

Price-Bandwidth Dynamics for WSPs in Heterogeneous Wireless Networks

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Abstract

This paper presents a comprehensive approach to spectrum pricing and bandwidth management for wireless service providers (WSPs) in heterogeneous wireless networks. Most approaches to spectrum management focus on revenue maximization for the WSPs. However, the key issue of the competitive edge held by a WSP over the others (i.e., its market share) is hardly addressed. The market shares of the WSPs depend on the prices they advertise and the bandwidths they provide. We develop a three phase game between WSPs. The first phase called the WSP-WSP price game enables WSPs determine the optimal price they must advertise. In the second phase, called the WSP-WSP bandwidth game, the WSPs use the Nash equilibrium of the WSP-WSP price game, to determine the optimal bandwidths they should advertise in order to maximize their market share. Finally, in the third phase, we develop a WSP-WSP trading game model the fact that users that start off with a WSP can not only demand bandwidth from that WSP, but also demand bandwidth from other WSPs in order to make best use of the available bandwidth at all the WSPs.

Keywords: Heterogeneous wireless networks, market share, bandwidth procurement, pricing, spectrum sharing

1. Introduction

Heterogeneous wireless networks have attained popularity because they allow mobile device connection to different access networks [1]. This flexibility was further enhanced by the advent of dynamic spectrum access (DSA) [2] based cognitive radio networks [3] provides users greater flexibility in choosing any access network belonging to any service provider. Multiple wireless service providers (WSPs) dynamically provide services to end users with varying quality-of-service (QoS) requirements over heterogeneous wireless access networks, thus providing an open market environment

(“mobile-bazaar” [4]). This dynamic trading of radio spectrum brings about new challenges because service providers can set different prices and advertise different amount of bandwidths, thus creating more choices for users.

Most of the current literature on spectrum pricing and bandwidth management in heterogeneous wireless networks focus on the fairness for the users or throughput maximization or the revenue earned by the WSPs [9]-[13]. However, in a competitive environment, the WSPs not only desire large revenues but also wish to hold a competitive edge over the other WSPs [5]. This competitive edge is determined by the *market share*, which is defined as *the percentage of an industry or market’s total sales (or revenue) that is earned by a particular firm. Market share of a firm is calculated by taking the firm’s revenue over a period and dividing it by the total revenue generated by all the firms in the market, over the same period. This metric is used to give a general idea of the relative impact a firm has on the market, with respect to its competitors [5].* Moreover, the market share metric also is less dependent upon macro-environmental variables such as the state of the economy or changes in tax policy [5]. We therefore study management of resources and price advertisements by the WSPs so that they obtain best market share. There have been studies in economics *e.g.*, [6]-[8], that have emphasized on market share as the objective than just the revenues. These include applications like airline strategies [6], or incentive-based market shares [7] and more recently, for motion pictures [8]. However, to the best of our knowledge, a comprehensive study on heterogeneous wireless network environments that addresses the pricing policies and resource allocation to maximize the market shares of the WSPs has not been done yet.

This paper presents such an attempt to study the market share maximization for WSPs in heterogeneous wireless networks. The market share of the WSPs depend on their revenues, which, in turn, depends on the amount of bandwidth available at the WSPs and the price per unit bandwidth set by the WSPs. Hence, the objective of the WSPs is to advertise the right price per unit bandwidth as well as the right amount of bandwidth, which will result in a revenue that maximizes their market share. These objectives are achieved in two phases.

- i. First, the WSPs set the price per unit bandwidth that maximizes their determining the optimal revenues which provide the best profit margins for the WSPs while not losing out customers (or users) to other WSPs. This is achieved by formulating a non-cooperative *WSP-WSP price game*, whose Nash equilibrium provides the required optimal prices.
- ii. In the second phase, WSPs make use of the optimal price per unit bandwidth given by the Nash equilibrium of the *WSP-WSP price game*, to determine the optimal bandwidth that they must advertise, in order to yield the best market share. A non-cooperative *WSP-WSP bandwidth game* is developed to achieve this, whose Nash equilibrium yields the required solution. At this stage, the inter-dependency between the Nash equilibrium of the *WSP-WSP price game* and that of the *WSP-WSP bandwidth game*, is also established.

Finally, we discuss a third phase which accounts for the fact that users who choose a particular WSP to begin with, may later decide to either continue staying with the

same WSP or move to another WSP depending on the experienced congestion and on the price advantages they obtain. We determine the optimal amount of bandwidth expended by WSPs for users who continue to stay and that for the users who move from other WSPs, by developing a non-cooperative *WSP-WSP trading game*.

The rest of the paper is organized as follows. The current related literature is presented in Section 2. Section 3 describes problem definition, the non-cooperative games between the WSPs to advertise the prices and bandwidth and to determine the amount of bandwidth expended on users that stay with and those that move from other WSPs. Sections 4 and 5 provide the numerical results and conclusions, respectively.

2. Related Work

Spectrum trading and pricing in heterogeneous wireless networks has been researched widely, e.g., [9]-[13]. Pricing was introduced to control the selfish behavior of users and preventing from hoarding resources [9]. Huang and Mang [10] present a bandwidth management and disposition method using an upgrade and downgrade rank for users depending on their requests and the bandwidth they actually obtain. Bandwidth reservation using multiple classes of utility functions for multiple traffic classes was proposed in [12]. Bandwidth reservation for multiple classes of traffic with varying quality-of-service (QoS) requirements was studied in [11]. A fairness index based on the moment of skewness of the throughputs was studied and bandwidth was allocated to maximize the fairness index. In both the approaches mentioned above, users requesting low priority classes of traffic were starved off bandwidth to allocate bandwidth to those requesting traffic of higher priority class.

Networks with service providers have typically been analyzed from the service providers' perspective to determine the optimal pricing policies. In such competitive markets, the presence of multiple players (internet service providers (ISPs) or wireless service providers (WSPs)) results in an inter-dependence between the strategies of the players. As an example, Shakkottai and Srikanth [14] demonstrated a price war between multiple internet service providers. This inter-dependence motivated the study of game theoretic models for resource allocation (e.g., [15]-[17]) and to determine the pricing policies (e.g., [18]-[30] and the references therein).

A two-tier non-cooperative game theoretic auction model for differentiated services on the Internet, was presented in [18]. In [19], Yaiche *et al* present a Nash bargaining solution in a cooperative game, that maximizes the revenue for the network in the presence of elastic traffic demands from the users. A Nash bargaining spectrum pricing solution for determining the optimal relay-transmitter pair in power constrained wireless networks was presented in [20]. Altman *et al* [21] formulated a Stackelberg game between a service provider and users with differentiated services. In [22], Furuhata *et al* proposed a game among competing retailers and studied the impact of the game with respect to the allocation strategies applied between retailers and suppliers.

Further studies on price and bandwidth competition can be found in [23]-[29]. Jia and Zhang [23] proposed a non-cooperative game with complete information among the WSPs to model the price and bandwidth competition. Rouskas *et al* [24] developed an extensive form game to maximize the revenue of the service providers as well as model the satisfaction of the current and future users and the reputation of the service

providers. Niyato *et al* present an auction model for network selection in IEEE 802.22 based DSA networks in [25]. The pay off for each user depends on the frame error rate driven QoS requirements. Zhang and Zhang [26] proposed a cooperative game among the secondary users acting as relay nodes for the primary users. In [27], Sengupta and Chatterjee combined a non-cooperative game with a fractional knapsack approach to maximize the revenue for the WSPs. Wu *et al* [28] presented a multi-winner auction game with semi-definite programming for auctioning spectrum that can be shared by multiple secondary users in DSA networks. Flat-rate and usage based pricing strategies were discussed in [29].

The studies listed above focus on the revenue generated by the WSPs. The market share which denotes the competitive edge of a WSP over others was not discussed. We discuss the one of the first approaches to spectrum pricing and bandwidth management in heterogeneous wireless networks that maximize their market share. The market share depends on the revenue generated by the WSPs, which, in turn, depends on the price set by the WSPs and the bandwidth available at the WSPs. We present a two phase game to maximize the market share. In the first phase, the WSPs determine the optimal price they must advertise and in the second phase, the WSPs determine the optimal bandwidth they must advertise in order to maximize their market shares.

3. Price and Bandwidth Management

3.1. Problem Definition

Consider a scenario with N competing WSPs, w_1, w_2, \dots, w_N , each providing services over several heterogeneous wireless networks. As an example, a service provider could provide users with access over a 3G wireless network, a Wi-Fi network, a WiMAX access network, or an IEEE 802.22 WRAN. It is also possible that the network under consideration is a DSA based cognitive radio network and the users are secondary users trying to choose between different service providers. Each WSP advertises the available bandwidth and the price per unit bandwidth. The prices and bandwidths advertised (which is also the bandwidth procured) by the service providers should result in a revenue that maximizes their net utility, i.e., their market share for the cost they incur. The objectives of advertising the right price and bandwidth are then obtained as follows.

