (g - 3)/2 \rfloor$ the following statements hold:

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(i) $|W_i| = |W| = \delta$, and every $a \in W_i$ is incident to a unique edge of $W$.

(ii) Every vertex $z \in V(H_i)$ such that $d(z, W_i) = \mu_i$ has $\deg(z) = \delta$.

(iii) For every $a \in W_i$ there exists a vertex $x \in V(H_i)$ such that $d(x, W_i) = \delta_i$ and $N_{(g-3)/2}(x) \cap W_i = \{a\}$. Further, $N(x)$ can be labeled as $\{u_1, u_2, \ldots, u_\delta\}$, and $W_i$ can be labeled as $\{a_1, a_2, \ldots, a_\delta\}$, where $a_1 = a$, so that $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ and $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ for every $k > 1$. Consequently $|N_{(g-3)/2}(x) \cap W_i, W_{i+1}| = 1$ and $|N_{(g-3)/2}(u_k) \cap W_i, W_{i+1}| = 1$ (with subscripts taken mod 2). See Figure 2.

![Figure 1: Lemma 4.](image)

**Proof:**

(i) Since $\mu_i = (g - 3)/2$, $d(x, W_i) \leq \mu_i = (g - 3)/2 < (g - 1)/2$ for all $x \in V(H_i)$. Hence from Proposition 3 (a), it follows that $|W_i| \geq \delta$, yielding $|W_i| = \delta$ because $|W_i| \leq |W| \leq \delta$. Observe that $\delta = |W_i| = |W|$ means that $|N(a) \cap W_{i+1}| = 1$ for each vertex $a \in W_i$ (taking the subscripts mod 2).

(ii) First observe that $\mu_i = (g - 3)/2 \geq 1$ since $g \geq 5$. Let us define the following partition of $N(v)$ for all $v \in V(H_i)$

$$S^{-}(v) = \begin{cases} 
\{z \in N(v) : d(z, W_i) = d(v, W_i) - 1\} & \text{if } v \notin W_i; \\
W_{i+1} \cap N(v) & \text{if } v \in W_i.
\end{cases}$$

$$S^{+}(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i) + 1\}$$

$$S^{-}(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i)\}.$$
and taking into account (1) we have
\[ N(z) = S^+(z) \cup S^-(z); \]
\[ |N_{(g-3)/2}(S^+(z)) \cap W_i| \geq |S^+(z)|; \]
\[ |N_{(g-5)/2}(S^-(z)) \cap W_i| \geq |S^-(z)|; \]
\[ N_{(g-3)/2}(S^+(z)) \cap N_{(g-5)/2}(S^-(z)) = \emptyset, \]

because otherwise cycles of length less than the girth \( g \) appear. Since
\[ \delta \leq \text{deg}(z) = |S^+(z)| + |S^-(z)| \]
\[ \leq |N_{(g-3)/2}(S^+(z)) \cap W_i| + |N_{(g-5)/2}(S^-(z)) \cap W_i| \]
\[ \leq |W_i| = \delta \]

it follows that \( \delta = \text{deg}(z) \). Therefore item (ii) holds.

(iii) First let us prove that there exists an edge \( zz' \) such that \( d(z, W_i) = d(z', W_i) = (g-3)/2 \). Otherwise, \( S^-(z) = \emptyset \) for all \( z \) with \( d(z, W_i) = (g-3)/2 \). This implies that for all \( u \in N(z), u \in S^-(z) \) and \( S^w(S^+(u)) = \emptyset \). Further, \( |N_{(g-5)/2}(u) \cap W_i| = 1 \) for all \( u \in N(z) \), because \( \delta = |W_i| = \sum_{u \in N(z)} |N_{(g-5)/2}(u) \cap W_i| \geq \delta \). Hence \( |S^-(u)| = 1 \), and so \( |S^+(u)| + |S^-(u)| = \text{deg}(u) - 1 \geq 2 \). Suppose that \( |S^+(u)| \geq 1 \) for some \( u \in N(z) \). Then as \( N_{(g-3)/2}(z) \cap W_i \) and \( N_{(g-5)/2}(S^+(u)) \cap W_i \) are two vertex disjoint sets we have \( |W_i| \geq |N_{(g-3)/2}(z) \cap W_i| + |N_{(g-5)/2}(S^+(u)) \cap W_i| \geq 1 + \delta + 1 \) which is a contradiction because \( |W_i| = \delta \). Then we must assume that for all \( u \in N(z), |S^+(u)| = \text{deg}(u) - 1 \geq \delta - 1 \geq 2 \). Let \( t \in S^+(u) - z \), according to our first assumption \( S^w(t) = \emptyset \) meaning that \( N(t) = S^-(t) \). Since \( t \) has the same behavior as \( z \) we have \( W_i = N_{(g-3)/2}(S^-(z)) = N_{(g-3)/2}(S^-(t)) \), and as \( 2 < \delta \leq \text{deg}(z) = \text{deg}(t) \), there exist cycles through \{\( z, u, t, w \)\} for some \( w \in W_i \) of length less than \( g \) which is a contradiction.

