On Asymptotic Ensemble Weight Enumerators of Multi-Edge Type Codes

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Abstract—In this paper, we investigate the asymptotic ensemble weight enumerators of multi-edge type codes whose component codes are arbitrary block codes. Two forms of asymptotic growth rate of codewords, corresponding to the primal and dual problems, are obtained. Furthermore, for the codewords of small linear-sized weights, we develop a simplification method to restrict the search space of the primal problem and study the optimality conditions of the dual problem, giving a first-order approximation of the growth rate and a condition of exponentially few small weight codewords.

I. INTRODUCTION

Low-density parity-check (LDPC) codes were proposed by Gallager [1] in the 1960s. In [1] the ensemble weight enumerators of regular codes are also studied. Tanner decoded the decoding of LDPC codes using a Tanner graph [4] and proposed generalized LDPC (GLDPC). In [8], doubly generalized LDPC (DGLDPC) codes were introduced. Recently, many structured LDPC code ensembles with some specifications and constraints of nodes and edges have been proposed, such as protograph-based LDPC [11] and low-density generator-matrix codes [19]. Most classes of LDPC codes can be unified within the framework of multi-edge type (MET) LDPC codes [2], for which the connection between variable and check nodes of various degrees is restricted.

Code performance highly depends on its weight distribution, and the number of small weight codewords affects the BER at high SNR. As an asymptotic analysis is a tractable method for capacity-approaching codes, we are interested in the asymptotic properties of the average weight enumerators when the code length goes to infinity. For unstructured codes, ensemble weight distributions of irregular LDPC codes were developed in [9]. This work was generalized to GLDPC in [5], [6], [7] and to DGLDPC codes in [15] and [16]. For structured codes, the asymptotic weight distributions of protograph GLDPC and DGLDPC codes were studied in [20] and [14], respectively. The weight distributions of MET LDPC codes were investigated in [3]. The asymptotic analysis of weight distributions was presented, and a first-order approximation was given.

In order to generalize MET LDPC codes to GLDPC and DGLDPC codes, we replace the repetition codes at variable nodes and single-parity-check (SPC) codes at check nodes with arbitrary block codes. Based on the optimization theory and Karush-Kuhn-Tucker (KKT) conditions, we obtain two forms of asymptotic weight distributions, the primal and the dual, which are then used to improve the computation of weight distributions and investigate the properties of small weight codewords. A first-order approximation of the growth rate of codewords is also provided.

II. PRELIMINARIES

Given the code length \( n \), suppose that there are \( n_N \) component nodes and \( n_E \) connecting edges in the graph \( G = (N, E, E') \), which consists of the set of nodes \( N \) and two sets of connecting edges \( E \) and \( E' \). We do not assume that the graph is bipartite, although one can modify it to be bipartite by properly inserting weight-2 repetition codes between component nodes and then reconnecting the edges. Each node represents a specific block code, and each edge connects two bits of different nodes such that the binary values of these two bits are the same. There are \( |E| \) edgetypes in \( E \), \( |E'| \) edgetypes in \( E' \), and \( |N| \) node-types in \( N \). They are classified into several types, according to the type of codes and their neighbors. On one side of type-\( j \) edges with \( j \in E \), these edges are connected to the socket \( (i, j) \) of type-i nodes with \( i \in N \), and on the other side all type-\( j \) edges are permuted by an interleaver to become type-\( j' \) edges with \( j' \in E' \). Thus we have \( |E'| = |E| \).

Classifying edges of \( G \) into \( E \) and \( E' \) is arbitrary. For each type-i node, the socket \( (i, j) \) connected to type-j edges is formed by grouping \( l_{i,j} \) bits of the corresponding code, without overlapping between sockets, so that the code length of type-i nodes is

\[
L_i = \sum_{j=1}^{\frac{|E|}{l_{i,j}}} l_{i,j} + \sum_{j'=1}^{\frac{|E'|}{l_{i,j}'}} l_{i,j}'.
\]