- I The WSPs should first determine the optimal prices they must advertise to obtain the best profit margins while still not losing out to other WSPs due to price competition.
- II The WSPs should then procure bandwidths that provides them with the best market share for the cost they incur. They advertise this bandwidth to the users.
- III Users then choose a particular WSP based on the advertised prices and bandwidths. Later, these users can decide to stay with the same WSP or switch to another WSP depending on the congestion levels at the WSPs and the prices advertised. WSPs should then manage their bandwidths optimally between users that decide to switch from other WSPs and those that initially chose the same WSP and decided to stay.

It is of interest to achieve the objectives listed above in three phases, by formulating three non-cooperative games between the WSPs.

- (i) The first phase or the first non-cooperative game is the *WSP-WSP price game* in which WSPs determine the optimal prices they advertise. This achieves objective I.
- (ii) Then a *WSP-WSP bandwidth game* is developed (in the second phase), in which WSPs determine the optimal bandwidths they procure and advertise, to achieve objective II. The net-utility function and the Nash equilibrium of the WSP-WSP bandwidth game will depend on the Nash equilibrium of the WSP-WSP price game.
- (iii) Finally, in the third phase, we develop a *WSP-WSP trading game* in which we address objective III. WSPs determine the optimal bandwidths they expend to serve users that stay with them and users that switch from other WSPs, so that the users obtain best payoff. The objective and the Nash equilibrium of this game will depend on the Nash equilibrium of the WSP-WSP price game and the WSP-WSP bandwidth game.

Remark: It is observed that in general, users choose service providers based on additional factors like user demand, signal propagation characteristics, user density, etc. Incorporating all these factors results in a very complex formulation and is beyond the scope of this paper. Moreover, the focus of this paper is on price and bandwidth advertisement by the WSPs. It is noted that the price per unit bandwidth and the total available bandwidth advertised by WSPs should be independent of the signal quality experienced by the users. In our analysis here, we focus on WSPs whose signal qualities are all acceptable to the user. The WSP-WSP trading game discussed in Section 3.4 takes into account users' demand and bandwidth availability at the WSPs.

3.2. *WSP-WSP price game*

In this subsection, we present the analysis for the optimal prices that the WSPs should advertise. The WSPs have an incentive in advertising as large a price per unit bandwidth as they desire because that gives them higher revenue (and higher profit margin) for the same bandwidth used by the users. However, when WSPs increase the price indefinitely, it causes users to leave the WSP and choose another WSP that offers a lesser price per unit bandwidth, i.e., due to price competition. Hence, WSPs must advertise a price which is not only large enough so that they obtain large revenue and profit but also low enough so that they not to lose users to other WSPs. This is achieved by defining suitable value functions for the WSPs that depend on the advertised price per unit bandwidth.

Users choose WSP i over WSP j , for two reasons, 1) when they find another WSP i offers a lower price than WSP j and 2) an aversion that occurs due to previous bad experience (like loss of quality-of-service (QoS) due to congestion) with the WSP j . Let p_i denote the price per unit bandwidth advertised by the i^{th} WSP and let θ_i denote

the aversion of a user to the i^{th} WSP¹. The value, V_i , perceived by the i^{th} WSP, which is a function of the advertised price per unit bandwidth, p_i , should satisfy the following properties [5]

- V_i should be an increasing function of p_i , with $V_i(0) = 0$. This is because, each WSP wishes a larger profit margin and therefore perceives larger value when the price per unit bandwidth is larger. Also a WSP that does not price users at all should perceive zero value.
- V_i should be a concave function to satisfy the law of diminishing marginal utility [5].
- V_i should also be a function of the price advertised by the other WSPs because the overall market share of a WSP is also dependent on the price advertised by other WSPs. Moreover, V_i should be a decreasing function of p_j because an increased p_j results in an increased revenue for the j^{th} WSP thereby decreasing the market share of the i^{th} WSP.

In order to satisfy the requirements mentioned above, we formulate V_i to be²

$$V_i = \Omega_i \ln \left(1 + \frac{p_i}{\alpha_i} \right), \quad (1)$$

where Ω_i is a scaling factor to make the formulation generalized (a special case being $\Omega_i = 1$),

$$\alpha_i \triangleq \delta_i + \theta_i, \text{ where} \quad (2)$$

$$\delta_i \triangleq \sum_{j \neq i} p_j. \quad (3)$$

The expressions in (1) and (2) can be interpreted as follows. The term, $\frac{p_i}{\alpha_i} = \frac{p_i}{\sum_{j \neq i} p_j + \theta_i}$, is like the signal-to-interference noise ratio (SINR) [31]. The term p_i in the value, V_i is like a “signal” which enhances the value of the i^{th} WSP, the term $\sum_{j \neq i} p_j$ denotes the market share competition from other WSPs which is like “interference” to the i^{th} WSP and the term θ_i , which is the aversion of a user to the i^{th} WSP is like the “white-noise”. Then the expression in (1) is like the Shannon’s channel capacity function [32].

Since V_i is an increasing function of p_i , it is maximized when $p_i \rightarrow \infty$. This indicates that WSPs obtain best benefit by indefinitely increasing the prices they advertise, In order to prevent such an indefinite increase in p_i , we introduce a penalty function, $\mu_i(\mathbf{p})$ for each WSP. The penalty, μ_i , incurred by the i^{th} WSP should increase when p_i increases. However, if the i^{th} WSP increases the price per unit bandwidth in response to the increase in the price advertised by another WSP, then the penalty incurred should be lower than that incurred by a WSP that increases the advertised price unilaterally even when the other WSPs do not do so. In other words, for the same advertised price,

¹The set of variables and notations used in this paper are listed in Table 1.

²This choice for V_i is not unique. Discussion on other possible value functions are beyond the scope of this paper.

p_i , the i^{th} WSP should incur lesser penalty if some other WSP increases its advertised price. One possible penalty function is³

$$\mu_i = \frac{p_i^2}{p_i + \alpha_i} = \frac{p_i^2}{p_i + \sum_{j \neq i} p_j + \theta_i}. \quad (4)$$

The net value perceived by the i^{th} WSP, \hat{V}_i , can then be written according to (5), from

$$\begin{aligned} \hat{V}_i &= V_i - \mu_i = \Omega_i \ln \left(1 + \frac{p_i}{\alpha_i} \right) - \frac{p_i^2}{p_i + \alpha_i} = \\ &\Omega_i \ln \left(1 + \frac{p_i}{\sum_{j \neq i} p_j + \theta_i} \right) - \frac{p_i^2}{p_i + \sum_{j \neq i} p_j + \theta_i} \end{aligned} \quad (5)$$

which it is noted that $\lim_{p_i \rightarrow \infty} \hat{V}_i = -\infty$. This happens because of the exponent, 2 for p_i in (4). The significance of this, is that it prevents WSPs from increasing the price indefinitely.

The i^{th} WSP then advertises a price p_i^* that maximizes the net value, \hat{V}_i . The net value perceived by the i^{th} WSP depends on the price strategy of *all the WSPs* and hence, the net value maximization problem can be modeled as a non-cooperative game. This game is called the *WSP-WSP price game*. The set of players are the WSPs and the set of strategies is the prices the WSPs advertise. We now present the analysis to obtain the unique Nash equilibrium of the WSP-WSP price game. It is observed from (5) that

$$\frac{\partial \hat{V}_i}{\partial p_i} = \frac{\Omega_i}{(p_i + \alpha_i)} + \frac{\alpha_i^2}{(p_i + \alpha_i)^2} - 1 \quad (6)$$

$$\text{and } \frac{\partial^2 \hat{V}_i}{\partial p_i^2} = -\frac{\Omega_i}{(p_i + \alpha_i)^2} - 2\frac{\alpha_i^2}{(p_i + \alpha_i)^3} < 0. \quad (7)$$

Therefore, \hat{V}_i is a concave function of p_i and the Nash equilibrium is unique if it exists [5]. The Nash equilibrium strategy, \mathbf{p}^* is obtained by solving the system of equations, $\left. \frac{\partial \hat{V}_i}{\partial p_i} \right|_{p_i=p_i^*} = 0, \forall i$, which, from (6), can be equivalent to solving the system of quadratic equations,

$$(p_i^* + \alpha_i)^2 - \Omega_i(p_i^* + \alpha_i) - \alpha_i^2 = 0, \quad \forall i, \quad (8)$$

$$\text{yielding } p_i^* = \alpha_i \left[\frac{(\xi_i - 2) + \sqrt{\xi_i^2 + 4}}{2} \right] = \alpha_i \lambda_i, \quad (9)$$