Hence we may assume that there exists an edge \( zz' \) such that \( d(z, W_i) = d(z', W_i) = (g-3)/2 \). Since \( N_{(g-5)/2}(S^-(z)) \cap W_i, N_{(g-5)/2}(S^-(z')) \cap W_i \) and \( N_{(g-3)/2}(S^w(z') - z) \cap W_i \) are three pairwise disjoint sets because \( g \geq 5 \), and taking into account (1) we have
\[ \delta = |W_i| \geq |N_{(g-5)/2}(S^-(z)) \cap W_i| + |N_{(g-5)/2}(S^-(z')) \cap W_i| + |N_{(g-3)/2}(S^w(z') - z) \cap W_i| \]
\[ \geq |S^-(z)| + |S^-(z')| + |S^w(z') - z| \]
\[ = \text{deg}(z) - 1 + |S^-(z)| \geq \delta. \]
Therefore, all inequalities become equalities, i.e., $|S^-(z)| = 1 = |N_{(g-5)/2}(S^-(z)) \cap W_i|$. So $S^-(z) = \{z_1\}$ and $N(z) = z_1 = S^+(z)$ yielding a partition of $W_i$:

$$W_i = (N_{(g-5)/2}(z_1) \cap W_i) \cup (\cup_{z' \in N(z) - z_1} N_{(g-3)/2}(z') \cap W_i),$$

because for all $z' \in N(z) - z_1$ the sets $N_{(g-3)/2}(z') \cap W_i$ and the set $N_{(g-5)/2}(z_1) \cap W_i$ are mutually disjoint. Thus, $|N_{(g-3)/2}(z') \cap W_i| = 1$ for all $z' \in N(z) - z_1$. Therefore, for every vertex $a \in W_i$ there exists a vertex $x \in (N(z) - z_1) \cup \{z\}$ such that $d(x, W_i) = d(x, a) = (g - 3)/2$ and $N_{(g-3)/2}(x) \cap W_i = \{a\}$. Furthermore, since every vertex $z' \in N(z) - z_1$ has the same behavior as $z$, $N(x)$ can be labeled as $\{u_1, u_2, \ldots, u_k\}$, and $W_i$ can be labeled as $\{a_1, a_2, \ldots, a_3\}$, where $a_1 = a$, so that $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ and $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ for every $k > 1$. Finally, using (i) we obtain $|[N_{(g-3)/2}(x) \cap W_i, W_i+1]| = 1$ and $|[N_{(g-3)/2}(u_k) \cap W_i, W_i+1]| = 1$, which finishes the proof. \[\square\]

A semiregular cage is known to be maximally edge-connected [3]. Now, we are ready to prove that semiregular cages with odd girth are edge-superconnected. As will be seen, Hall’s Theorem is a key point of this study. Recall that if $S$ is a set of vertices in a graph $G$, the set of all neighbors of the vertices in $S$ is denoted by $N(S)$.

**Theorem 5 (Hall’s Theorem)** A bipartite graph with bipartition $(X_1, X_2)$ has a matching which covers every vertex in $X_1$ if and only if

$$|N(S)| \geq |S|$$

for all $S \subset X_1$.

Using Hall’s Theorem Jiang [14] proved the following result.

**Lemma 6** [14] Let $G$ be a bipartite graph with bipartition $(X_1, X_2)$ where $|X_1| = |X_2| = r$. If $G$ contains at least $r^2 - r + 1$ edges, then $G$ contains a perfect matching.

The following lemma is an stronger version of Lemma 6, which is also proved using Hall’s Theorem.

**Lemma 7** Let $\mathcal{B}$ be a bipartite graph with bipartition $(X_1, X_2)$ where $|X_1| = |X_2| = r$. If $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$, then $\mathcal{B}$ contains a perfect matching.