Furthermore, each type-i node has \( s_i \) transmitted bits. The relation between nodes and edges is specified by the multivariate polynomial as follows:

\[
L(v, y, z) = \sum_{i=1}^{\frac{|N|}{l_{i,j}}} \lambda_i v^{s_i} \prod_{j=1}^{\frac{|E|}{l_{i,j}}} y_j^{l_{i,j}} \prod_{j'=1}^{\frac{|E'|}{l_{i,j}'}} z_j^{l_{i,j}'},
\]  

where \( v = (v_1, v_2, ..., v_{|E|}), \) \( y = (y_1, y_2, ..., y_{|E|}), \) \( z = (z_1, z_2, ..., z_{|E'|}), \) and \( \lambda_i n_N \) is the number of type-i nodes. In the code book \( C_i \) of type-i nodes, the type-k codewords with multiplicity \( a_{i,k} \) has weight \( t_{i,j} \) on the socket \( (i, j) \) with \( j \in E \), weight \( t_{i,j}' \) on the socket \( (i, j) \) with \( j \in E' \), and weight \( b_{ij} \) on transmitted bits. The
transmitted bits
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transmitted bits

Fig. 1. The graph of an MET code.

type-0 codeword is the zero codeword with $a_i^0 = 1$. The code book $C_i$ is specified by the weight enumerating function:

$$f_i(V, Y, Z) = 1 + \sum_{k=1}^{[C_i]} a_i^k V^k Y^j Z^j$$

(2)

where $[C_i]$ is the number of codeword-types of type-$i$ nodes, $Y = (Y_1, Y_2, ..., Y_{[E]})$, and $Z = (Z_1, Z_2, ..., Z_{[E]})$.

Example 1: The protograph base matrix $B_3$ is defined in [13] as:

$$B_3 = \begin{bmatrix} 2 & 1 & 2 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(3)

for which we label the variable nodes from the leftmost column as type-1 to type-6, and label the check nodes from upmost row as type-7 to type-9. The edges from type-1 to type-6 are labeled as type-$i$ edges, and those from type-7 to type-9 are labeled as type-$j$ edges, for $i = 1, 2, ..., 9$.

Example 2: The coefficients of $L(v, y, z)$ for the MET code of the graph depicted in Fig. 1 are $\omega_1 = 1/6$, $\omega_2 = 1/6$, $\omega_3 = 1/2$, and $\omega_4 = 1/6$. Since the entries for type-$i$ nodes are degree-$3$ repetition codes, and type-$j$ nodes are $[7, 4]$ hamming codes, we have $f_3(V, Y, Z) = 1 + V Y_1 Z_7$ and $f_4(V, Y, Z) = 1 + V Y_1 Z_7$.

III. WEIGHT ENUMERATORS OF MET CODES

Suppose that we define the codeword vectors $x = [x_1, x_2, ..., x_{[N]}]^T$. Letting the proportion of “ones” in type-$j$ edges be denoted by $u_j = \Omega_j / n E_j$, since $u_j = u_j'$, we have

$$\omega_j = \frac{1}{\sigma_j} \sum_{i=1}^{[C_i]} \lambda_i \sum_{k=1}^{[C_i]} t_{i,j} (a_i^k x_i^k)$$

(8)

$$\omega_{j'} = \frac{1}{\sigma_j} \sum_{i=1}^{[C_i]} \lambda_i \sum_{k=1}^{[C_i]} t_{i,j'} (a_i^k x_i^k)$$

(9)

for all $i \in E$ and $j' \in E'$. Define the weight vectors $w = [w_1, w_2, ..., w_{[N]}]^T$ and $w' = [w_1', w_2', ..., w_{[N]}']^T$, such that $w = T x$ and $w' = T' x$, where the $j$-th row of $T$ is $1/\sigma_j \lambda_i a_i^j$, and the $j'$-th row of $T'$ is $1/\sigma_j \lambda_i a_i^{j'}$. The normalized codeword weight $d$ can be computed by:

$$d = \frac{1}{\tau} \sum_{i=1}^{[N]} \lambda_i \sum_{k=1}^{[C_i]} b_i^k (a_i^k x_i^k)$$