³There can be other choices for $\mu(\mathbf{p})$, the discussion of which are beyond the scope of this paper.

where $\xi_i = \Omega_i/\alpha_i$ and $\lambda_i \triangleq \left[\frac{(\xi_i-2)+\sqrt{\xi_i^2+4}}{2} \right]$. From (2), (3) and (9),

$$p_i^* - \sum_{j \neq i} \lambda_i p_j^* = \lambda_i \theta_i, \quad \forall i, \quad (10)$$

which, in turn, can be written as

$$\mathbf{A} \mathbf{p}^* = \mathbf{D}_\lambda \boldsymbol{\theta}, \quad (11)$$

where $\mathbf{D}_\lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ and $\boldsymbol{\theta} = [\theta_i]_{1 \leq i \leq N}$ and

$$\mathbf{A} = \begin{pmatrix} 1 & -\lambda_1 & -\lambda_1 & \cdots & -\lambda_1 \\ -\lambda_2 & 1 & -\lambda_2 & \cdots & -\lambda_2 \\ -\lambda_3 & -\lambda_3 & 1 & \cdots & -\lambda_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\lambda_N & -\lambda_N & -\lambda_N & \cdots & 1 \end{pmatrix}. \quad (12)$$

The WSP-WSP price game has a unique feasible Nash equilibrium if and only if there exists a positive vector⁴, \mathbf{p}^* , that satisfies (11). In order to present the necessary and sufficient conditions to obtain a unique Nash equilibrium for the WSP-WSP price game, we provide the following definition.

Definition 1. [33] An $n \times n$ matrix $\mathbf{B} = [b_{ij}]_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}}$ is called a \mathcal{Z} -matrix if the off-diagonal elements are non-positive, i.e., $b_{ij} \leq 0, \forall i \neq j$. A \mathcal{Z} -matrix, \mathbf{B} , is called an \mathcal{M} -matrix if \mathbf{B}^{-1} is non-negative.

A comprehensive study on \mathcal{M} -matrices (including the following lemma) can be found in [34].

Lemma 1. [34] The following statements are equivalent for any $n \times n$ \mathcal{Z} -matrix, \mathbf{B} .

1. \mathbf{B} is an \mathcal{M} -matrix.
2. \mathbf{B}^{-1} exists and is positive.
3. All principal and leading principal minors of \mathbf{B} are positive.
4. \exists a positive vector \mathbf{y} such that $\mathbf{B}\mathbf{y}$ is positive.

It is observed that the matrix, \mathbf{A} in (12) is a \mathcal{Z} -matrix according to Definition 1. Lemma 1 can then be used to provide the following theorem, which provides a necessary and sufficient condition for the existence of a unique Nash equilibrium for the WSP-WSP price game.

Theorem 1. *The WSP-WSP price game has a unique Nash equilibrium if and only if \mathbf{A} in (12) is an \mathcal{M} -matrix.*

⁴A vector or a matrix is **positive** if all elements in the vector or matrix are positive. Negative, non-negative and non-positive vectors and matrices can be defined in a similar manner.

Proof: The necessary part follows from 4) in Lemma 1. The sufficient part is established as follows. Rewriting (11),

$$\mathbf{p}^* = \mathbf{A}^{-1} \mathbf{D}_\lambda \boldsymbol{\theta}. \quad (13)$$

When \mathbf{A} is an \mathcal{M} -matrix, \mathbf{A}^{-1} is non-negative, *i.e.*, all the terms in the right hand side of (13) are non-negative and hence, \mathbf{p}^* is non-negative. The uniqueness follows from the fact that \mathbf{p}^* is obtained from (13) by inversion of a non-singular matrix. \square

While, Theorem 1 does establish conditions on \mathbf{A} to result in a unique Nash equilibrium, it is still not evident at this point, how the values of λ_i 's (and in turn, ξ_i 's and Ω_i 's) affect the existence of a unique feasible Nash equilibrium for the WSP-WSP price game. The following lemma and theorem provide conditions on these parameters that will result in a unique Nash equilibrium for the WSP-WSP price game.

Lemma 2. Sherman-Morrison Formula [33] Consider an $n \times n$ non-singular matrix, \mathbf{B} . Let \mathbf{g} and \mathbf{h} be $n \times 1$ vectors, such that $\mathbf{h}^T \mathbf{B}^{-1} \mathbf{g} \neq -1$ ($[\cdot]^T$ represents transpose of a vector or matrix). Then

$$(\mathbf{B} + \mathbf{g} \mathbf{h}^T)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1} \mathbf{g} \mathbf{h}^T \mathbf{B}^{-1}}{1 + \mathbf{h}^T \mathbf{B}^{-1} \mathbf{g}}. \quad (14)$$

Theorem 2. \mathbf{A} in (12) is an \mathcal{M} -matrix if and only if

$$\sum_{i=1}^N \frac{\lambda_i}{1 + \lambda_i} < 1. \quad (15)$$

Proof: Note that the matrix, \mathbf{A} in (12) can be written as

$$\mathbf{A} = \tilde{\mathbf{D}}_\lambda - \mathbf{1} \mathbf{h}^T, \quad (16)$$

where $\tilde{\mathbf{D}}_\lambda = \text{diag}(1 + \lambda_1, 1 + \lambda_2, \dots, 1 + \lambda_N)$, $\mathbf{1}$ is the vector of all ones and $\mathbf{h}^T = [\lambda_1 \ \lambda_2 \ \dots \ \lambda_N]$. Using the Sherman-Morrison formula (*i.e.*, (14) in Lemma 2),

$$\mathbf{A}^{-1} = \tilde{\mathbf{D}}_\lambda^{-1} + \frac{\tilde{\mathbf{D}}_\lambda^{-1} \mathbf{1} \mathbf{h}^T \tilde{\mathbf{D}}_\lambda^{-1}}{1 - \mathbf{h}^T \tilde{\mathbf{D}}_\lambda^{-1} \mathbf{1}}. \quad (17)$$

Since $\tilde{\mathbf{D}}_\lambda^{-1}$ is a diagonal matrix with non-negative elements, the elements of \mathbf{A}^{-1} , given by (17), are all non-negative, *i.e.*, \mathbf{A} is an \mathcal{M} -matrix, if and only if $1 - \mathbf{h}^T \tilde{\mathbf{D}}_\lambda^{-1} \mathbf{1} < 1$, which simplifies to (15). \square

Theorems 1 and 2 together imply that the condition in (15) is the necessary and sufficient condition for the existence of a unique Nash equilibrium for the WSP-WSP price game. Theorem 2 also indicates that the values of Ω_i 's should be chosen so that λ_i 's satisfy (15). It is observed that if $0 < \xi_i \ll 1, \forall i$, then the condition in (15) is more likely to be satisfied. The Nash equilibrium prices, \mathbf{p}^* , obtained by solving (13) are then advertised by the WSPs. In the following subsection, we discuss the optimal bandwidths that should be advertised by the WSPs.

3.3. WSP-WSP bandwidth game

The set of players in the WSP-WSP bandwidth game is the set of WSPs and the strategy set for each WSP is the amount of bandwidth procured and advertised by each WSP. The net utility perceived by the i^{th} WSP is the difference between its market share and its cost incurred. According to the definition of market share in [5] mentioned in Section 1, it is the ratio of the revenue earned by the i^{th} WSP and the sum total of the revenues earned by all the WSPs. Each WSP procures and advertises a bandwidth, b_i . The i^{th} WSP incurs a cost c_i to procure a unit of bandwidth. The i^{th} WSP earns revenue, $\omega_i = p_i b_i$. The value of p_i is the optimal p_i^* , given by the Nash equilibrium of the WSP-WSP price game⁵. The utility obtained by the i^{th} WSP, U_i , is its market share, given by $U_i = \omega_i / \sum_{j=1}^N \omega_j$. Thus, the net utility of the i^{th} WSP, \hat{U}_i , can be written as

$$\hat{U}_i = U_i - c_i b_i = \frac{p_i b_i}{\sum_{j=1}^N p_j b_j} - c_i b_i. \quad (18)$$

It is observed that the net utility obtained by the i^{th} WSP not only depends on the strategy of the i^{th} WSP (*i.e.*, b_i), but also on the strategies of all the other WSPs (*i.e.*, $b_j, j \neq i$). This results in the non-cooperative game of complete information between the WSPs called the *WSP-WSP bandwidth game*. The optimal $b_i, \forall i$ (termed as b_i^*), which is determined by maximizing \hat{U}_i in (18), is then the Nash equilibrium of the WSP-WSP bandwidth game.