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**Proof:** Let \( B = (X_1, X_2) \) be a bipartite graph with \(|X_1| = |X_2| = r\), \( \delta(B) \geq 1 \) and \(|E(B)| \geq r^2 - r\). We shall apply Hall’s Theorem to prove the lemma; we shall show that for a subset \( S \subseteq X_1 \), \(|N(S)| \geq |S|\). Notice that if \(|S| = 1\), then \(|N(S)| \geq 1 = |S|\) because \( \delta(B) \geq 1 \); and if \( S = X_1 \), \( N(S) = X_2 \) because \( \delta(B) \geq 1 \) implies that each vertex \( u \in X_2 \) must have a neighbor in \( S \), hence \(|S| = |N(S)|\).

Therefore we continue the proof reasoning by contradiction and so assuming that \( 1 \leq |N(S)| < |S| = t \leq r - 1 \). Then the number of edges in \( B \) is at most

\[
|E(B)| = [|S, N(S)] + [|X_1 \setminus S, X_2]| \leq t(t-1) + (r-t)r,
\]

and by hypothesis \(|E(B)| \geq r^2 - r\). Thus \( r^2 - r \leq t(t-1) + (r-t)r \), yielding \( 0 \leq (t-r)(t-1) \), which is an absurdity because \( 1 < t < r \). Therefore \(|N(S)| \geq |S|\) for all \( S \subseteq X_1 \), and by Hall’s Theorem the lemma follows. \(\Box\)

**Theorem 8** Let \( G \) be a \((\{d, d+1\}; g)\)-cage with odd girth \( g \geq 5 \), and \( d \geq 3 \). Then \( G \) is edge-superconnected.

**Proof:** Let us assume that \( G \) is a non edge-superconnected \((\{d, d+1\}; g)\)-cage, and we will arrive at a contradiction. To this end, let us take a minimum nontrivial edge-cut \( W = [W_0, W_1] \subseteq E(G) \) such that \(|W| \leq \delta\).

Let \( G - W = H_0 \cup H_1 \), and let \( W_i \subseteq V(H_i) \) be the set of vertices of \( H_i \) which are incident with some edge in \( W \), \( i = 0, 1 \). From Proposition 3 it follows that \( \mu_i = \max\{d(x, W_i) : x \in V(H_i)\} \geq (g - 3)/2 \), \( i = 0, 1 \). Let \( x_i \in V(H_i) \cap N_{p_i}(W_i) \). As \( G \) is a \((\{d, d+1\}; g)\)-cage, the diameter is at most \( diam(G) \leq g \) by Theorem 2, so we get the following chain of inequalities:

\[
g \geq diam(G) \geq d(x_0, x_1) \geq d(x_0, W_0) + 1 + d(x_1, W_1) = \mu_0 + 1 + \mu_1 \geq g - 2.
\]

If we assume henceforth \( \mu_0 \leq \mu_1 \) (without loss of generality), then either \((g - 3)/2 \leq \mu_0 \leq (g + 1)/2\), or \( \mu_0 = \mu_1 = (g - 1)/2 \). We proceed to study each one of these cases.

In what follows, let \( X_0, X_1 \) be two subsets of \( V(G) \) such that \(|X_0| = |X_1|\). Let \( B_\Gamma \) denote the bipartite graph with bipartition \((X_0, X_1)\) and \( E(B_\Gamma) = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \geq g - 1\} \), where \( \Gamma \) is a certain subgraph of \( G \).

**Case (a):** \( \mu_0 = (g - 3)/2 \).
From Lemma 4 (i), \(|W_0| = d = |V|\) so that each vertex of \(W_0\) is incident to a unique edge of \(W\), yielding that every vertex \(a \in W_0\) has \(\text{deg}_{H_0}(a) \in \{d - 1, d\}\). Also by Lemma 4 (ii), every vertex \(x \in N((g-3)/2) \cap V(H_0)\) has \(\text{deg}(x) = d\). And by Lemma 4 (iii), for every \(a \in W_0\) there exists a vertex \(x_0 \in N((g-3)/2) \cap V(H_0)\) such that \(N(x_0) = \{u_1, u_2, \ldots, u_d\}\) and \(W_0 = \{a_1, a_2, \ldots, a_d\}\), where \(a_i = a\), in such a way that \(d(u_1, a_1) = d(u_1, W_0) = (g - 5)/2, d(u_j, W_0) = d(u_j, a_j) = (g - 3)/2\), and by (ii), \(\text{deg}(u_j) = d\) for every \(j \geq 2\). This implies that \(d_{G-x_0}(u_1, a_j) \geq (g - 1)/2\) for all \(j \geq 2\), because the shortest \((u_1, a_j)\)-path in \(G - x_0\), the shortest \((u_j, a_j)\)-path in \(G\), and the path \(u_jx_0u_1\) in \(G\) of length two, form a closed walk containing a cycle. Reasoning analogously, \(d_{G-x_0}(u_j, a_1) \geq (g + 1)/2\) for all \(j \geq 2\) and \(d_{G-x_0}(u_j, a_i) \geq (g - 1)/2\) for \(j \neq i, j, i \in \{2, \ldots, d\}\). Furthermore, \([N((g-3)/2) \cap W_0, W_1] = \{a_1b_1\}\) for some \(b_1 \in W_1\).