(10)

or equivalently $d = B x$, where $B = \sum_{i=1}^{[N]} \lambda_i \sum_{k=1}^{[C_i]} a_i^k I(x_i^k)$. The AWE of the ensemble with the graph $G = (N, E, E')$, $L(v, y, z)$, and $F(V, Y, Z)$ can be obtained by minimizing the cost function $R(x, w)$ as:

$$\gamma(d) = -\alpha \min_{x(8), (9), (10)} R(x, w)$$

(11)

with $R(x, w) = \sum_{i=1}^{[N]} \lambda_i \sum_{k=0}^{[C_i]} a_i^k I(x_i^k) + \sum_{j=1}^{[E]} \sigma_j H(w_j)$, where the entropy function is given as $H(\psi) = -\psi \ln \psi - (1-\psi) \ln (1-\psi)$, $\sigma_j = n E_j / n N$, and $\alpha = n N / n$. This is the
primal problem of AWE, which was also studied in [11] and [20].

It should be noted that (11) is not necessarily a convex problem, but we can split it into two parts such that the front part is an entropy maximization problem and can be solved by the dual problem [18]. Suppose that the dual variables are \( u, p, q = [u_1, p_2, ..., p_{|E|}]^T \), and \( q = [q_1, q_2, ..., q_{|E|}]^T \), where \( u \) corresponds to the constraint (10), \( p_j \) corresponds to the constraint (8), and \( q_j \) corresponds to the constraint (9), such that the dual problem can be expressed as:

\[
\begin{align*}
&\min_{x, \gamma, \bar{u}, \bar{p}, \bar{q}} \sum_{i=1}^{|N|} \lambda_i \sum_{k=0}^{|C_i|} a_i^k I(x_i^k) \\
= & \max_{u, p, q} \frac{d \tau u}{u, p, q} + \sum_{j=1}^{|E|} w_j \sigma_j p_j + \sum_{j=1}^{|E'|} w_j \sigma_j q_j \\
- & \sum_{i} \lambda_i \log \left( 1 + \sum_{k=1}^{|C_i|} a_i^k \exp(ub_i^k) + \sum_{j=1}^{|E|} p_j t_{i,j}^k + \sum_{j=1}^{|E'|} q_j t_{i,j}'^k \right) \\
= & \max_{u, p, q} d \tau u + w^T \bar{I}_\sigma (p + q) - \sum_{i} \lambda_i \log f_i(e_i^u, e_i^p, e_i^q) \\
= & \max_{u, p, q} D(d, w, u, p, q),
\end{align*}
\]

where \( D(d, w, u, p, q) \) is the objective function, \( \tau = \sum_{i=1}^{|N|} = \lambda_i \sigma_i, \sigma = [\sigma_1, \sigma_2, ..., \tau] \), and \( \bar{I}_\sigma \) is a diagonal matrix with \( \bar{\sigma} \) on the main diagonal. Since this is an unconstrained problem, the KKT condition is given by:

\[
\begin{align*}
&d \tau = \sum_{i=1}^{|N|} \lambda_i \frac{\partial f_i(e_i^u, e_i^p, e_i^q)}{\partial u} f_i(e_i^u, e_i^p, e_i^q) = 0, \\
&w_j \sigma_j = \sum_{i=1}^{|N|} \lambda_i \frac{\partial f_i(e_i^u, e_i^p, e_i^q)}{\partial p_j} f_i(e_i^u, e_i^p, e_i^q), \\
&w_j \sigma_j = \sum_{i=1}^{|N|} \lambda_i \frac{\partial f_i(e_i^u, e_i^p, e_i^q)}{\partial q_j} f_i(e_i^u, e_i^p, e_i^q),
\end{align*}
\]

for all \( j \in E \) and \( j' \in E' \). Since the dual problem (12) and its optimality conditions (13), (14), and (15) are identical to Hayman Formula [9] and Theorem 2 in [10], the primal (11) is valid. Since from (12) \( \nabla_w D(d, w, u, p, q) = I_\sigma (p + q) \), the optimal solution must satisfy

\[
\frac{\partial R(x, w)}{\partial w_j} = \sigma_j (p_j + q_{j'}) + \sigma_j H'(w_j) = 0,
\]

where \( H'(\psi) = -\ln \psi + \ln(1 - \psi) \). It leads to the condition for the optimality of (11) given by:

\[
w_j = 1/(1 + \exp(-p_j - q_{j'})).
\]