Applying the first order necessary condition to (18), b_i^* is obtained as the solution to

$$\left. \frac{\partial \hat{U}_i}{\partial b_i} \right|_{b_i=b_i^*} = \frac{\sum_{\substack{j=1 \\ j \neq i}}^N p_j b_j^*}{\left(\sum_{j=1}^N p_j b_j^* \right)^2} - c_i = 0, \quad \forall i \quad (19)$$

subject to the constraints $b_i^* \geq 0, \forall i$. From (19), we obtain $\frac{\partial^2 \hat{U}_i}{\partial b_i^2} = -\frac{2 \sum_{\substack{j=1 \\ j \neq i}}^N p_i p_j b_j}{\left(\sum_{j=1}^N p_j b_j \right)^3} < 0, \forall i$, when $b_i \geq 0$. Thus, \hat{U}_i is a concave function of b_i and b_i^* , which solves (19) subject to $b_i^* \geq 0, \forall i$, is a local as well as a global maximum point. In other words, according to [35], the WSP-WSP bandwidth game has a unique Nash equilibrium, $\mathbf{b}^* = [b_1^* \ b_2^* \ \dots \ b_N^*]^T$, obtained by numerically solving the system of N non-linear equations specified by (19). However, to study the effect of the incurred costs and the advertised prices on the bandwidth strategies of the WSPs, it is desired to obtain an expression that relates the vectors, $\mathbf{b}^*, \mathbf{p} = [p_i]_{1 \leq i \leq N}$ and $\mathbf{c} = [c_i]_{1 \leq i \leq N}$. Re-writing (19),

$$\left(\sum_{j=1}^N \omega_j^* \right)^2 - \beta_i \sum_{\substack{j=1 \\ j \neq i}}^N \omega_j^* = 0, \quad \forall N, \quad (20)$$

⁵Throughout Section 3.3, the value of p_i is the optimal p_i^* obtained according to the analysis in Section 3.2.

where $\beta_i \triangleq p_i/c_i$. Eqn. (20) can be written as

$$(\boldsymbol{\omega}^*)^T \mathbf{1} \mathbf{1}^T \boldsymbol{\omega}^* \mathbf{1} - \mathbf{D}_\beta (\mathbf{d} \mathbf{d}^T - \mathbf{I}) \boldsymbol{\omega}^* = \mathbf{0}, \quad (21)$$

where \mathbf{D}_β is the diagonal matrix $\text{diag}(\beta_1, \beta_2, \dots, \beta_I)$, $\mathbf{1}$ is the column vector in which all entries are one, $\mathbf{0}$ is the column vector in which all entries are zero and \mathbf{I} is the identity matrix. The optimal \mathbf{b}^* can therefore be obtained by solving for the optimal $\boldsymbol{\omega}^*$ that satisfies (21).

It can be easily verified the vectors, $\mathbf{y}_1 = \left[\frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \frac{1}{\sqrt{N}} \quad \dots \quad \frac{1}{\sqrt{N}} \right]^T$ and for $j = 2, 3, \dots, N$, $\mathbf{y}_j = \left[y_{1j} \quad y_{2j} \quad y_{3j} \quad \dots \quad y_{(N-1)j} \quad y_{Nj} \right]^T$, where

$$y_{kj} = \begin{cases} \frac{1}{\sqrt{j(j-1)}} & k < j \\ -\frac{j-1}{\sqrt{j(j-1)}} & k = j \\ 0 & k > j, \end{cases} \quad (22)$$

form a set of ortho-normal eigen vectors to the matrix, $\mathbf{d} \mathbf{d}^T$. The eigen value corresponding to \mathbf{y}_1 is N and those corresponding to $\mathbf{y}_2, \dots, \mathbf{y}_N$ are 0s. Let $\mathbf{P} = [\mathbf{y}_1 | \mathbf{y}_2 | \dots | \mathbf{y}_N]$. Then, \mathbf{P} is an orthogonal matrix and by orthogonality transformation,

$$\mathbf{P}^T \mathbf{1} \mathbf{1}^T \mathbf{P} = \mathbf{D} = \text{diag}(N, 0, 0, \dots, 0). \quad (23)$$

Let $\mathbf{x} = [x_1 \quad x_2 \quad x_3 \quad \dots \quad x_{N-1} \quad x_N]^T$. Since the eigen vectors of a matrix form a basis for the N -dimensional sub-space [33], the vector, $\boldsymbol{\omega}^*$, can be written as $\boldsymbol{\omega}^* = \mathbf{P} \mathbf{x}$. Combining this with (21) and (23), we obtain $\mathbf{x}^T \mathbf{D} \mathbf{x} \mathbf{1} - \mathbf{D}_\beta (\mathbf{1} \mathbf{1}^T - \mathbf{I}) \mathbf{P} \mathbf{x} = \mathbf{0}$, *i.e.*,

$$N x_1^2 - \beta_1 \frac{(N-1)}{\sqrt{N}} x_1 + \sum_{k=2}^N \frac{\beta_1}{\sqrt{k(k-1)}} x_k = 0, \quad (24)$$

$$N x_1^2 - \beta_j \frac{(N-1)}{\sqrt{N}} x_1 - \frac{(j-1)\beta_j}{\sqrt{j(j-1)}} x_j + \sum_{k=j+1}^N \frac{\beta_j}{\sqrt{k(k-1)}} x_k = 0, \quad j = 2, 3, \dots, N. \quad (25)$$

From (24) and (25), for $2 \leq k \leq N$, x_k can be written in terms of x_1 as in (26). From (24) and (26), the vector, \mathbf{x} , can be obtained in terms of $\beta_1, \beta_2, \dots, \beta_N$, according to (27). Using the facts $\boldsymbol{\omega}^* = \mathbf{P} \mathbf{x}$, $\omega_i^* = p_i b_i^*$ and $\beta_i = \frac{p_i}{c_i}$ in (27), the unique Nash equilibrium of the WSP-WSP bandwidth game can be obtained as

$$b_i^* = \frac{(N-1)}{c_i} \frac{c_i}{p_i} \left[\frac{\sum_{j=1}^N \frac{c_j}{p_j} - (N-1) \frac{c_i}{p_i}}{\left(\sum_{j=1}^N \frac{c_j}{p_j} \right)^2} \right]. \quad (28)$$

$$\frac{x_k}{\sqrt{k(k-1)}} = \frac{Nx_1^2}{k(k-1)} \left[\frac{k}{\beta_k} + \sum_{j=k+1}^N \frac{1}{\beta_j} \right] - \frac{x_1}{\sqrt{N}} \frac{N(N-1)}{k(k-1)} \quad (26)$$

$$\frac{x_k}{\sqrt{k(k-1)}} = \frac{(N-1)^2}{k(k-1)} G^{-1} \left[G^{-1} \left(\frac{k}{\beta_k} + \sum_{j=k+1}^N \frac{1}{\beta_j} \right) - 1 \right], \quad x_1 = \frac{N-1}{\sqrt{N}} G^{-1}, \quad G \triangleq \sum_{j=1}^N \frac{1}{\beta_j} \quad (27)$$

Note that the unique Nash equilibrium \mathbf{b}^* , is feasible, *i.e.*, $b_i^* > 0, \forall i$ if and only if

$$(N-1) \frac{c_i}{p_i} < \sum_{j=1}^N \frac{c_j}{p_j}. \quad (29)$$

Thus, the following theorem can be stated.

Theorem 3. The WSP-WSP bandwidth game has a unique feasible Nash equilibrium if and only if (29) is satisfied.

Theorem 3 has the following interesting implications.

- The Nash equilibrium depends on the ratio $\frac{c_i}{p_i}, \forall i$. The condition in Theorem 3 indicates that it is desired that all WSPs should maintain a low value of $\frac{c_i}{p_i}$, which indicates cost incurred per unit advertised price. Intuitively this means that in order to obtain better benefits, WSPs should either incur low cost or set a high price (*i.e.*, obtain large profit).
- If one WSP has a very high value of $\frac{c_i}{p_i}$, it causes an infeasible equilibrium for that WSP. This is because, the particular WSP will not be able to receive enough revenue to offset the cost and eventually has to leave the market.
- From (20), the optimal revenue of a service provider depends on the $\frac{c_i}{p_j}$ of *all the service providers*. This is intuitively correct in a competitive market because the success of a firm not only depends the firm's strategy to manage its resources and revenue but also on the strategy of its competitors on terms of how they manage their resources and revenues.
- Also, (28) suggests that if $\frac{c_i}{p_i} = \frac{c_j}{p_j}, \forall i, j$, then $b_i^* = \frac{N-1}{N^2 c_i}$, *i.e.*, when all service providers incur same cost per unit revenue, then the service provider that incurs lesser cost procures more bandwidth for the same cost. This allows the WSP to advertise more bandwidth, which is also intuitively correct.