**Subcase (a.1):** \(\mu_1 = (g + 1)/2\).

Let \(x_1 \in V(H_1)\) be any vertex such that \(d(W_1, x_1) = (g+1)/2\). Let \(X_0 = \{u_2, \ldots, u_d\} \cup \{x_0\}\) and \(X_1 = \{v_1, v_2, \ldots, v_d\} \subseteq N(x_1)\). As \(d(u_i, W_0) = (g - 3)/2\) for \(i \geq 2\) and \(d_{G-x_1}(W_1, N(x_1)) \geq (g - 1)/2\), then \(d_{G-x_1}(X_0, X_1) \geq g - 1\), so \(|E(B_{\Gamma})| = d^2\), where \(\Gamma = G - x_1\). Clearly \(B_{\Gamma}\) is a complete bipartite graph, so there is a perfect matching \(M\) which covers every vertex in \(X_0\) and if \(\text{deg}(x_1) = d\), also covers \(N(x_1)\). Hence, in this case the graph \(G^* = (G - \{x_1\} - \{x_0u_d\}) \cup M\) has girth at least \(g\) and the vertices \(u_2, \ldots, u_{d-1}\) have degree \(d+1\) in \(G^*\) as they had degree \(d\) in \(G\); for the same reason \(x_0\) and \(u_d\) have degree \(d\) in \(G^*\). The remaining vertices have the same degree they had in \(G\). As \(G^*\) is a \((\{d, d+1\}; g^*)\)-graph with girth \(g^* \geq g\) and \(|V(G^*)| < |V(G)|\), we get a contradiction to the monotonicity Theorem 1. If \(\text{deg}(x_1) = d + 1\), since \(d_{G^*}(u_d, v_{d+1}) \geq g - 1\) where \(v_{d+1} \in N(x_1) \cap X_1\), we can add the new edge \(u_dv_{d+1}\) to \(G^*\) without decreasing the girth. Then \(G^* \cup \{u_dv_{d+1}\}\) gives us again a contradiction.

**Subcase (a.2):** \(\mu_1 = (g - 3)/2\).

By Lemma 4, given \(b_1 \in W_1\) there exists \(x_1 \in V(H_1) \cap N((g-3)/2)(W_1)\) of \(\text{deg}(x_1) = d\) such that \(N(x_1) = \{v_1, v_2, \ldots, v_d\}\), \(W_1 = \{b_1, b_2, \ldots, b_d\}\) and each vertex of \(W_1\) is incident to a unique edge of \(W\), hence \(W = \{a_1b_1, a_2b_2, \ldots, a_db_d\}\). Also, \(d(b_1, v_1) = d(W_1, v_1) = (g - 5)/2\), and \(d(W_1, v_j) = d(b_j, v_j) = (g - 3)/2\) for every \(j \geq 2\) and besides \(\text{deg}(v_j) = d\). Then \(d(x_0, x_1) = d(x_0, a_1) + 1 + d(b_1, x_1) = g - 2\), and if \(g = 5\) it is easy to see that the shortest \((x_0, x_1)\)-path of length three is unique, clearly \(x_0a_1b_1x_1\).
Now let $\Gamma = G - \{x_0, x_1\}$. We have
\[
d_{\Gamma}(u_1, N(x_1) - v_1) = \min\{d_{\Gamma}(u_1, a_j) + 1 + d_{\Gamma}(b_j, N(x_1) - v_1), j \geq 2\}
\]
\[
d_{\Gamma}(u_1, a_j) + 1 + d_{\Gamma}(b_j, N(x_1) - v_1), j \geq 2
\]
\[
\geq \min\{\frac{g - 5}{2} + 1 + \frac{g + 3}{2}, \frac{g - 1}{2} + 1 + \frac{g - 3}{2}\} = g - 1,
\]
since $d_{\Gamma}(b_j, v_j) \geq (g + 1)/2$ for all $j \geq 2$, because the shortest $(b_j, v_j)$-path in $\Gamma$, the shortest $(b_j, v_1)$-path in $\Gamma$, and the path $v_j x_1 v_1$ in $G$ of length two, form a closed walk containing a cycle. Reasoning in the same way, it follows for all $j \geq 2$ that
\[
d_{\Gamma}(u_j, N(x_1) - v_j) = \min\{d_{\Gamma}(u_j, a_h) + 1 + d_{\Gamma}(b_h, N(x_1) - v_j), h \neq j\}
\]
\[
+ d_{\Gamma}(b_h, N(x_1) - v_j), h \neq j\}
\]
\[
\geq \min\left\{\begin{array}{l}
\frac{g + 3}{2} + 1 + \frac{g - 1}{2}, \frac{g - 1}{2} + 1 + \frac{g - 3}{2} \\
\frac{g - 3}{2} + 1 + \frac{g - 1}{2}, \frac{g + 1}{2} + 1 + \frac{g - 5}{2}
\end{array}\right\}
\]
\[
= g - 1.
\]
Analogously, $d_{\Gamma}(N(x_0) - u_1, v_1) \geq g - 1$ and $d_{\Gamma}(N(x_0) - u_j, v_j) \geq g - 1$ for all $j \geq 2$. Let $X_0 = N(x_0)$ and $X_1 = N(x_1)$. The bipartite graph $B_\Gamma = (X_0, X_1)$ has $|E(B_\Gamma)| = d^2 - d$ and $\deg_{B_\Gamma}(w) \geq 1$ for all $w \in X_0 \cup X_1$. From Lemma 7, there is a perfect matching $M$ between $X_0 = N(x_0)$ and $X_1 = N(x_1)$. Hence $G^* = (G - \{x_0, x_1\}) \cup M$ is a $(\{d, d + 1\}; g^*)$-graph (because every vertex in $G^*$ has the same degree it had in $G$ and the removed vertices $x_0, x_1$ had degree $d$, as well as the vertices $u_j, v_k$ for every $j, k \geq 2$) with $g^* \geq g$ and $|V(G^*)| \leq |V(G)|$, which contradicts the monotonicity Theorem 1, and we are done.