Substituting (20) into (14) and (15) yields

\[
\begin{align*}
\sigma_j &= \sum_{i=1}^{|N|} \lambda_i \frac{\partial f_i(e_i^u, e_i^p, e_i^q)}{\partial p_j} f_i(e_i^u, e_i^p, e_i^q), \\
\sigma_j &= \sum_{i=1}^{|N|} \lambda_i \frac{\partial f_i(e_i^u, e_i^p, e_i^q)}{\partial q_j} f_i(e_i^u, e_i^p, e_i^q).
\end{align*}
\]

In addition, from (12) and (19), we have the derivative of the AWE with respect to \( d \) equal to

\[
\gamma'(d) = -\alpha \left( \frac{\partial D(d, w, u, p, q)}{\partial d} \right) + \sum_{j=1}^{|E|} \frac{\partial R(x, w)}{\partial w_j} \frac{\partial w_j}{\partial d}
\]

\[
= -\alpha \tau u.
\]

Theoretically, the optimal solution \((u^*, p^*, q^*)\) can be obtained by solving (21) and (22), and \( w \) can be obtained by (14) or (15). However, it is so impractical that we can optimize (12) with a fixed \( w \) and then update \( w \) by gradient descent from (19) iteratively for a local search. Depending on the graph structure, more complex searching algorithms, such as genetic algorithm or differential evolution, may be required to find the global optimum. The dual problem was provided as an alternative to the primal to obtain the AWE in [12] and [14]. Its properties were also studied in several papers including [9], [10], [16], [15], and [3].

**Example 1 continued:** For the variable nodes, we denote the proportion of the only nonzero codeword occurring in type-4 nodes as \( x_4 \). For the check nodes, we denote the proportion of the codeword with weight 2 on the 4th and 5th columns occurring in type-7 nodes as \( x_{4,7} \). Also we denote the proportion of ones in type-(4,7) edges as \( w_{4,7} \). The ratios \( \sigma_j \) are \( 1/9 \) and \( \sigma(4,7) = 2/9 \).

**IV. AWE OF CODEWORDS WITH SMALL LINEAR-SIZED WEIGHTS**

In this section we focus on the AWE of codewords with small weights linear to the code length \( n \), i.e., \( d \to 0 \). In this special case, we can simplify the AWE both from the primal and dual. First, we restrict the search space of the primal. Then the optimality conditions (21) and (22) of the dual can be reduced to obtain the properties of the AWE.
A. Restricted Search Space of the Primal

Ignoring temporarily the constraint (10), since \( x > 0 \) and \( T x = T' x \), we have \( A x = 0 \) with \( A = T - T' \). It follows that the conic hull \( \{ x > 0 | x \in A^+ \} \) is the set of feasible codeword vectors \( x \) which can be decomposed as \( x = H m \), where the positive vector \( m = [m_1, m_2, \ldots]^T \), and the positive matrix \( H = [h_1, h_2, \ldots] \) are the basis of the set. The cost function becomes

\[
R_T (x) = R_T (H m) = R(H m, TH m). \tag{24}
\]

Let \( m_\ell \) be the \( \ell \)-th entry of \( m \) and \( h_\ell \) be the \( \ell \)-th column of \( H \). Considering (7), the directional derivative along \( h_\ell \) can be obtained by:

\[
\nabla_{h_\ell} R_T (x) = \frac{\partial R_T (x)}{\partial h_\ell} = \sum_{i=1}^{[N]} \lambda_i \left( - \sum_{k=1}^{[C_i]} h_{i,\ell}^k \ln x_i^k + \sum_{k=1}^{[C_i]} h_{i,\ell}^k a_i^k \ln x_i^k \right) + \sum_{j=1}^{[E]} \sigma_j r_j,\ell H' (w_j),
\]

where \( h_{i,j}^k \) is an entry of \( H \), and \( r_j,\ell \) is an entry of \( r_\ell = Th_\ell \). We also define \( g_\ell = Bh_\ell \).