Remark 1: From Theorem 3, it appears that WSPs have a greater benefit in increasing p_i , *i.e.*, pricing users high. However, one must also bear in mind, the analysis in Section 3.2, which prevented WSPs from increasing the prices indefinitely. The condition in Theorem 3 provides another reason for WSPs to perceive larger value when pricing users higher. Here, the reason for pricing higher is not just incentive driven (*i.e.*, larger

profit margin) but also penalty driven (*i.e.*, danger of having to leave the market if p_i is too low).

Remark 2: Theorem 1 and 2 provide necessary and sufficient conditions for the existence of a feasible Nash equilibrium for the WSP-WSP price game. These theorems only impose conditions on Ω_i to result in $p_i^* > 0$. Theorem 3 imposes additional conditions on p_i^* and in turn, additional conditions on Ω_i to result in the existence of a unique feasible Nash equilibrium for the WSP-WSP bandwidth game in addition to the existence of a unique feasible Nash equilibrium for the WSP-WSP price game.

The price and bandwidth strategies of the WSPs indicate what the WSPs stand to obtain *if all their bandwidth are used up by the users belonging to them*. However, the users have a conflicting requirement. If all the bandwidth provided by the i^{th} WSP is used up by all the users, then the users connected to the i^{th} WSP experience high level of congestion i^{th} WSP and therefore risk experiencing lack of sufficient bandwidth for any future requests. Users would then like to choose WSPs so that they experience lesser congestion. However, this may cause them to pay a different price. In other words, users typically start off connecting to a specific WSP and then either decide to stay with the same WSP or switch to another WSP. WSPs should then manage their advertised/procured bandwidth, *from the users' perspective*. In other words, the i^{th} WSP should optimally allocate its bandwidth, b_i^* , among users that start with WSP i and stay with WSP i and those that start with other WSPs and later move to WSP i , so that the users perceive best payoff. In the following subsection, we discuss a *WSP-WSP trading game* to achieve this objective.

3.4. WSP-WSP trading game

Initially, users can choose WSPs based on advertised bandwidth and advertised price. However, it may be possible that due to increasing congestion at a WSP, users risk the possibility of finding insufficient bandwidth or loss of QoS for any future requests. Then, WSPs should also be allowed to trade spectrum with other WSPs to take care of the varying congestion levels. As an example, AT&T and T-Mobile struck a temporary roaming deal to jointly service users due to high call volume during hurricane Sandy [36]. Then users either decide to stick to the same WSP or move to some other WSP. Here by “sticking” to the same WSP, we do not imply physical stationarity but only imply the fact that the user may continue with the same WSP despite moving physically to another location. Similarly, by “moving to another WSP” we imply that a user connects to a different WSP even though the physical location may not necessarily change. This is represented in Fig. 1. This scenario is more likely in DSA based cognitive radio networks because of the high level of flexibility available to users in simultaneously choosing multiple WSPs as long as spectrum is available. In other words, users that begin connecting to the i^{th} WSP, not only utilize the bandwidth available at the i^{th} WSP, but can also utilize the bandwidth available at other WSPs.

Let s_i denote the amount of bandwidth expended towards users who decide to stick to the i^{th} WSP and $m_{i,j}$ denote the amount of bandwidth expended towards users who initially start with the i^{th} WSP and then choose to move to the j^{th} WSP. The i^{th} WSP must determine the optimal s_i and the optimal $m_{k,i}$, $k \neq i$. However, the i^{th} WSP must determine these quantities from *the users' perspective*. Therefore, it is of interest to obtain a payoff function from the users' perspective, *i.e.*, one which denotes the

$$\pi_i = (b_i - s_i - \sum_{k \neq i} m_{ki})s_i + \sum_{j \neq i} (b_j - s_j - \sum_{k \neq j} m_{kj})m_{ij} - p_i s_i - \sum_{k \neq i} p_k m_{ik} \quad (30)$$

net satisfaction obtained by the users. As mentioned in the previous subsection, the requirements of the users are contrary to the requirements of the WSPs. In particular, WSPs desire that all the bandwidth available with them be used by the users, while users desire that a small portion of the available bandwidth is used up, so that a large portion of the available bandwidth remains unused, whereby a user can experience lesser congestion, so that any future bandwidth requests may be met more easily. As an example, consider a user, u_i , who desires 10 Kbps of bandwidth from a WSP. If the total available bandwidth at the WSP is 10 Mbps and all but 15 Kbps are already used by other users, then the user, u_i experiences larger congestion for the same bandwidth demand, than when 1 Mbps of bandwidth is left unused.

Users starting with WSP i may choose to demand bandwidth from WSP $j \neq i$ under two circumstances, (i) when the congestion perceived at WSP j is lower for the same advertised price and/or (ii) the price paid at WSP j is lower for the same congestion level. The congestion level at WSP i is lesser if the residual or remaining bandwidth at WSP i is larger. The payoff function at WSP i from the users' perspective, π_i , should then satisfy the following properties

1. For the same bandwidth, s_i , a user served by WSP i should experience larger payoff, π_i when the congestion at WSP i is lesser, i.e., the residual bandwidth at WSP i is larger. The residual bandwidth at WSP i is the difference between the advertised bandwidth, b_i (given by computing b_i^* yielded by the WSP-WSP bandwidth game) and the bandwidth, already used up at the i^{th} WSP.
2. For the same bandwidth, m_{ij} , a user belonging to WSP i demands from WSP j , the payoff, π_i , should be larger when the congestion at WSP j is lower, i.e., the residual bandwidth at WSP j is larger.
3. The payoff, π_i should be larger when the price paid by the user at WSP i , p_i , (obtained by solving the WSP-WSP price game) is lower.
4. The payoff, π_i , should be larger when the price paid at WSP j , p_j , is lower.

Based on the above considerations, the payoff, π_i , for a users belonging to the i^{th} WSP can be written according to (30). The first two terms in (30) represent the increasing payoff due to lower congestion (larger residual bandwidths) and the last two terms represent the increasing payoff due to lower prices⁶. Note that the i^{th} WSP is interested in determining the optimal bandwidth, s_i^* , expended towards users who stay with it and the optimal bandwidth, m_{ji}^* , $j \neq i$, expended towards users that move into it from

⁶Throughout this subsection, the value of p_i is that given by p_i^* obtained from the WSP-WSP price game and the value of b_i is given by that of b_i^* obtained from the WSP-WSP bandwidth game.

other WSPs. However, since this has to be done from the users' perspective, one has to determine the optimal demand, s_i^* , users demand from the i^{th} WSP and the optimal demand, m_{ij}^* , users who decide to move from the i^{th} to the j^{th} WSP place at the j^{th} WSP, which can be obtained by solving

$$\left. \frac{\partial \pi_i}{\partial s_i} \right|_{s_i=s_i^*} = 0, \quad \left. \frac{\partial \pi_i}{\partial m_{ij}} \right|_{m_{ij}=m_{ij}^*} = 0, \quad \forall i, j. \quad (31)$$

Applying the condition in the above for π_i given by (30), results in a set of linear equations,

$$2s_i^* + \sum_{k \neq i} m_{ki}^* = b_i - p_i, \quad \forall i \text{ and} \quad (32)$$

$$2m_{ij}^* + s_j^* + \sum_{\substack{k \neq i \\ k \neq j}} m_{kj}^* = b_j - p_j - p_i, \quad \forall i, j \neq i. \quad (33)$$

The following definitions, lemmas and theorem will be used to solve the set of linear equations in (32) and (33).

Definition 2. [33] An $n \times n$ **permutation matrix**, \mathcal{Q} , is an $n \times n$ matrix whose entries are all 0's and 1's, such that every row and column has exactly one 1 in it.

Definition 3. An $n \times n$ permutation matrix, \mathbf{Q}_k , is called the k^{th} **circulant matrix**, if \mathbf{Q}_k is obtained by k cyclic permutations of the rows of the $n \times n$ identity matrix, \mathbf{I}_n .