Subcase (a.3): $\mu_1 = (g - 1)/2$. In this case we distinguish two other possible subcases.

Subcase (a.3.1): There exists $x_1 \in V(H_1) \cap N(g_{-1}/2)(W_1)$ such that $d(b, v) \leq (g - 1)/2$ for all $b \in W_1$ and for all $v \in N(x_1)$.

Then, every $b \in W_1$ has $\deg_{H_1}(b) = \deg(x_1) \in \{d, d + 1\}$ because $d(b, v) \leq (g - 1)/2$ and $|N(g_{-3}/2)(v) \cap N(b)| \leq 1$ for all $v \in N(x_1)$ (otherwise
cycles of length less than $g$ appear). Hence $\text{deg}(x_1) = d$ and $\text{deg}(b) = d + 1$ for all $b \in W_1$. Thus $N(x_1) = \{v_1, \ldots, v_d\}$ and $W = [W_0, W_1]$ is a matching, i.e., $W = \{a_1b_1, \ldots, a_d b_d\}$. Therefore the subgraph $H_1$ gives a contradiction unless $H_1$ is $d$-regular. In this case let us consider the graph $\hat{G} = (G - x_1 - W) \cup \{a_1 v_1, \ldots, a_d v_d\}$ which clearly has girth at least $g$. Moreover $\text{deg}_{\hat{G}}(b_i) = \text{deg}(b_i) - 1 = d'$ and every vertex different from $b_1$ has the same degree it had in $G$. Thus we may suppose that $\hat{G}$ is $d'$-regular because otherwise $\hat{G}$ would be a $(\{d, d + 1\}; g^*)$-graph with girth $g^* \geq g$ and smaller than $G$, a contradiction. Moreover, we may assume that $d_{H_1}(b_1, v_1) = (g - 3)/2$ and $d_{H_1}(b_1, N(x_1) - v_1) = (g - 1)/2$. Thus we have

$$d_{\hat{G}}(b_1, u_2) \geq \min\{d_{H_1}(b_1, v_2) + |\{v_2 a_2\}|$$

$$+ d_{H_1}(a_2, u_2); d_{H_1}(b_1, v_1) + |\{v_1 a_1\}| + d_{H_1}(a_1, u_2)\}$$

$$\geq \min\{\frac{g - 1}{2} + 1 + \frac{g - 3}{2}; \frac{g - 3}{2} + 1 + \frac{g + 1}{2}\}$$

$$= g - 1,$$

which implies that we can add to $\hat{G}$ the edge $u_2 b_1$ to obtain a graph without decreasing the girth $g$. As this new graph is smaller than $G$ and has degrees $\{d, d + 1\}$ we get a contradiction to the monotonicity Theorem 1, and we are done.