The optimization algorithm searches for an optimal \( m \) to minimize the cost function \( R_T (x) \). From optimization theory, the KKT condition is [18]

\[
\nabla_{h_\ell} R_T (x) = g_\ell \mu, \tag{25}
\]

for any \( H \) and all \( \ell \), with the constraint \( BH m = \sum_{\ell} g_\ell m_\ell = d \), where \( \mu \) is a constant. We assume that \( g_\ell > 0 \) for all \( h_\ell \). As \( d \to 0 \), there must exist some entries in \( m \) with \( m_\ell = o(d) \). Therefore we can assume that all entries of \( m \) are \( o(d) \) before we find an optimal \( m \). Although \( H \) is arbitrary, it should be noted that a basis vector \( h_\ell \) can have \( \nabla_{h_\ell} R_T (x) \) approaching \( \infty \), \( -\infty \), or converging to a constant. Hence, we introduce the following theorem.

Theorem 1: Assuming that all entries of \( m \) are \( o(d) \), the directional derivative \( \nabla_{h_\ell} R_T (x) \) converges to a constant if providing \( x = h_\ell \), the number of nonzero codewords occurring in \( N \) is equal to the number of ones in \( E \).

Proof: When \( d \to 0 \) and \( m_\ell = o(d) \) for all \( \ell \), the entries of \( x \) and \( w \) are all \( o(d) \) such that

\[
\nabla_{h_\ell} R_T (x) = \sum_{i=1}^{[N]} \lambda_i \left( \sum_{k=1}^{[C_i]} h_{i,\ell}^k a_i^k \ln x_i^k \right) - \sum_{j=1}^{[E]} \sigma_j r_j,\ell \ln w_j + o(d) \text{.}
\]

Multiplying by \( n_N \) and letting \( x = h_\ell \), we need to prove

\[
\sum_{i=1}^{[N]} \lambda_i n_N \sum_{k=1}^{[C_i]} a_i^k x_i^k \sum_{j=1}^{[E]} n_{E_j} w_j = 0 \text{.} \tag{26}
\]

The expression \( a_i^k h_{i,\ell}^k \) denotes the proportion of type-\( k \) codewords occurring in type-\( i \) nodes. Therefore the left-hand side (LHS) sums of (26) the number of nonzero codewords over all \( i \in N \). On the right-hand side (RHS), \( r_j,\ell \) denotes the proportion of ones in type-\( j \) edges. As a result, this side sums the number of ones over all \( j \in E \).

The next two lemmas follow straightforwardly.

Lemma 2: If (26) does not hold, \( \nabla_{h_\ell} R_T (x) = o(\ln d) \to \pm\infty \). If the LHS of (26) is strictly larger than the RHS, \( \nabla_{h_\ell} R_T (x) < 0 \). Otherwise if the LHS is strictly smaller than the RHS, \( \nabla_{h_\ell} R_T (x) > 0 \).

Lemma 3: Suppose that the weight of type-\( k \) codewords is defined as \( t_k^\ell = \sum_{i=1}^{[E]} n_{E_i} r_{i,j}^k \). Providing \( \nabla_{h_\ell} R_T (x) \) converges to a constant, if no codeword with weight 1 exists in \( C_i \) for all \( i \in N \), then only codewords with weight 2 are involved in \( h_\ell \), that is, \( h_{i,\ell}^k > 0 \) if \( t_k^\ell = 2 \).

Proof: Assuming \( x = m_\ell \), the RHS of (26) is computed as:

\[
\sum_{j=1}^{[E]} n_{E_j} r_{j,\ell} = \frac{1}{2} \left( \sum_{j=1}^{[E]} \Omega_j + \sum_{j=1}^{[E']} \Omega_j^\prime \right) = \frac{1}{2} \sum_{i=1}^{[N]} \frac{1}{\sigma_i} \sum_{k=1}^{[C_i]} a_i^k n_{N} h_{i,\ell}^k \text{.} \tag{27}
\]

Compared to the LHS of (26), if \( t_k^\ell > 1 \), then \( t_k^\ell = 2 \).