The expressions in (32) and (33) can be combined to yield matrix equation,

$$\mathcal{A}\mathcal{S} = \mathcal{P}, \quad (34)$$

where, \mathcal{A} , \mathcal{S} and \mathcal{P} are defined according to (35) and (36), in which, \mathbf{I}_N is the $N \times N$ identity matrix, \mathbf{Q}_j is the j^{th} circulant matrix (defined in Definition 3) and $\mathbf{1}$ is the vector in which all entries are 1.

$$\mathcal{A} = \begin{bmatrix} 2\mathbf{I}_N & \mathbf{Q}_{N-1} & \mathbf{Q}_{N-2} & \cdots & \mathbf{Q}_2 & \mathbf{Q}_1 \\ \mathbf{Q}_1 & 2\mathbf{I}_N & \mathbf{Q}_{N-1} & \cdots & \mathbf{Q}_3 & \mathbf{Q}_2 \\ \mathbf{Q}_2 & \mathbf{Q}_1 & 2\mathbf{I}_N & \cdots & \mathbf{Q}_4 & \mathbf{Q}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{Q}_{N-2} & \mathbf{Q}_{N-3} & \mathbf{Q}_{N-4} & \cdots & 2\mathbf{I}_N & \mathbf{Q}_{N-1} \\ \mathbf{Q}_{N-1} & \mathbf{Q}_{N-2} & \mathbf{Q}_{N-3} & \cdots & \mathbf{Q}_1 & 2\mathbf{I}_N \end{bmatrix}. \quad (35)$$

In [37], we had solved an expression similar to (34). Using our analysis in [37], s_i^* and m_{ij}^* can be obtained as

$$s_i^* = \frac{b_i - p_i + \sum_{k \neq i} p_k}{N+1}, \quad \forall i \text{ and} \quad (38)$$

$$m_{ij}^* = \frac{b_j - Np_i - p_j + \sum_{\substack{k \neq i \\ k \neq j}} p_k}{N+1}, \quad \forall i, j \neq i. \quad (39)$$

$$m_{ji}^* = \frac{b_i - Np_j - p_i + \sum_{\substack{k \neq i \\ k \neq j}} p_k}{N+1}, \quad \forall i, j \neq i. \quad (40)$$

The following observations can be made from (38)-(40).

$$\mathcal{S} \triangleq \begin{bmatrix} \mathbf{s} \\ \mathbf{m}_1 \\ \mathbf{m}_2 \\ \vdots \\ \mathbf{m}_{N-1} \\ \mathbf{m}_N \end{bmatrix} \text{ and } \mathcal{P} \triangleq \begin{bmatrix} \mathbf{b} - \mathbf{p} \\ \mathbf{b}_{-1} - \mathbf{p}_{-1} - p_1 \mathbf{1} \\ \mathbf{b}_{-2} - \mathbf{p}_{-2} - p_2 \mathbf{1} \\ \vdots \\ \mathbf{b}_{-(N-1)} - \mathbf{p}_{-(N-1)} - p_{N-1} \mathbf{1} \\ \mathbf{b}_{-N} - \mathbf{p}_{-N} - p_N \mathbf{1} \end{bmatrix} \quad (36)$$

$$\mathbf{s} \triangleq \begin{bmatrix} s_1^* \\ s_2^* \\ \vdots \\ s_{i-1}^* \\ s_i^* \\ \vdots \\ s_N^* \end{bmatrix}, \quad \mathbf{m}_i \triangleq \begin{bmatrix} m_{i1}^* \\ m_{i2}^* \\ \vdots \\ m_{ii-1}^* \\ m_{ii+1}^* \\ \vdots \\ m_{iN}^* \end{bmatrix}, \quad \mathbf{b}_{-i} \triangleq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{i-1} \\ b_{i+1} \\ \vdots \\ b_N \end{bmatrix} \text{ and } \mathbf{p}_{-i} \triangleq \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{i-1} \\ p_{i+1} \\ \vdots \\ p_N \end{bmatrix} \quad (37)$$

- For a fixed b_i , when p_i is larger, s_i^* decreases and when p_i is smaller, s_i^* increases. This implies that for the same available bandwidth, when WSPs advertise larger price users tend not to stick to the i^{th} WSP and when the advertised price is smaller, then users tend to stick to the same WSP.
- For a fixed b_i , when p_k , $k \neq i$, is larger, s_i^* increases and when p_k is smaller, s_i^* decreases. This implies that for the same advertised bandwidth, if other WSPs advertise a larger price then users tend to stick to the same WSP.
- For a fixed p_i , when b_i is larger, s_i^* increases and when b_i is smaller, s_i^* decreases. This implies that for the same advertised price, when WSPs advertise larger available bandwidth, users stick to the WSP. When both b_i and p_i increases one cannot conclusively say what the users will tend to do.
- For a fixed b_j if p_i is large and p_j is small, then m_{ij}^* increases. In other words, when the advertised bandwidths are fixed, then users tend to move from a more expensive WSP to a less expensive one.
- For a fixed value of the advertised prices, m_{ij}^* increases if b_j increases. Therefore, users tend to move to WSPs who provide maximum bandwidth. Again, if price as well as bandwidth increase or decrease, one cannot conclusively say whether users will stick to a WSP or move to another WSP. Even in the events users move to another WSP, one cannot determine which particular WSP the users will be inclined to move to.

Remark 3: The interference conditions experienced by the users at different locations may also cause degradation in the signal quality due to propagation and interference conditions, which, in turn, affects the available bandwidth at each WSP. However, it is noted that the WSP-WSP trading game is modeled to yield the best utilization of the bandwidth available at all the WSPs. Hence, the computations are done assuming best signal quality experience for the users. When the same computations are to be made by the individual users, then b_i is replaced by \hat{b}_i , where \hat{b}_i represents the bandwidth

degraded by the poor signal quality experienced by the users. If a user cannot connect to WSP i due to policy reasons or due to very poor signal quality, \hat{b}_i is taken to be zero.

Remark 4: The games in Sections 3.2, 3.3 and 3.4 have to adapt to changes in the system. WSPs learn from the amount of bandwidth actually expended (*i.e.*, s_i^* and $m_{j_i}^*$), to modify Ω_i in (5). This, in turn, modify the values of ξ_i and λ_i . However, Ω_i is modified so that λ_i 's satisfy the condition in Theorems 1 and 2 and the optimal p_i^* satisfy the condition in Theorem 3. The modified Ω_i 's then result in different advertised price and different advertised bandwidth, according to which users make the choice for the appropriate WSP.

4. Results and Discussion

We consider a system with $N = 10$ WSPs. We discuss the numerical results in two parts. First, we compare the proposed game theoretic mechanism with other mechanisms used in the literature and demonstrate the effectiveness of our proposed game theoretic scheme. In order to compare, we consider two widely used pricing mechanisms- namely uniform pricing [10], where in, all WSPs advertise the same price and make the same revenue and hence obtain same market share. Another scheme we consider is that of proportional bandwidth advertisement [14], where in, revenue is proportional to the advertised price and in turn, advertised bandwidth is proportional to advertised price. The costs, c_i , $1 \leq i \leq N = 10$, were generated according to a uniform distribution. Fig. 2 presents the net utility obtained by WSPs under the proposed game theoretic mechanism and the other mechanisms, namely, uniform and proportional, mentioned above. The largest net utility obtained by a WSP across all the schemes is normalized to one. The WSPs are indexed in the decreasing order of c_i 's, *i.e.*, $c_i < c_j$ if $i > j$. As expected, WSPs whose incurred cost is lower, experience better net utility. It is also observed that the proposed mechanism yields 55% (for WSP 10) to 72% additional net utility (for WSP 1) compared to the other schemes.

It is also important to address the variable demands of users. In order to study the effect of growing demands we consider statistical models for different traffic classes as described in [11] and [12]. We consider two cases for users who do not find sufficient bandwidth-(*a*) a loss model where in, traffic that does not find sufficient bandwidth is *dropped* and (*b*) a delay model where in, traffic that does not find sufficient bandwidth is *buffered* to be transmitted later. We perform 100000 simulation experiments on UBUNTU linux platform to measure the ‘‘churning rate’’, defined as the probability that the fraction of dropped traffic exceeds 0.1% for the loss model and the fraction of traffic whose delay exceeds 10% of its mean delay. Fig. 3 presents the comparison of churning rates (fraction of dissatisfied users) experienced by different WSPs. It is observed that the proposed game theoretic framework results in almost equal churning rate for all WSPs. However, the proportional and uniform schemes result in low churning rate for WSPs who advertise high price (and those who attract small number of users) but a high churning rate for WSPs advertising low price where more users are likely to connect. Over all, the proposed scheme results in a reduced churning for the WSPs.