**Subcase (a.3.2):** For all $z \in V(H_1) \cap N_{(g-1)/2}(W_1)$ there exists $v \in N(x_1)$ and $b \in W_1$ such that $d(b, v) \geq (g + 1)/2$.

Let $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$, $v_1 \in N(x_1)$ and $b^* \in W_1$ be such that $d(b^*, v_1) \geq (g + 1)/2$. By Lemma 4, there exists a unique edge $a^* b^* \in W$ to which the vertex $a^* \in W_0$ is incident, and there exists a vertex $x^* \in V(H_0)$ of $\text{deg}(x^*) = d$ such that $d(x^*, W_0) = d(x^*, a^*) = (g - 3)/2$ and $N_{(g-3)/2}(x^*) \cap W_0 = \{a^*\}$. Further, $N(x^*)$ can be labeled as $\{z_1, z_2, \ldots, z_d\}$, and $W_0$ can be labeled as $\{a_1, a_2, \ldots, a_d\}$, where $a_1 = a^*$, so that $N_{(g-3)/2}(z_1) \cap W_i = \{a_1\}, N_{(g-3)/2}(z_k) \cap W_i = \{a_k\}$ and $\text{deg}(z_k) = d$ for every $k > 1$. Furthermore, $[N_{(g-3)/2}(x^*) \cap W_0, W_1] = \{a_1 b^*\}$

Let $\Gamma = G - \{x^*, x_1\}$. We obtain

$$d_{\Gamma}(z_1, v_1)$$

$$= \min\{d_{\Gamma}(z_1, a_1) + 1 + d_{\Gamma}(b^*, v_1); d_{\Gamma}(z_1, a_j) + 1 + d_{\Gamma}(b', v_1), j \geq 2, a_j b' \in W\}$$

$$\geq \min\{\frac{g - 5}{2} + 1 + \frac{g + 1}{2}; \frac{g - 1}{2} + 1 + \frac{g - 3}{2}\} = g - 1.$$
Moreover, \(d_{H_0}(z_k, W_0) = (g - 3)/2\) for all \(z_k \in N(x^*) - z_1\) and for \(k > 1\) there exists a unique vertex say \(b_k \in W_1\) for which \(a_kb_k \in W\). As for each \(b \in W_1\), \(|N_{(g-3)/2}(b) \cap N(x_1)| \leq 1\) (otherwise cycles of length less than \(g\) appear) we may denote by \(v_k\) the vertex in \(N(x_1) - v_1\) such that \(d(b_k, v_k) = (g - 3)/2\), if any. Thus we obtain

\[
d_{r}(z_k, N(x_1) \setminus \{v_1, v_k\}) = d(z_k, a_k) + 1 + d(b_k, N(x_1) \setminus \{v_1, v_k\}) \geq g - 1.
\]

Let us consider \(X_0 = N(x^*) - z_1\) and \(X_1 \subseteq N(x_1) - v_1\), with \(|X_1| = d - 1\). It is clear that \(|deg_{(r)}(z_k)| \geq d - 2 \geq 1\) for all \(z_k \in N(x^*) - u_1\) yielding \(|E(B_r)| \geq (d - 2)(d - 1) = (d - 1)^2 - (d - 1)\).

First, suppose that \(|deg_{(r)}(v)| \geq 1\) for all \(v \in N(x_1) - v_1\). From Lemma 7, there is a matching \(M\) which covers every vertex in \(N(x^*) - z_1\) and every vertex in \(N(x_1) - v_1\) if \(deg(x_1) = d\). In this case \(G^* = (G - \{x^*, x_1\}) \cup M \cup \{z_1v_1\}\) is a graph with girth \(g^* \geq g\) and smaller than \(G\) whose vertices have the same degree they had in \(G\); thus \(G^*\) is a \((\{d, d + 1\}; g^*)\)-graph and we are done. Thus suppose that \(deg(x_1) = d + 1\) and that after adding the matching \(M \cup \{z_1v_1\}\) to \(G - \{x^*, x_1\}\) the vertex \(v_{d+1} \in (N(x_1) - v_1) - X_1\) remains of degree \(d - 1\). By Lemma 4 every \(z_k, k > 1\), has degree \(d\) in \(G\), and we have proved that \(d(z_k, N(x_1) - \{v_1, v_k\}) \geq g - 1\). Then we add one extra edge \(z_kv_{d+1}\) to \(G^*\) obtaining a new \((\{d, d + 1\}; g^*)\)-graph with \(g^* \geq g\) and smaller than \(G\), a contradiction to the monotonicity Theorem 1, so we are done.