If \( \nabla_{h_\ell} R_T (x) \to \pm\infty \), \( \nabla_{h_\ell}^2 R_T (x) = o(d^{-2}) \). From [18], if there exists an \( h_\ell \) with \( \nabla_{h_\ell} R_T (x) \to -\infty \), we have \( m_\ell = o(d) \), and the AWE is \( \gamma (d) = o(d \ln d) > 0 \) as \( d \to 0 \). If there are only \( h_\ell \) with \( \nabla_{h_\ell} R_T (x) \to \infty \), the AWE is \( \gamma (d) = o(d \ln d) < 0 \), because there must exist some entries in \( m \) with \( m_\ell = o(d) \). Otherwise, if there exists no \( h_\ell \) with \( \nabla_{h_\ell} R_T (x) \to -\infty \), but there exists some \( h_\ell \) with a constant derivative, we are required to search for a new basis \( H \) of the intersection of the set \( \{ x > 0 | x \in A^+ \} \) and the hyperplane described by (26) and (27). This set \( \{ x > 0 | m > 0, x = H m \} \) is denoted by \( S(H) \). After restricting the search space to \( S(H) \), in order to satisfy (21) and (22) we set \( x_i^k = o(d^\epsilon) \) with some \( \epsilon > 1 \) if the corresponding row of \( H \) is a zero row, such that \( \mu \) can be solved by (25) analytically or numerically by setting \( m_\ell = o(d) \). Since \( \sum_{\ell} g_\ell m_\ell = d \), with the minimal solution of \( \mu \) denoted by \( \mu^* \), from (25) the AWE can be obtained as:

\[
\gamma (d) = -\alpha \left( R_T (0) + \sum_{\ell} \nabla_{h_\ell} R_T (x) m_\ell^* + o(d^\epsilon) \right) = -\alpha \sum_{\ell} \nabla_{h_\ell} R_T (x) m_\ell^* + o(d^\epsilon) \tag{28}
\]

If there exists a \( h_\ell \) such that \( g_\ell = 0 \), we must set \( d = 0 \) and solve (11). If we have \( \gamma (0) \to 0 \), the procedure of obtaining the AWE remains the same. Otherwise if \( \gamma (0) = o(1) > 0 \), this MET code cannot be used since the zero codeword may correspond to not only zero but nonzero input sequences. Moreover, for the unstructured codes considered in [15] and [16], it is proved that when \( d \) is small, the codewords with the smallest weight over all component codes occur in the nodes more frequently than other codewords with larger weights, and are “dominant,” since they determine the AWE of small weight codewords. For MET codes, the codewords with nonzero corresponding rows of \( H \) occur in the nodes more frequently and
are dominant, and those with zero rows are “nondominant.” The unstructured codes can be regarded as a special class of MET codes so that one can consider the techniques used for MET codes in this paper as a generalization of those used for the unstructured codes in [15] and [16].

Example 1 continued: By observing the protograph base matrix $B_A$, there exists only one codeword vector $h_1$ as the basis vector in $S(\hat{H})$ with a constant derivative, which is on four edge types in $E$: $(4, 7), (5, 7), (4, 8)$, and $(5, 8)$, involving type-$4$, type-$5$, type-$7$, and type-$8$ nodes. Normalizing $h_1$ such that $g_i = 1$, we have $m_1 = d/2$ and $x_1^A = x_2^A = x_3^{(4, 5)} = x_5^{(4, 5)} = d/2$, such that

$$\nabla_{h_1} R_T(x) = \ln \frac{\sum_{j=1}^{L} x_2^A x_7^A (4, 5) x_3^A (4, 5)}{u_1^{(4, 7)} u_2^{(5, 7)} u_3^{(4, 8)} u_4^{(5, 8)}} = 4 \ln(d/2) - 4 \ln(d/2) = 0 ,$$

so $\gamma(d) = o(d^2)$.