We now focus on the revenue generated by WSPs and bandwidth available to the users for various distribution of costs. It is noted from the analysis in Section 3.3,

that the revenue of WSP i , ω_i , has two components. Specifically, $\omega_i = p_i b_i$, where p_i is the advertised price and b_i is the advertised bandwidth. Therefore we study the behavior of the advertised price and advertised bandwidth. The performance metrics from the users' perspective would be the bandwidth available to the users they stay with the same WSP they start with and that available to users that move from one WSP to another. The costs, c_i , are generated at random according to (i) a uniform distribution, (ii) an exponential distribution and (iii) a normal distribution. We conduct about 100000 experiments on UBUNTU linux platform to compute the (average) variance in the following parameters

- the prices advertised by the WSPs⁷,
- the advertised bandwidths,
- the percentage of bandwidth expended towards users that stay with the WSP,
- the percentage of bandwidth expended towards users that move from other WSPs and
- the percentage of bandwidth left unused.

The behavior of these parameters provide indications on the consumer behavior and variations in the market.

Fig. 4 shows the behavior of the variance in the advertised price (Fig. 4(a)) and the advertised bandwidth (Fig. 4(b)) as a function of the variance in the advertised cost. It is observed that the variance in advertised price and bandwidth increase when the variance in the incurred cost increases. It is observed that the normal distribution exhibits largest variance in the price as well as the bandwidth. The reason for this is that, for the same variance, the Gaussian random process can have more samples in a wider dynamic range [38].

The reason for this behavior is as follows. A larger variation in the cost, c_i among the WSPs indicate a scenario in which some WSPs can procure bandwidths at very low prices while some are required to pay high prices. This results in some WSPs advertising very low prices and high bandwidths (particularly, the ones that incur low costs) and some advertising high prices and low bandwidths (the WSPs incurring higher costs). This causes more users to choose certain WSPs (the ones which advertise lower price) over others.

Fig. 5 shows the variation of the percentage in bandwidth expended towards users that stay with the same WSP, for the uniform, exponential (Fig. 5(a)) and the normal distribution (Fig. 5(b)). Fig. 6 depicts the percentage of bandwidth expended towards users moving from other WSPs or "roaming users", for the uniform, exponential (Fig. 6(a)) and the normal distribution of costs (Fig. 6(b)). It is observed that in the cases of the uniform and exponential distributions, the WSPs expend about 10% bandwidth towards users that stay and about 90% bandwidth towards roaming users. At low variances, the normal distribution of costs result in users expending most of the bandwidth towards users that stay. This is because, for low variance in costs, the Ω_i 's also did not

⁷Note that the price, p_i does not directly depend on c_i . However, to satisfy the condition in Theorem 3, we vary, Ω_i in (1).

vary much and hence, almost all users advertised comparable prices and bandwidths. Users did not have any incentive to move. However, at higher variances, the Gaussian distribution results in almost no bandwidth for users that stay and all bandwidth is only for roaming users. This also confirms the normal strategy in wireless networks that most revenue is generated by roaming users [39].

Finally, we also show the amount of bandwidth left unused by the WSPs in Fig. 7, for the uniform, exponential (Fig. 7(a)) and the normal distribution (Fig. 6(a)). It is observed that the percentage of bandwidth left unused is very negligible (less than 4%) for all the distributions. *This shows that our proposed scheme allows WSPs to earn maximum revenue (by using up all the available bandwidth, while yet providing highest utility for the users (because the used bandwidth was computed from the users' perspective).*

5. Conclusion

We presented a comprehensive approach to price and bandwidth management for wireless service providers in cognitive heterogeneous networks, which maximizes the market share for the WSPs. We also determined the optimal allocation of the bandwidth towards users who switch from other WSPs and users who stay with the same WSP. Some of the key results we observed include

- The proposed game theoretic framework yields 55-72% additional net utility to WSPs compared to other well known schemes.
- The proposed scheme results in lower overall churning rate compared to other well known schemes.
- Larger variation in costs result in very large price variations and hence more roaming users.
- WSPs and users find best benefit when more bandwidth is expended towards roaming users.
- The proposed mechanism encourages users to use as much bandwidth as possible to enjoy the best QoS while it results in maximum revenue for the WSPs that results in optimal market share.

The application of our analysis for cognitive radio networks for various primary secondary relationships, is a topic for future investigation.

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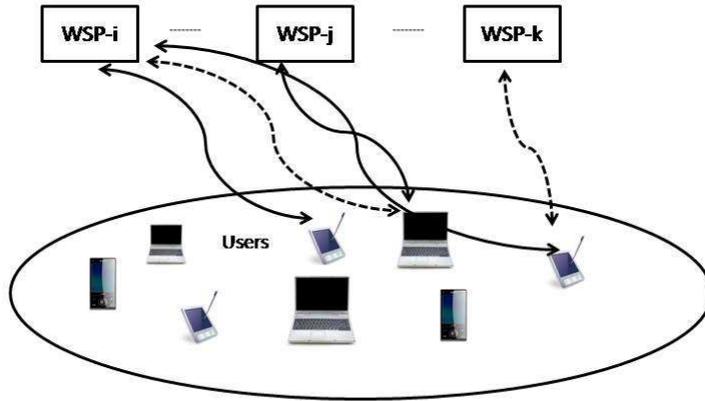
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Table 1: Terms/Variables/Notations used in the analysis in this paper.

Term	Description
N	Number of wireless service providers (WSPs)
θ_i	Aversion of a user to the WSP i
p_i	Price per unit bandwidth advertised by the WSP i
δ_i	A term representing $\sum_{j \neq i} p_j$
V_i	Value perceived by i^{th} WSP while advertising p_i
b_i	Bandwidth advertised by the i^{th} WSP
\hat{V}_i	Net value perceived by i^{th} WSP while advertising p_i
α_i	A term denoting $\delta_i + \theta_i$
U_i	Utility perceived by i^{th} WSP while advertising b_i
ω_i	Revenue earned by WSP i
\hat{U}_i	Net utility perceived by i^{th} WSP while advertising b_i
Ω_i	A scaling factor
c_i	Cost per unit bandwidth incurred by the i^{th} WSP
ξ_i	A term denoting Ω_i/α_i
π_i	Payoff obtained by users served by i^{th} WSP
$\mathbf{1}$	The all-one vector
s_i	Amount of bandwidth expended by the i^{th} WSP for users starting with and staying with the i^{th} WSP
\mathbf{I}	The identity matrix
m_{ij}	Amount of bandwidth expended by the j^{th} WSP for users moving from the i^{th} WSP to the j^{th} WSP
β_i	A term denoting p_i/c_i



↔ **Current WSP**
 ⇄ **Previous WSP**

Figure 1: Users staying with the same WSP or moving to another WSP. Here, there are users that stay with WSP i , users that move from WSP i to WSP j and users that move from WSP k to WSP i .

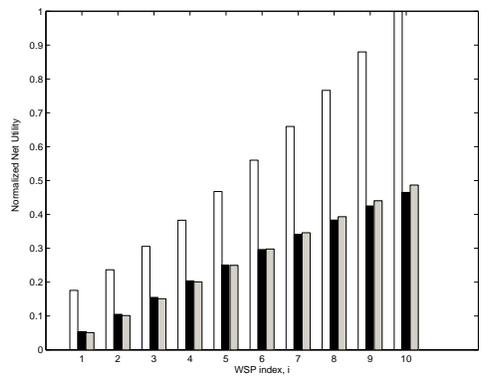
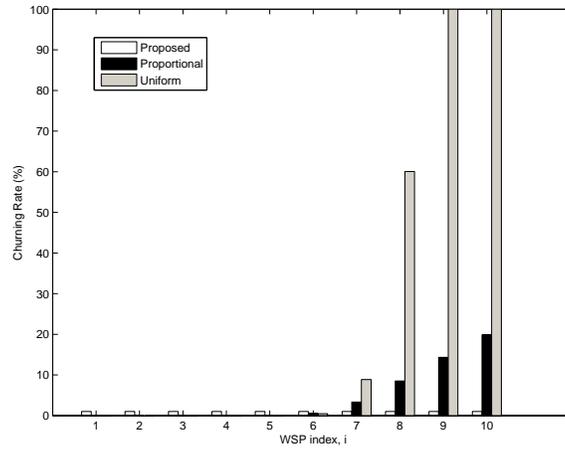
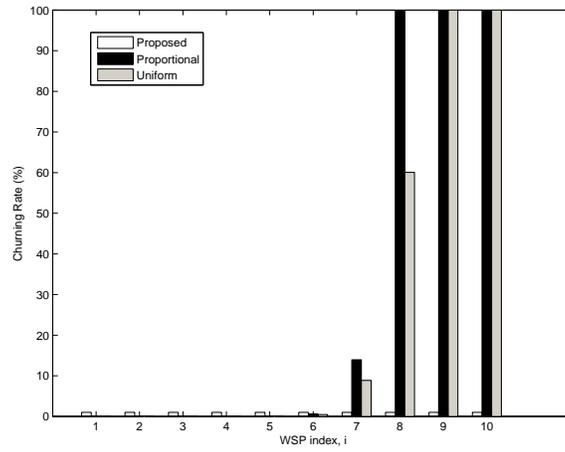


Figure 2: Net utility obtained by the WSPs when price and bandwidth are advertised according to the proposed scheme in comparison with other known schemes. The highest net utility obtained across all schemes is normalized to 1. WSPs are indexed so that $c_i < c_j$ when $i > j$.