Therefore we must suppose that there exists \(v_2 \in N(x_1) - v_1\) such that \(|deg_{(r)}(v_2)| = 0\). This implies that \(d(v_2, b) = (g - 3)/2\) for all \(b \in W_1 - b^*\), hence \(d(v_1, W_1 - b^*) = (g - 1)/2\) for all \(v \in N(x_1) - v_2\). First suppose that \(d(v_2, b^*) \geq (g + 1)/2\); then \(d_{r}(z_1, v_2) \geq g - 1\), \(d_{r}(z_k, N(x_1) - v_2) = g - 1\) for all \(k \geq 2\), thus we consider the set \(X_1 \subseteq N(x_1) - v_2\) with \(|X_1| = d - 1\).

It is clear that \(|deg_{(r)}(w)| \geq d - 1\) for all \(w \in X_0 \cup X_1\). Using Lemma 7 and reasoning as before we get a contradiction. Therefore we must suppose that \(d(v_2, b^*) \leq (g - 1)/2\). Since \(N(x_1) - v_2 \subseteq N_{(g - 1)/2}(W_1) \cap V(H_1)\) we have by hypothesis that for all \(v \in N(x_1) - v_2\) there exists \(\tilde{v}_1 \in N(v)\) and \(b^* \in W_1\) such that \(d(b^*, \tilde{v}_1) \geq (g + 1)/2\). As the behavior of any \(v \in N(x_1) - v_2\) is the same as vertex \(x_1\), reasoning as before we get a contradiction unless for all \(v \in N(x_1) - v_2\) there exists \(\tilde{v}_2 \in N(v) - \tilde{v}_1\) such that \(|deg_{(r)}(\tilde{v}_2)| = 0\) satisfying \(d(\tilde{v}_2, b) = (g - 3)/2\) for all \(b \in W_1 - b^*\) and \(d(\tilde{v}_2, b^*) \leq (g - 1)/2\). Therefore we conclude that every vertex \(b \in W_1\) has
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\( \text{deg}_{H_1}(b) = \text{deg}(x_1) \in \{d, d + 1\} \). Now considering the same graph as in Subcase (a.3.1) we get a contradiction.

Case (b): \( \mu_0 = \mu_1 = (g - 1)/2 \).

Let \( x_0 \in V(H_0) \) and \( x_1 \in V(H_1) \) satisfy \( d(x_i, W_i) = (g - 1)/2, i = 0, 1 \).

First of all note that there must exist a vertex in \( N(x_0) \) of degree \( d \), otherwise \( G - x_0 \) would be either a \( \{d, d + 1\} \)-graph or a \( d \)-regular graph.

In the former case we get a contradiction because \( G - x_0 \) is smaller than \( G \) and has girth at least \( g \). And in the latter case we consider the graph \( (G - x_0) \cup \{u_i x_1\} \) with \( u_i \in N(x_0) \), which gives again a contradiction. Similarly, note that there must exist a vertex in \( N(x_1) \) of degree \( d \).

Suppose that \( \text{deg}(x_0) = \text{deg}(x_1) = r \) with \( r \in \{d, d + 1\} \). Let \( X_0 = N(x_0), X_1 = N(x_1) \) and \( \Gamma = G - \{x_0, x_1\} \). Define \( A = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_{\Gamma}(u_i, v_j) \leq g - 2\} \) and consider \( B_{\Gamma} = K_{[X_0], [X_1]} - A \). Note that every \( (u_i, v_j) \)-path in \( G \) goes through an edge of \( W \). Therefore every edge in \( W \) gives rise to at most one element in \( A \), otherwise \( G \) would contain a cycle of length at most \( 2(g - 3)/2 + 2 = g - 1 \). Hence \( |A| \leq |W| \leq d \) and \( |E(B_{\Gamma})| = |K_{r,r} - |A| \geq r^2 - d \).

If \( r = d + 1 \) then \( |E(B_{\Gamma})| = (d + 1)^2 - d = d^2 + d + 1 \) and by Lemma 6, the graph \( B_{\Gamma} \) contains a perfect matching \( M \). Therefore the graph \( G' = G - \{x_0, x_1\} \cup M \) has fewer vertices than \( G \) and girth at least \( g \) producing a contradiction unless \( G' \) is regular of degree \( d \). In this case we consider the graph \( G'' = G' \cup \{uv\} \) where \( u \in N(x_0) \) such that \( d(u, W_0) = (g - 1)/2 \) (such a vertex must exist because \( \text{deg}(x_0) = d + 1 \) and \( |W_1| \leq d \) and \( v \in N(x_1) \) such that \( uv \notin M \). As \( G'' \) is a \( \{d, d + 1\}; g \)-graph with fewer vertices than \( G \) and girth \( g \) a contradiction is again obtained.