B. Optimality Conditions of the Dual

Although one can restrict the search space of the primal problem to obtain the AWE of every MET code, examining the optimal conditions of the dual problem provides other interesting properties of the AWE, e.g., the conditions of exponentially few codewords with linear small weights, as derived in [15], [16], and [3]. Since the proportion of type-$k$ codewords occurring in type-$i$ nodes is equal to the ratio of the $k$-th term to the sum of all terms, given by:

$$x_k^i = \frac{\exp(\sum_{j=1}^{E_k} p_{ij} l_{ij} + \sum_{j=1}^{E_k} p_{ij} l_{ij} + \sum_{j=1}^{E_k} p_{ij} l_{ij})}{f_i(e, P, e^q)} ,$$

the $k$-th term of $f_i(e, P, e^q)$ is related to type-k codewords. After restricting the search space, the terms of $f_i(e, P, e^q)$ related to nondominant $x_k^i = o(d^2)$ can be ignored. We define $\phi_i(V, Y, Z)$ as the modified weight enumerating function of type-$i$ nodes after eliminating the nondominant terms in $f_i(V, Y, Z)$. Then we define $\Phi(V, Y, Z) = \sum_{i=1}^{N_2} \gamma_i \phi_i(V, Y, Z)$, such that (21) and (22) can be rewritten as:

$$\hat{Z} = I_\gamma^{-1} \nabla_{\hat{Y}} \Phi(\hat{V}, \hat{Y}, \hat{Z})$$

$$\hat{Y} = I_\gamma^{-1} \nabla_{\hat{Z}} \Phi(\hat{V}, \hat{Y}, \hat{Z})$$

with $\hat{V} = e^u$, $\hat{Y} = (e^{P_1}, e^{P_2}, ..., e^{P_{E_1}})$, and $\hat{Z} = (e^{P_1}, e^{P_2}, ..., e^{P_{E_1}})$. Equations (31) and (32) can be considered as a fixed point problem. With $K = (\hat{Z}, \hat{Y}, \hat{V})$, let (31) and (32) be expressed as $K = \Theta_{\Phi}(K)$. Assuming the solution of $K$ converging to $K^*$ as $d \to 0$, we are required to find a fixed point approaching $K^*$. Let $J(\cdot)$ denote the Jacobian matrix. The existence of a fixed point is determined by the following theorem.

Theorem 4: (Fixed Points) There exists a fixed point $\hat{K}$ such that $|\hat{K} - K^*| = o(d)$ if and only if $J(\Theta_{\Phi}(K))|_{K = K^*}$ has an eigenvalue equal to 1.

Proof: There exists some $\hat{K}$ such that

$$\Theta_{\Phi}(\hat{K}) = \Theta_{\Phi}(K^*) + J(\Theta_{\Phi}(K))|_{K = K^*} (\hat{K} - K^*) + o(d^2)$$

$$= K^* + J(\Theta_{\Phi}(K))|_{K = K^*} (\hat{K} - K^*) + o(d^2)$$

$$= \hat{K} + o(d^2)$$

if and only if $J(\Theta_{\Phi}(K))|_{K = K^*}$ has an eigenvalue equal to 1.

The Jacobian matrix can be evaluated by:

$$J(\Theta_{\Phi}(K)) = \begin{bmatrix} P & Q \\ M & G \end{bmatrix}$$

with

$$P_{i,j} = \frac{1}{\sigma_i} \frac{\partial^2 \Phi(\hat{V}, \hat{Y}, \hat{Z})}{\partial Z_i \partial Y_j}$$

$$Q_{i,j} = \frac{1}{\sigma_i} \frac{\partial^2 \Phi(\hat{V}, \hat{Y}, \hat{Z})}{\partial Y_i \partial Z_j}$$

$$M_{i,j} = \frac{1}{\sigma_i} \frac{\partial^2 \Phi(\hat{V}, \hat{Y}, \hat{Z})}{\partial Y_i \partial Y_j}$$

$$G_{i,j} = \frac{1}{\sigma_i} \frac{\partial^2 \Phi(\hat{V}, \hat{Y}, \hat{Z})}{\partial Z_i \partial Z_j}$$