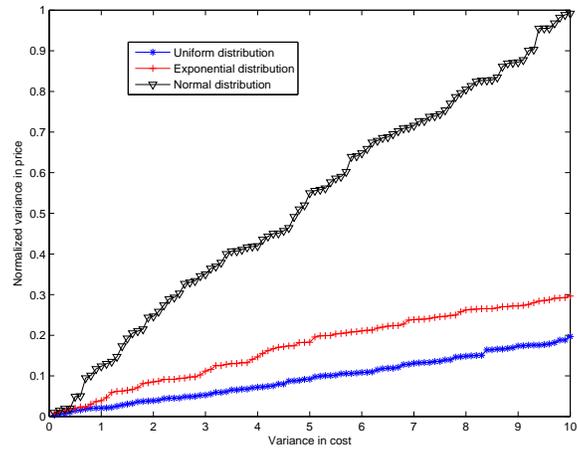


(a) Loss model

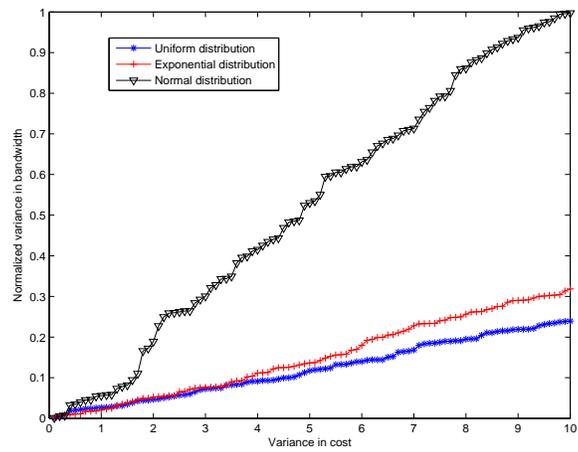


(b) Delay model

Figure 3: Churning rate experienced by WSPs. WSPs are indexed so that $c_i < c_j$ for $i > j$.

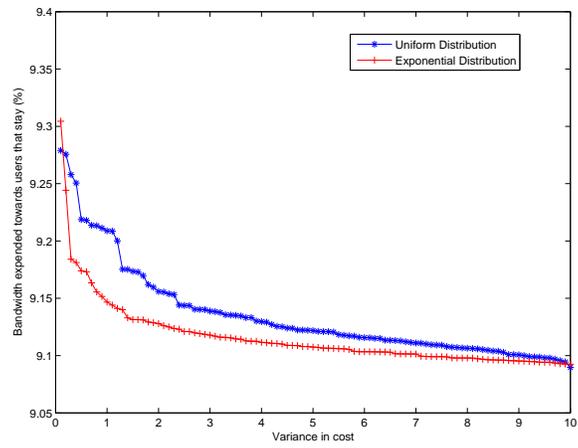


(a) Advertised price

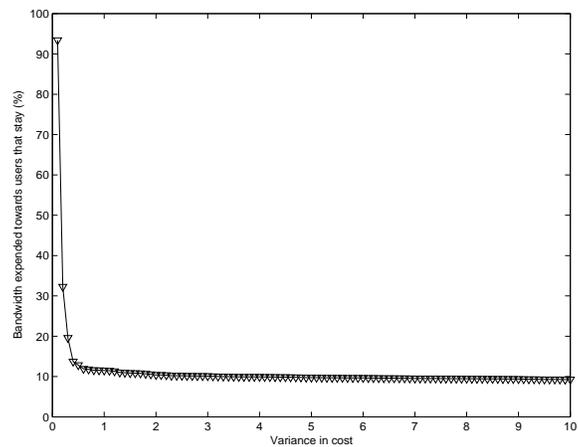


(b) Advertised bandwidth

Figure 4: Variance in the price and bandwidth with respect to the variance in the incurred cost.

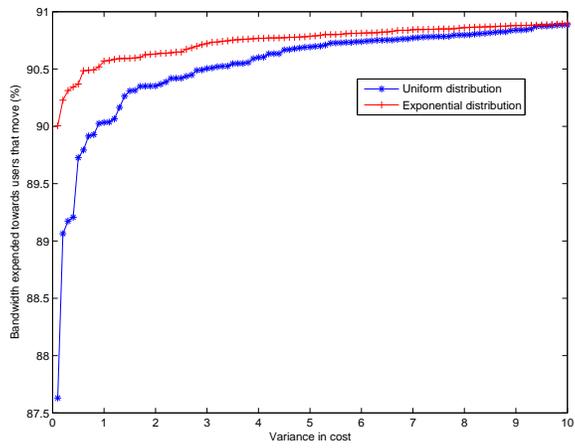


(a) Uniform and Exponential

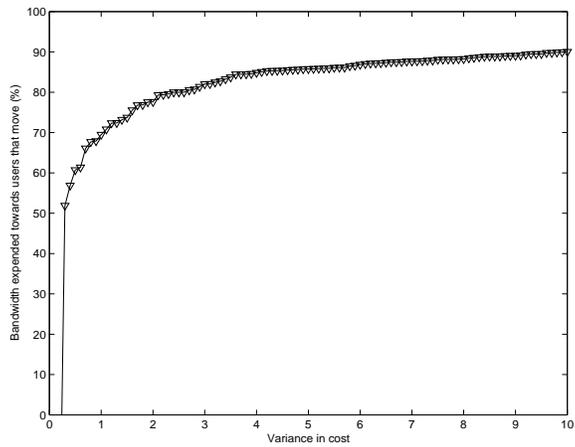


(b) Normal distribution

Figure 5: Variance in the percentage of bandwidth expended towards users that stay with the same WSP.

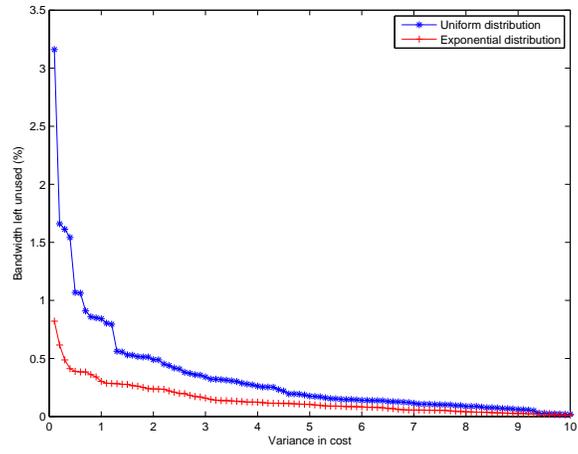


(a) Uniform and Exponential

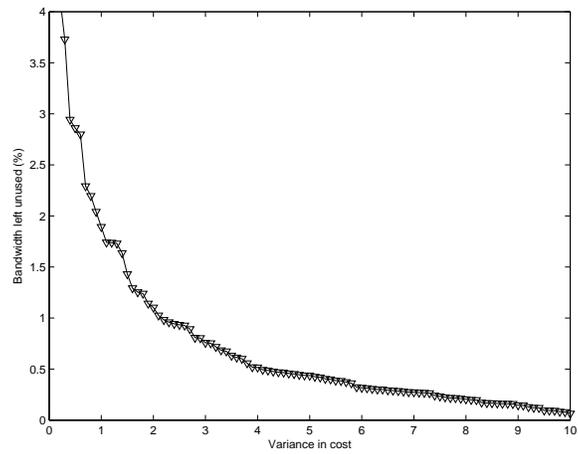


(b) Normal distribution

Figure 6: Variance in the percentage of bandwidth expended towards users that move from other WSPs.



(a) Uniform and Exponential



(b) Normal distribution

Figure 7: Variance in the percentage of bandwidth left unused.