Suppose \( r = d \). If \( \text{deg}_{B_{\Gamma}}(z) \geq 1 \) for all \( z \in B_{\Gamma} \), then by Lemma 7 there exists a perfect matching \( M \) between \( X_0 \) and \( X_1 \); reasoning as before we obtain again a contradiction. Hence, we may assume that \( \text{deg}_{B_{\Gamma}}(u_1) = 0 \) for some \( u_1 \in X_0 \). This implies that \( d_{\Gamma}(u_1, v_j) = g - 2 \) for all \( v_j \in N(x_1) \), or equivalently \( d_{\Gamma}(v_j, W_1) = (g - 3)/2 \) for all \( v_j \in N(x_1) \). From this, and because \( g \geq 5 \), we get \( |W_1| \geq |N(x_1)| = d \), yielding \( |W_1| = d \) (since \( d = |W| \geq |W_1| \)), and also \( N_{(g-3)/2}(v_j) \cap W_1 = \{b_j\} \) for all \( v_j \in N(x_1) \). That is, \( |N(b_j) \cap W_0| = 1 \) for every \( b_j \in W_1 \). Also we have \( N_{(g-1)/2}(u_1) \cap W_1 = W_1 \), hence \( N_{(g-3)/2}(u_1) \cap W_0 = W_0 \) and thus \( d(u_1, W_0) = (g - 1)/2 \) for \( i \geq 2 \).

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Let $u_k \in N(x_0)$, $k \geq 2$, define $\Gamma_k = G - \{u_k, x_1\}$ and consider the sets

$$X_k = \begin{cases} N(u_k) & \text{if } \deg(u_k) = d; \\ N(u_k) - x_0 & \text{if } \deg(u_k) = d + 1; \end{cases}$$

$$X_1 = N(x_1);$$

$$A_k = \{z_i v_j : z_i \in X_k, v_j \in X_1, d_{\Gamma_k}(z_i, v_j) \leq g - 2\}.$$ 

Let $B_{\Gamma_k} = K_{|X_k|, |X_1|} - A_k$.

If $\deg_{B_{\Gamma_k}}(z) \geq 1$ for all $z \in X_k$, we get a perfect matching $M$ between $X_k$ and $N(x_1)$ by Lemma 7; if $\deg(u_k) = d$ the graph $\Gamma_k \cup M$ yields a contradiction; if $\deg(u_k) = d + 1$ the graph $\Gamma_k \cup M \cup \{x_0 v_j\}$, where $v_j$ is a vertex of $N(x_1)$ with degree $d$, yields again a contradiction. Therefore we can suppose that for every $u_k \in N(x_0) - u_1$ there exists $\hat{z}_k \in N(u_k)$ such that $d_{\Gamma_k}(\hat{z}_k, v_j) = g - 2$ for all $v_j \in N(x_1)$. Hence, $N_{(g-3)/2}(\hat{z}_k) \cap W_0 = W_0$, that is $d_{\Gamma_0}(\hat{z}_k, a_j) = (g - 3)/2$ for each $a_j \in W_0$. Therefore $\deg_{H_0}(a_j) = d$, $\deg(a_j) = d + 1$ and $[W_0, W_1]$ is a matching (recall that $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$). We can now use the same graph $\tilde{G} = (G - \{x_0\} - W) \cup \{b_1 u_1, \ldots, b_d u_d\}$ as used in Case (a.3.2), arriving again at a contradiction.

The only remaining case occurs when $x_0$ and $x_1$ have different degrees. Let us suppose $\deg(x_0) = d$ and $\deg(x_1) = d + 1$. As $\deg(x_1) = d + 1 > |W_1|$, there exists, say $v_{d+1} \in N(x_1)$, such that $d(v_{d+1}, W_1) = (g - 1)/2$. We proceed as before, with the sets $X_0 = N(x_0)$ and $X_1 = N(x_1) - v_{d+1}$, finding a graph $G'$ with fewer vertices and the same girth and degrees as $G$, except for the vertex $v_{d+1}$. Recall that there must exist a vertex $y \in N(x_0)$ such that $\deg(y) = d$. Then we construct the graph $G^* = G' \cup \{y v_{d+1}\}$, which is a new $\{d, d + 1\}$-graph with girth $g$, arriving at a contradiction. This ends the proof of the theorem. \(\square\)

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