where $M$, $G$, $P$, and $Q$ are $|E| \times |E|$ matrices. The eigenvalues of $J(\Theta_{\Phi}(K))|_{K = K^*}$ depend only on $\hat{V}$. It follows that $\hat{V}$ can be solved by $\det(J(\Theta_{\Phi}(K))|_{K = K^*} - I_{2|E|}) = 0$, where $I_{2|E|}$ is a $2|E| \times 2|E|$ identity matrix. For evaluating (33), we are not required to decrease the dimension of $S(\hat{H})$ as small as possible, but at least all $h_\ell$ with a constant $\nabla_{h_\ell} R_T(x)$ should be contained in $S(\hat{H})$, such that replacing $\Phi(\hat{V}, \hat{Y}, \hat{Z})$ with $\Lambda(\hat{V}, \hat{Y}, \hat{Z})$ in (33) gives the same $J(\Theta_{\Phi}(K))|_{K = K^*}$. Due to Lemma 2, if there is no weight-1 codeword, eliminating the terms of $f_1(V, Y, Z)$ related to the codewords with weights more than 2 yields $J(\Theta_{\Phi}(K))$ depending only on $\hat{V}$. If only unpunctured MET LDPC codes are considered, $P = 0$ and $G = 0$, so that (33) is reduced to an eigenvalue problem with constant $Q$ and $M = \hat{M} \hat{V}$ with $[\hat{M}]_{i,j} = [\hat{M}]_{i,j} |_{\gamma = 1}$, given by:

$$\hat{Z} = \hat{V} \hat{M} \hat{Q} \hat{Z}$$

which identical to (13) in [3]. Finally, suppose $u^* = \ln \hat{V}^*$, where $\hat{V}^*$ is the minimal solution of $\hat{V}$. Ignoring the higher order terms in the above calculation, from (23) we have

$$\gamma(d) = -\alpha \tau d u^* + o(d^3) ,$$

with some $\epsilon > 1$. Furthermore, there exist some $d^* > 0$ such that if $d^* > 0$, there are exponentially few codewords of linear weights $dn$ for $d < d^*$. Also note that (35) must be equal to (28), so $\gamma d u^* = \mu^*$.

Example 1 continued: This code has weight-1 variable nodes so that Lemma 2 is not applicable, but considering $S(\hat{H})$, we have $\Phi(\hat{V}, \hat{Y}, \hat{Z}) = \frac{1}{2} \hat{V} Y_{(4, 7)} Y_{(5, 8)} + \frac{1}{2} \hat{V} Y_{(5, 7)} Y_{(5, 8)} + \frac{1}{2} \hat{Z}_{(7, 4)} Z_{(8, 4)} + \frac{1}{2} \hat{Z}_{(7, 5)} Z_{(8, 5)}$. With $\sigma_j = 1/9$ for $j = (4, 7), (4, 8), (5, 7),$ and $(5, 8)$, the Jacobian matrix is given by

$$Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE "GLOBECOM" 2009 proceedings.
in the column and row orders as (7, 4), (8, 4), (7, 5), and (8, 5),

\[
M = \begin{bmatrix}
0 & \bar{V} & 0 & 0 \\
\bar{V} & 0 & 0 & 0 \\
0 & 0 & 0 & \bar{V} \\
0 & 0 & \bar{V} & 0
\end{bmatrix}
\]

in the column and row orders as (4, 7), (4, 8), (5, 7), and (5, 8), \( P = 0 \), and \( G = 0 \). Since \( MQ = \bar{V} M \bar{I}_4 \), it is also reduced to an eigenvalue problem with four eigenvalues equal to 1. Thus \( u^* = 0 \), and \( \gamma(d) = o(d^*) \).

\[\square\]

**Example 3:** Consider a protograph-based DGLDPC code of the graph depicted in Fig. 2 with the generator matrix of type-1 to type-4 nodes as \( \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \) and type-5 nodes of (24, 23) SPC codes. The AWE curves of the DGLDPC, protograph (3, 24) LDPC, and protograph (4, 32) LDPC codes are shown in Fig. 3. For DGLDPC, \( d^* = 0.00246 \). For (3, 24) and (4, 32) LDPC codes, the values of \( d^* \) are 0.000224 and 0.002911, respectively. \[\square\]

**V. Conclusion**

We obtained the primal and dual problems of the AWE of MET codes. For small weight codewords, an optimization technique based on the primal and properties of the dual were presented. A condition for exponentially few small weight codewords was then derived. In this paper, a general framework to follow in order to obtain the AWE of MET codes was developed. While numerical techniques have been suggested to evaluate the general equations derived, closed form expressions based on these equations may be found if a particular class of codes is considered.

**References**